RELATIVE INTEGRAL FUNCTORS FOR SINGULAR FIBRATIONS AND SINGULAR PARTNERS

DANIEL HERNÁNDEZ RUIPÉREZ, ANA CRISTINA LÓPEZ MARTÍN,
AND FERNANDO SANCHO DE SALAS

ABSTRACT. We study relative integral functors for singular schemes and characterise those which preserve boundness and those which have integral right adjoints. We prove that a relative integral functor is an equivalence if and only if its restriction to every fibre is an equivalence. This allows us to construct a non-trivial auto-equivalence of the derived category of an arbitrary genus one fibration with no conditions on either the base or the total space and getting rid of the usual assumption of irreducibility of the fibres. We also extend to Cohen-Macaulay schemes the criterion of Bondal and Orlov for an integral functor to be fully faithful in characteristic zero and give a different criterion which is valid in arbitrary characteristic. Finally, we prove that for projective schemes both the Cohen-Macaulay and the Gorenstein conditions are invariant under Fourier-Mukai functors.

CONTENTS

INTRODUCTION

The relevance of derived categories and Fourier-Mukai functors in birational geometry is nowadays well known [8, 18]. One of the most important problems in this context is the minimal model problem. There are examples proving that derived categories have a nice behavior under some birational transformations as blow-ups, flips

and flops. Furthermore, Bondal and Orlov conjectured [5] that each flip (resp. flop) relating two smooth varieties X and X^+ should induce a fully faithful functor (resp. an equivalence) from the bounded derived category of coherent sheaves on X^+, $D_c^b(X^+)$, to the corresponding bounded derived category $D_c^b(X)$. Since varieties with singularities have to be allowed in the minimal model programme, one of the main problems this programme leads to is the study of derived categories for singular projective varieties. However, not too much attention has been paid to singular varieties in the literature on the topic. The reason may be that many of the fundamental results rely deeply on smoothness.

In dimension three, the difficulties arising when one deals with Gorenstein schemes can be circumvent using a smoothing approach [13]. Threefold Gorenstein terminal singularities are isolated hypersurface singularities ([21]) and a hypersurface singularity can be considered as a special fiber of a smooth fourfold. Thus, one can get some information about the derived categories in dimension 3 from the derived categories of the corresponding smooth fourfold. The results in [13] can be generalized to Q-Gorenstein teminal threefolds (see [1, 18, 19]). In this situacion, the associated Gorenstein stack allows to reduce the problem to the Gorenstein case. Nevertheless, since in higher dimension there are not local models for quotient singularities and these seem to be more rigid, this smoothing idea does not work and new results are needed.

Following a completely different path, we started in [16] the study of derived categories and fully faithful integral functors for schemes with Gorenstein singularities or fibered in schemes of that kind, and generalised to that situation the characterisation of fully faithfulness originally proved by Bondal and Orlov [6] in the smooth case.

The aim of this paper is twofold. On the one hand, we give a very general result that characterizes when relative integral functors are equivalences and that allow to reduce the problem to the absolute setting. This gives in particular a construction of a non-trivial invertible integral functor for a general genus one fibration. Notice that we adopt here a slightly different definition of a genus one fibration; actually we mean a flat Gorenstein morphism whose fibres are curves of arithmetic genus 1 and have trivial dualising sheaf; the last condition is a consequence of the others when the fibres are reduced. In the case of dimensions two or three, we are then allowing all types of Kodaira fibre, thus getting rid of the usual assumption of irreducible fibres.

Moreover, for fully faithfulness we are now able to consider more general singularities, namely Cohen-Macaulay schemes, both in the absolute and in the relative case. The case of positive characteristic, never studied before, is considered as well. On the second hand, we also tackle the question of what geometric information of the scheme can be recovered from its derived category. For projective schemes, we give an affirmative answer for both the Cohen-Macaulay and the Gorenstein properties.

The paper is organised as follows:

Section 1 is a repository of formulas used throughout the work and collected here for the reader's convenience.

In Section 2 we study integral functors, which we define directly in the relative situation, and determine conditions for them to map bounded complexes to bounded complexes and to have right adjoints with the same property. The study requires the notions of complexes of relative finite projective dimension and of relative finite homological dimension, which are equivalent in the case of projective morphisms as we proved in [16]. Among the results in Section 2 we can mention Proposition 2.10, where we prove that if the integral functor defined by a kernel $\mathcal{K}^\bullet \in D_c^b(X \times Y)$ induces an

equivalence between the bounded derived categories of X and Y, then the kernel has to be of finite homological dimension over both X and Y and the same happens to the kernel $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times Y}}(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$ of the right adjoint.

