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Optimal low symmetric dissipation Carnot engines and refrigerators

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A unified optimization criterion for Carnot engines and refrigerators is proposed. It consists of maximizing the product of the heat absorbed by the working system times the efficiency per unit time of the device, either the engine or the refrigerator. This criterion can be applied to both low symmetric dissipation Carnot engines and refrigerators. For engines the criterion coincides with the maximum power criterion and then the Curzon-Ahlborn efficiency $\eta_{\rm CA} = 1 - \sqrt{T_c/T_h}$ is recovered, where T_h and T_c are the temperatures of the hot and cold reservoirs, respectively [Esposito, Kawai, Lindenberg, and Van den Broeck, Phys. Rev. Lett. **105**, 150603 (2010)]. For refrigerators the criterion provides the counterpart of Curzon-Ahlborn efficiency for refrigerators $\varepsilon_{\rm CA} = [1/(\sqrt{1-(T_c/T_h)}] - 1$, first derived by Yan and Chen for the particular case of an endoreversible Carnot-type refrigerator with linear (Newtonian) finite heat transfer laws [Yan and Chen, J. Phys. D: Appl. Phys. **23**, 136 (1990)].

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The publication in 1975 of Curzon and Ahlborn's pioneering work [1] opened the perspective of establishing more realistic theoretical bounds for real energy converters, and gave rise to the birth and development of finite-time thermodynamics (FTT), a branch of thermodynamics devoted to extend classical reversible thermodynamics to include more realistic finite-time and finite-size (irreversible) processes. The main goal of FTT is to ascertain the best operating mode of heat devices with finite-time cycles. Basically, finite-rate constraints arising from several sources of irreversibility are modeled and then a suitable functional is optimized with respect to the involved parameters. In principle, one has the freedom to choose such a functional. This has lead to the proposal of a great variety of criteria based on thermodynamic, economic, compromise, and sustainability considerations [2–11].

A class of limit models into the FTT framework is the so-called endoreversible (or exoirreversible) models, in which all the accounted irreversibilities come from the couplings between the working system and the external heat reservoirs, whereas the Clausius equality (i.e., reversibility) holds for the cyclic system. These endoreversible models have suffered some criticisms precisely due to the internal reversible assumption, which is contradictory to the existence of external finite-area exchangers interacting with the internal working fluid across finite-temperature gaps [12] (see also Ref. [13]). In this sense the proposal of models, accounting for the irreversibility of real processes without the consideration of the endoreversibility hypothesis and capable of being analyzed by thermodynamic optimization methods, is especially valuable. Along this line, recently Esposito *et al.* [14] have proposed a model for low-dissipation Carnot engines which do not make use of the endoreversibility hypothesis. In this model the entropy generation in each heat-exchange process is assumed to be inversely proportional to the time duration of the process, and the reversible regime is approached in the limit of infinite times. They show for this model that the maximum power regime allows to recover the Curzon-Ahlborn efficiency $\eta_{\rm CA}$, when symmetric dissipation is considered, but without the requirement of assuming any specific-heat transfer law or the linear-response regime since the derivation is independent of external temperature values. We recall that symmetric low dissipation refers to an equal amount of heat dissipated at each thermal bath provided there is an equal time duration at each heat transfer.

Another important shortcoming of the endoreversible models is its lack of generality. This problem remains an open question provided that the optimization of the refrigeration power for an endoreversible Carnot refrigerator, with linear (Newtonian) finite-time heat transfers, cannot allow to obtain an analogous expression for the efficiency of the refrigerator to Curzon-Ahlborn's value for endoreversible heat engines. So, a number of different optimization criteria has been proposed for these kinds of models. Yan and Chen [15] reported an optimization study taking as the target function $\epsilon \dot{Q}_c$, where $\dot{Q}_c$ is the cooling power of the refrigerator, and ε is the usual coefficient of performance (COP) for refrigerators $\varepsilon = \dot{Q}_c/\dot{W}$, where $\dot{W}$ denotes the power imput. The optimized coefficient of performance they obtain depends only on $\tau \equiv T_c/T_h$ and was later independently reported by Velasco *et al.* [16,17] using a maximum per unit time COP, and very recently by Allahverdyan *et al.* [18] from a quantum model with two n-level systems interacting via a pulsed external field in the classical limit. Nevertheless, within the linear irreversible thermodynamics formalism the analysis of a specific working regime gave a different optimized coefficient of performance [19]. However, it has been claimed that none of those optimized COP could be considered as the equivalent to the Curzon-Ahlborn efficiency for endoreversible refrigeration cycles [15–19].