The main result of this Section is however Proposition 2.15 that proves, in great generality, that a relative integral functor is fully faithful (or an equivalence) if and only if the absolute integral functors induced on the fibers are fully faithful (or equivalences). This result could be useful for the study of relative Fourier-Mukai transforms between the derived categories of the total spaces of two very general fibrations. The reason is that it allows to pass from a relative to an absolute situation where more things are known. For instance in the absolute setting, there is an important class of well-known Fourier-Mukai functors: the twist functors along spherical objects that were firstly introduced by Seidel and Thomas in [26]. As a direct corollary of this proposition, we construct a non-trivial auto-equivalence of the derived category of an arbitrary genus one fibration. The result is the following:

Theorem (Proposition 2.16). *Let S be an algebraic scheme, $X \to S$ a genus one fibration, and $\mathcal{I}_\Delta$ the ideal sheaf of the relative diagonal. The relative integral functor*

$$\Phi^{\mathcal{I}_\Delta}_{X \to X} \colon D^b_c(X) \to D^b_c(X)$$

is an equivalence of categories.

This extends the result and the proof given in [16, Prop. 2.7] in two directions. First the characteristic of the base field is arbitrary. Second, all kind of possible fibres of a genus one fibration are allowed; in particular, for dimensions two or three they are all the Kodaira fibres. Moreover, no condition is imposed on either the base S or the total space X of the fibration.

When the fibration has only integral fibres, Proposition 2.16 gives a short proof of the invertibility of the usual elliptic integral functor. We then expect that Proposition 2.16 could be a useful tool for the study of the moduli spaces of relatively semistable sheaves on $X \to S$ with respect to a suitable relative ample divisor following [3, 2, 9, 15], and for the study of the derived category of X generalising [10].

In Section 3 we give criteria to characterise fully faithful integral functors. We first consider the case of characteristic zero, and extend to Cohen-Macaulay schemes the characterisation of fully faithful integral functors given by Bondal and Orlov. The characterisation result is the following.

Theorem (Theorem 3.6). *Let X and Y be proper schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times Y)$ of finite homological dimension over both X and Y. Assume also that X is projective, Cohen-Macaulay and integral. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y} \colon D^b_c(X) \to D^b_c(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is strongly simple over X.*

This theorem also generalises the extension to varieties with Gorenstein singularities given in [16, Thm. 1.22]. The new theorem is also stronger in the Gorenstein case, because we do not need to assume any longer that Y is projective and Gorenstein.

As in the Gorenstein case, strong simplicity in the Cohen-Macaulay case (Definition 3.3) is defined in terms of locally complete intersection zero cycles instead of the structure sheaves of the closed points. In the smooth case, our definition is weaker that the usual one given by Bondal and Orlov, and then Theorem 3.6 improves the characterization of fully faithfulness of Bondal and Orlov. It should be noticed that we

give here a slightly different notion of strong simplicity than the one given in [16] for Gorenstein varieties. Since both characterise the fully faithfulness of the corresponding integral functors, they are equivalent in the Gorenstein case.

We also consider the case of positive characteristic which is different because Theorem 3.6 fails in that situation. We have modified the notion of strong simplicity to a new one, which also characterises fully faithfulness in arbitrary characteristic. The precise statement is:

Theorem (Theorem 3.8). *Let X and Y be proper schemes over an algebraically closed field of arbitrary characteristic, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times Y)$ of finite homological dimension over both X and Y. Assume also that X is connected, equidimensional, projective and Cohen-Macaulay. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y} \colon D^b_c(X) \to D^b_c(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ has the following properties:*

(1) *For every closed point $x \in X$ there is a l.c.i. zero cycle Z_x supported on x such that*

$$\operatorname{Hom}^i_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_{x_1}}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{x_2})) = 0$$

unless $x_1 = x_2$ and $0 \le i \le \dim X$.

(2) *There exists a closed point x such that at least one of the following conditions is fulfilled:*

(2.1) $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_X), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k$.

(2.2) $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k$ *for any l.c.i. zero cycle Z_x supported on x.*

(2.3) $\dim_k \operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x})) \le l(\mathcal{O}_{Z_x})$ *for any l.c.i. zero cycle Z_x supported on x, where $l(\mathcal{O}_{Z_x})$ is the length of $\mathcal{O}_{Z_x}$.*

Due to Proposition 2.15 and the properties of relative integral functors proved in Section 2, the extension of the above criteria to the relative setting is straightforward (see Theorem 3.11).

As an application we give a different proof of Proposition 2.16 which doesn't need to use the work of Seidel and Thomas about spherical objects.

The last Section 4 is devoted to the study of Fourier-Mukai partners of a given proper scheme X, that is, proper schemes Y with equivalent coherent bounded category (i.e, D-equivalent to X), and such that the equivalence is given by an integral functor. In the projective smooth case, the second condition is automatically fulfilled, due to Orlov's representation theorem [25]. However, the validity of Orlov's theorem for singular varieties is still unknown; then, in principle, two D-equivalent singular varieties might not be Fourier-Mukai partners.