The objective of this work is twofold. First we show that the optimization criterion reported by Yan and Chen [15] for refrigerators is exactly the same as that for the optimization of power output for heat engines, when both are properly considered in terms of the working systems and total cycle time. This equivalence is universal, i.e., independent of any particular model. Second, we apply this unified figure of merit to a symmetric dissipation Carnot refrigerator to obtain the counterpart of the Curzon-Ahlborn efficiency for refrigerators.

To attain these goals, we first address the problem of considering a unified optimization criterion for both heat engines and refrigerators, and we focus our attention on the common characteristic of every energy converter, the cyclic working system, instead of any specific coupling to external heat sources which can vary according to a particular arrangement. Thus, we introduce a different figure of merit χ, defined as the product of the converter efficiency z times the heat absorbed by the working system $Q_{\rm in}$, divided by the time duration of cycle $t_{\rm cycle}$:

$$\chi = \frac{z Q_{\rm in}}{t_{\rm cycle}}. \tag{1}$$

For a Carnot-type engine $Q_{\rm in} \equiv Q_{\rm h}$, with $Q_{\rm h}$ the amount of heat absorbed from the hot reservoir by the working system, and $z \equiv \eta \equiv -W/Q_{\rm h}$. Then, from (1),

$$\chi^{(\rm E)} = \frac{\eta Q_{\rm h}}{t_{\rm cycle}} = -\frac{W}{t_{\rm cycle}}, \tag{2}$$

which shows that $\chi^{(\rm E)}$ coincides with the power output of the engine. On the other hand, for a Carnot-type refrigerator $Q_{\rm in} \equiv Q_{\rm c}$, with $Q_{\rm c}$ the amount of heat evacuated to the cold reservoir by the working system, and $z \equiv \varepsilon \equiv Q_{\rm c}/W$. Then, from (1),

$$\chi^{(\rm R)} = \frac{\varepsilon Q_{\rm c}}{t_{\rm cycle}}, \tag{3}$$

which shows that $\chi^{(\rm R)}$ coincides with the figure of merit first proposed by Yan and Chen [15] for refrigerators. In this simple way, χ includes, under a unified expression, the power output for power cycles and the the optimization criterion $\epsilon \dot{Q}_c$ for refrigerator cycles. This is the first main result of this Rapid Communication.

Obviously, when applied to the low-dissipation heat engine model, at maximum power the χ-criterion recovers the Curzon-Ahlborn value if dissipation is symmetric, as reported in Ref. [14]. The second goal of this Rapid Communication is the analysis of low-dissipation Carnot refrigerators, under the above unified perspective, by studying the maximum χ regime with the same symmetric conditions. This extends the work [14] for heat engines to refrigerators, overcoming the drawbacks of the endoreversibility hypothesis. Our starting point will be a Carnot refrigerator, for which all processes are reversible and therefore have a infinite-time duration. Then, the entropy change of the Carnot refrigeration device in a cycle $\Delta S^{(C)}$ must be zero, determining

$$0 = \Delta S^{(C)} = \Delta S^{(C)}_{T_{\rm h}} + \Delta S^{(C)}_{T_{\rm c}} \Rightarrow \Delta S \equiv \Delta S^{(C)}_{T_{\rm h}} = -\Delta S^{(C)}_{T_{\rm c}}, \tag{4}$$

where $\Delta S^{(C)}_{T_{\rm h}}$ and $\Delta S^{(C)}_{T_{\rm c}}$ are the entropy changes of the hot and cold reservoirs, respectively. Moving away from reversibility, for a Carnot-like refrigerator the times $t_{\rm h}$ and $t_{\rm c}$, associated with the heat exchanges between the working system and the hot and cold reservoirs, respectively, are finite. As in Ref. [14] we only assume here that the entropy generation in each of these processes is inversely proportional to the time of the process without any additional hypothesis about heat transfer laws or external heat bath temperatures. Then we get

$$\Delta S_{T_{\rm h}} = \Delta S + \frac{\Sigma}{t_{\rm h}}, \qquad \Delta S_{T_{\rm c}} = -\Delta S + \frac{\Sigma}{t_{\rm c}}, \tag{5}$$

where $\Delta S_{T_{\rm h}}$ and $\Delta S_{T_{\rm c}}$ are the entropy changes of the hot and cold reservoirs, respectively for the real Carnot-type refrigerator. Equations (5) assume symmetric dissipation by considering the same constant Σ for both processes, and allow to recover the reversible refrigerator in the limits $t_{\rm h} \to \infty$ and $t_{\rm c} \to \infty$.

The entropy changes of the hot $\Delta S_{T_{\rm h}}$ and cold $\Delta S_{T_{\rm c}}$ reservoirs are expressed as

$$\Delta S_{T_{\rm h}} = -\frac{Q_{\rm h}}{T_{\rm h}}, \qquad \Delta S_{T_{\rm h}} = -\frac{Q_{\rm c}}{T_{\rm c}}. \tag{6}$$

Thus, from Eqs. (5) and (6) these amounts of heat can be written:

$$Q_{\rm h} = T_{\rm h}\left(-\Delta S - \frac{\Sigma}{t_{\rm h}}\right), \qquad Q_{\rm c} = T_{\rm c}\left(\Delta S - \frac{\Sigma}{t_{\rm c}}\right). \tag{7}$$

The first principle of thermodynamics provides the amount of work needed by the device in every cycle and is determined by

$$W = -Q_{\rm h} - Q_{\rm c}, \tag{8}$$

while its efficiency is then given by

$$\varepsilon = \frac{Q_{\rm c}}{-Q_{\rm h} - Q_{\rm c}}. \tag{9}$$

The substitution of (7) in (9) allows one to obtain the expression of the efficiency of our refrigeration device:

$$\varepsilon = \frac{T_{\rm c}\left(\Delta S - \frac{\Sigma}{t_{\rm c}}\right)}{-T_{\rm h}\left(-\Delta S - \frac{\Sigma}{t_{\rm h}}\right) - T_{\rm c}\left(\Delta S - \frac{\Sigma}{t_{\rm c}}\right)}, \tag{10}$$

and also, from Eqs. (7), and assuming that the time duration of the cycle is $t_{\rm cycle} = t_{\rm h} + t_{\rm c}$, the refrigeration power R will be given by

$$R \equiv \dot{Q}_{\rm c} \equiv \frac{Q_{\rm c}}{t_{\rm cycle}} = \frac{T_{\rm c}\left(\Delta S - \frac{\Sigma}{t_{\rm c}}\right)}{t_{\rm h} + t_{\rm c}}. \tag{11}$$

Equations (10) and (11) show that, by fixing the temperatures of the hot and cold reservoirs $T_{\rm h}$ and $T_{\rm c}$, the present model depends on four parameters: the entropy change of the reservoirs $\pm\Delta S$; the constant characterization of the entropy generation of the reservoirs caused by the irreversibilities Σ; and the time durations $t_{\rm h}$ and $t_{\rm c}$ associated with the heat exchanges between the working system and the hot and cold reservoirs, respectively.

Thus, the substitution of Eqs. (10) and (11) in (3) determines that $\chi^{(\mathrm{R})}$ is a function of these four parameters $\chi^{(\mathrm{R})}(t_{\mathrm{h}},t_{\mathrm{c}};\Sigma,\Delta S)$. The optimal $\chi^{(\mathrm{R})}$ regime is obtained by maximizing this function with respect to the time durations t_{h} and t_{c}. This requires to fulfill the following conditions:

$$\begin{aligned}&\left(\frac{\partial\chi}{\partial t_{\mathrm{c}}}\right)(t_{\mathrm{c}}=t_{\mathrm{c}_{\max\chi}},t_{\mathrm{h}}=t_{\mathrm{h}_{\max\chi}})=0,\\&\left(\frac{\partial\chi}{\partial t_{\mathrm{h}}}\right)(t_{\mathrm{c}}=t_{\mathrm{c}_{\max\chi}},t_{\mathrm{h}}=t_{\mathrm{h}_{\max\chi}})=0,\\&\left\{\left(\frac{\partial^2\chi}{\partial t_{\mathrm{c}}^2}\right)\left(\frac{\partial^2\chi}{\partial t_{\mathrm{h}}^2}\right)-\left[\left(\frac{\partial^2\chi}{\partial t_{\mathrm{c}}\partial t_{\mathrm{h}}}\right)\right]^2\right\}\\&\quad(t_{\mathrm{c}}=t_{\mathrm{c}_{\max\chi}},t_{\mathrm{h}}=t_{\mathrm{h}_{\max\chi}})>0,\\&\left(\frac{\partial^2\chi}{\partial t_{\mathrm{c}}^2}\right)(t_{\mathrm{c}}=t_{\mathrm{c}_{\max\chi}},t_{\mathrm{h}}=t_{\mathrm{h}_{\max\chi}})<0.\end{aligned}\tag{12}$$