It is known that smooth Fourier-Mukai partners share many geometrical properties. In the same vein, we prove in Section 4 that singular Fourier-Mukai partners also have many geometrical properties in common. Our main result in this direction is the following.

Theorem (Theorem 4.4). *Let X be a projective equidimensional Cohen-Macaulay scheme and Y a projective Fourier-Mukai partner of X. Then one has*

(1) *If Y is reduced, then Y is equidimensional and $\dim Y = \dim X$.*

(2) *If Y is equidimensional and $\dim Y = \dim X$, then Y is Cohen-Macaulay. Moreover, if X is Gorenstein, then Y is Gorenstein as well.*

Conventions. In this paper, scheme means separated scheme of finite type over an algebraically closed field k. By a Gorenstein or a Cohen-Macaulay morphism, we understand a flat morphism of schemes whose fibres are respectively Gorenstein or Cohen-Macaulay. For any scheme X we denote by $D(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. This is the essential image of the derived category of quasi-coherent sheaves in the derived category $D(\mathfrak{Mod}(X))$ of all $\mathcal{O}_X$-modules [4, Cor. 5.5]. Analogously $D^+(X)$, $D^-(X)$ and $D^b(X)$ will denote the derived categories of complexes which are respectively bounded below, bounded above and bounded on both sides, and have quasi-coherent cohomology sheaves. The subscript c will refer to the corresponding subcategories of complexes with coherent cohomology sheaves.

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1. Some basic formulas in derived category

We recall here some basic formulas which will be used in the rest of the paper.

If X is a scheme, there is a functorial isomorphism (in the derived category)

$$(1.1) \qquad \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{H}^\bullet)) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet, \mathcal{H}^\bullet)$$

where $\mathcal{F}^\bullet$, $\mathcal{E}^\bullet$ and $\mathcal{H}^\bullet$ are in $D(\mathfrak{Mod}(X))$ [27, Thm. A]. One also has a functorial isomorphism in $D(\mathfrak{Mod}(X))$

$$(1.2) \qquad \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet)$$

when either $\mathcal{F}^\bullet$ or $\mathcal{H}^\bullet$ has finite homological dimension. When $\mathcal{F}^\bullet$ is bounded above with coherent cohomology sheaves, $\mathcal{E}^\bullet$ is bounded below and $\mathcal{H}^\bullet$ has finite homological dimension, the formula is standard (cf. [14, Prop. II.5.14]). Since we have not found a reference for the unbounded case, we give a simple proof here: Let $\mathcal{E}^\bullet \to \mathcal{I}^\bullet$ be an injective resolution [27, Thm. 4.5] and $\mathcal{P}^\bullet \to \mathcal{H}^\bullet$ a flat resolution [27, Prop. 5.6]. One has morphisms of complexes

$$\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \otimes \mathcal{P}^\bullet \to \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet \otimes \mathcal{P}^\bullet) \to \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{J}^\bullet)$$

where $\mathcal{J}^\bullet$ is an injective resolution of $\mathcal{I}^\bullet \otimes \mathcal{P}^\bullet$. This proves the existence of a morphism

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet \to \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet) \,.$$

We now prove that this is an isomorphism if either $\mathcal{F}^\bullet$ or $\mathcal{H}^\bullet$ has finite homological dimension. This is a local question, so we may assume that $X = \operatorname{Spec} A$ is affine and that either $\mathcal{F}^\bullet$ or $\mathcal{P}^\bullet$ is a bounded complex of free A-modules of finite rank. In both cases we have an isomorphism of complexes

$$\mathcal{H}om^\bullet_A(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \otimes \mathcal{P}^\bullet \xrightarrow{\sim} \mathcal{H}om^\bullet_A(\mathcal{F}^\bullet, \mathcal{I}^\bullet \otimes \mathcal{P}^\bullet) \,.$$

In the first case, this proves directly the isomorphism (1.2). In the second case, we have only to take into account that since $\mathcal{P}^\bullet$ is a bounded complex of free modules of finite rank, the complex $\mathcal{I}^\bullet \otimes \mathcal{P}^\bullet$ is injective.

If $f\colon X \to Y$ is a morphism of schemes, the direct and inverse images are defined for unbounded complexes, $\mathbf{R}f_*\colon D(\mathfrak{Mod}(X)) \to D(\mathfrak{Mod}(Y))$, $\mathbf{L}f^*\colon D(\mathfrak{Mod}(Y)) \to D(\mathfrak{Mod}(X))$ and the latter is a right adjoint to the former [27, Thm. B]. They induce morphisms $\mathbf{R}f_*\colon D(X) \to D(Y)$, $\mathbf{L}f^*\colon D(Y) \to D(X)$ which are adjoint of each other as well. One has a "projection formula"

$$\mathbf{R}f_*(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet) \simeq \mathbf{R}f_*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet$$

for $\mathcal{F}^\bullet$ in $D(X)$ and $\mathcal{G}^\bullet$ in $D(Y)$ [24, Prop. 5.3].