These conditions give rise to a unique, physically acceptable solution:

$$\begin{aligned}t_{\mathrm{c}_{\max\chi}}&=\frac{2\Sigma}{\Delta S}\left(1+\frac{1}{\sqrt{1-\tau}}\right),\\t_{\mathrm{h}_{\max\chi}}&=\frac{2\Sigma}{\Delta S}\frac{1}{\sqrt{1-\tau}}.\end{aligned}\tag{13}$$

Thus, the efficiency at maximum χ is obtained by the substitution of (13) in (10), leading to

$$\varepsilon_{\max\chi}=\frac{\tau}{1-\tau+\sqrt{1-\tau}}=\frac{1}{\sqrt{1-\tau}}-1\equiv\varepsilon_{\mathrm{CA}}.\tag{14}$$

Equation (14) allows one to recover the τ-dependent optimal efficiency obtained by Yan and Chen [15] for endoreversible refrigerators with linear finite-rate heat transfer laws, but here it is obtained without any specific-heat law, under symmetric conditions and when the ratio of contact times is given by $t_{\mathrm{c}_{\max\chi}}/t_{\mathrm{h}_{\max\chi}}=\sqrt{1-\tau}+1=\frac{1}{1+\epsilon_{\max\chi}}+1$ (for heat engines this ratio is $t_{\mathrm{c}_{\max\chi}}/t_{\mathrm{h}_{\max\chi}}=\sqrt{\tau}=1-\eta_{\max\chi}$; see Ref. [14]).

Given that the result in Eq. (14) for refrigerators has been obtained with the same model, under the same symmetric conditions, and under the same optimization criterion as the Curzon-Ahlborn value, it is appealing to be considered as the genuine counterpart value for refrigerators. This is the second main result of this Rapid Communication.

We close by comparing the observed results with those predicted by Eq. (14). For this we rewrite Eq. (14) in terms of the Carnot efficiency $\varepsilon_{\mathrm{C}}=\tau/(1-\tau)$ as $\varepsilon_{\max\chi}(\varepsilon_{\mathrm{C}})=\sqrt{1+\varepsilon_{\mathrm{C}}}-1$ [for heat engines $\eta_{\max\chi}(\eta_C)=1-\sqrt{\tau}=1-\sqrt{1-\eta_C}$; see Ref. [14]]. Then, the following limits hold:

$$\begin{aligned}&\lim_{\varepsilon_{\mathrm{C}}\to0}\frac{\varepsilon_{\max\chi}(\varepsilon_{\mathrm{C}})}{\varepsilon_{\mathrm{C}}}=\frac{1}{2},\\&\lim_{\varepsilon_{\mathrm{C}}\to\infty}\frac{\varepsilon_{\max\chi}(\varepsilon_{\mathrm{C}})}{\varepsilon_{\mathrm{C}}}=0,\end{aligned}\tag{15}$$

and, provided that $\varepsilon_{\max\chi}/\varepsilon_{\mathrm{C}}$ is a monotonous decreasing function of ε_{C} and $0<\varepsilon_{\mathrm{C}}<\infty$, we finally obtain

$$0\leqslant\varepsilon_{\max\chi}(\varepsilon_{\mathrm{C}})\leqslant\tfrac{1}{2}\varepsilon_{\mathrm{C}}\equiv\varepsilon^{\sup}_{\max\chi}(\varepsilon_{\mathrm{C}}).\tag{16}$$

TABLE I. Theoretical and experimental data for a high-temperature refrigerator [20].

T_{c} (K)	T_{h} (K)	ε_{C}	$\varepsilon_{\mathrm{exp}}$	$\varepsilon_{\mathrm{exp}}^{-1}$	$\varepsilon_{\max\chi}^{-1}$	$(\varepsilon^{\sup}_{\max\chi})^{-1}$
283	293	28.300	14.085	0.071	0.227	0.071
283	298	18.867	11.111	0.090	0.289	0.106
283	303	14.150	9.009	0.111	0.346	0.141
283	308	11.320	7.407	0.135	0.398	0.177
283	313	9.433	6.135	0.163	0.448	0.212
273	293	13.650	8.333	0.120	0.354	0.147
273	298	10.920	7.407	0.135	0.408	0.183
273	303	9.100	6.135	0.163	0.459	0.220
273	308	7.800	5.102	0.196	0.509	0.256
273	313	6.825	4.255	0.235	0.556	0.293
263	293	8.767	6.135	0.163	0.471	0.228
263	298	7.514	5.000	0.200	0.521	0.266
263	303	6.575	4.545	0.220	0.571	0.304
263	308	5.844	3.610	0.277	0.619	0.342
263	313	5.260	2.950	0.339	0.666	0.380
253	293	6.325	4.292	0.233	0.586	0.316
253	298	5.622	3.610	0.277	0.634	0.356
253	303	5.060	3.021	0.331	0.684	0.395
253	308	4.60	2.538	0.394	0.732	0.435
253	313	4.217	2.183	0.458	0.779	0.474
233	293	3.883	1.792	0.558	0.827	0.515
233	298	3.585	1.471	0.680	0.876	0.558
233	303	3.329	1.203	0.831	0.925	0.601
233	308	3.107	0.982	1.018	0.974	0.644
233	313	2.913	0.784	1.275	1.022	0.687

Even though real refrigerators do not operate as a Carnot cycle and then the low-dissipation hypothesis may not be verified, a comparison of $\varepsilon_{\max\chi}(\varepsilon_{\mathrm{C}})$, $\varepsilon^{\sup}_{\max\chi}(\varepsilon_{\mathrm{C}})$, and the Carnot efficiency ε_{C} with observed efficiencies should be very valuable. In Fig. 1 these efficiencies versus the experimental data corresponding to a high-temperature refrigerator from Table I are represented. The plots of inverse efficiencies are widely used in studies of real irreversible refrigerators [20]. The figure shows that $\varepsilon^{\sup}_{\max\chi}$ is quite close to the experimental data. For low values of ε_{C}, which imply low values of τ and T_{c}, the optimum χ regime predicted efficiencies that fit better to experimental data. The results shown in Table I

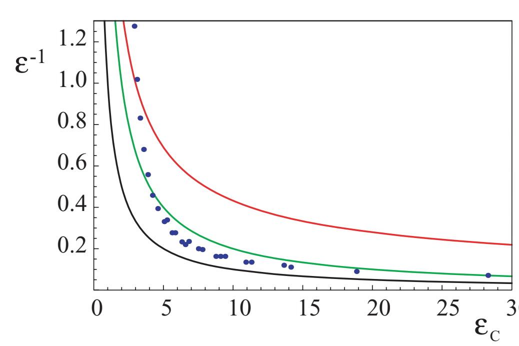

FIG. 1. (Color) $\varepsilon_{\max\chi}^{-1}$ (red curve), $(\varepsilon^{\sup}_{\max\chi})^{-1}$(green curve), $\varepsilon_{\mathrm{C}}^{-1}$ (black curve), and inverse efficiency of a real high-temperature refrigerator $\varepsilon_{\mathrm{exp}}^{-1}$ (blue dotted curve) (see Table I) vs the Carnot efficiency ε_{C}.

and Fig. 1 reveal that $\varepsilon_{\max\chi}(\varepsilon_C)$ allows for an accurate and easy estimation of high-temperature real refrigerators (as the $\eta_{CA} \equiv \eta_{\max\chi}$ value for power heat devices, see Table I and Fig. 1 in Ref. [14]).

In summary, we have presented a unified optimization criterion χ, which it is formulated in terms of the working system instead of the external coupling characteristics. It allows to recover the CA efficiency $\eta_{CA} = 1 - \sqrt{\tau}$ when it is applied to low-dissipation heat engines under symmetric conditions, and the result $\varepsilon_{\max\chi} = \frac{\tau}{1-\tau+\sqrt{1-\tau}} = \frac{1}{\sqrt{1-\tau}} - 1 \equiv \varepsilon_{CA}$ for low-dissipation refrigerators under the same symmetric conditions. The criterion accounts for the maximum power for heat engines and the product of $\epsilon \dot{Q}_c$ for refrigerators. Most important, all of these results have been obtained without invoking the endoreversibility assumption, or any specific-heat transfer law between the cyclic system and external heat bath couplings, and independently of the external temperatures values, i.e., beyond the linear-response regime usually considered both in the stochastic thermodynamics [10,21] and in the linear irreversible thermodynamics frameworks [22,23].

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