Let us consider a cartesian diagram of morphisms of algebraic varieties

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle\tilde{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

Then for any complex $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules there is a natural morphism

$$\mathbf{L}g^*\mathbf{R}f_*\mathcal{G}^\bullet \to \mathbf{R}\tilde{f}_*\mathbf{L}\tilde{g}^*\mathcal{G}^\bullet.$$

Moreover, if $\mathcal{G}^\bullet$ is in $D(X)$ and either f or g is flat, the above morphism is an isomorphism. This is the so-called "base-change formula" in the derived category. The flat base-change formula, i.e., when g is flat, is well-known [14, Prop. II.5.12]. If g is arbitrary and f is flat, the formula is proven in [2, Appendix A]. In this paper we only need the following very simple case.

Proposition 1.1. *Let us consider a diagram*

$$\begin{array}{ccc} X_y & \overset{j_{X_y}}{\hookrightarrow} & X \\ {\scriptstyle f_y}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \{y\} & \overset{j_y}{\hookrightarrow} & Y \end{array}$$

where f is a flat morphism of schemes, $y \in Y$ is a closed point and $X_y = f^{-1}(y)$ is the fibre. For every object $\mathcal{G}^\bullet$ in $D(X)$, there is a base-change isomorphism

$$\mathbf{L}j_y^*\mathbf{R}f_*\mathcal{G}^\bullet \simeq \mathbf{R}f_{y*}\mathbf{L}j_{X_y}^*\mathcal{G}^\bullet.$$

in the derived category.

Proof. It is enough to prove that the induced morphism

$$j_{y*}(\mathbf{L}j_y^*\mathbf{R}f_*\mathcal{G}^\bullet) \to j_{y*}(\mathbf{R}f_{y*}\mathbf{L}j_{X_y}^*\mathcal{G}^\bullet)$$

is an isomorphism in $D(Y)$. By the projection formula, the first member is isomorphic to $\mathcal{O}_y \overset{\mathbf{L}}{\otimes} \mathbf{R}f_*\mathcal{G}^\bullet$. The second member is isomorphic to $\mathbf{R}f_*j_{X_y*}\mathbf{L}j_{X_y}^*\mathcal{G}^\bullet$, which is isomorphic to $\mathbf{R}f_*(\mathcal{O}_{X_y} \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet)$ again by the projection formula. Moreover $\mathcal{O}_{X_y} \simeq \mathbf{L}f^*\mathcal{O}_y$ because f is flat, and then $\mathbf{R}f_*(\mathcal{O}_{X_y} \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet) \simeq \mathcal{O}_y \overset{\mathbf{L}}{\otimes} \mathbf{R}f_*\mathcal{G}^\bullet$ by the projection formula for f. $\qquad\square$

Let $f\colon X \to Y$ be a proper morphism of schemes. The relative Grothendieck duality states the existence of a functorial isomorphism in the derived category

$$(1.3) \qquad \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

for $\mathcal{G}^\bullet$ in $D(Y)$ and $\mathcal{F}^\bullet$ in $D(X)$ (see for instance [24]). By applying the derived functor of the global section functor, we obtain the *global duality formula*

$$(1.4) \qquad \operatorname{Hom}_{D(Y)}(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

In other words, the direct image $\mathbf{R}f_*\colon D(X) \to D(Y)$ has a right adjoint $f^!\colon D(Y) \to D(X)$. If $g\colon Y \to Z$ is another proper morphism, there is a natural functor isomorphism $(g \circ f)^! \simeq f^! \circ g^!$.

We shall call the objet $f^!\mathcal{O}_Y$ the *relative dualizing complex* of X over Y. When Y is a point, we also write $\mathcal{D}_X^\bullet$ instead of $f^!\mathcal{O}_Y$.

When f is flat, then it is Cohen-Macaulay if and only if the relative dualizing complex $f^!\mathcal{O}_Y$ is isomorphic to a single sheaf $\omega_{X/Y}$ placed at degree $-n$ (where n is the relative dimension of f). Moreover $\omega_{X/Y}$ is a line bundle if and only if f is a Gorenstein morphism.

We finish this part by recalling some properties of Grothendieck duality which we shall use in this paper.

Firstly, if $f\colon X \to Y$ is a finite morphism, then $f_*f^!\mathcal{G}^\bullet \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(f_*\mathcal{O}_X, \mathcal{G}^\bullet)$. Secondly, Grothendieck duality is compatible with flat base-change, that is, if $g\colon Z \to Y$ is a flat morphism and $f_Z\colon Z \times_Y X \to Z$ is the induced morphism, then $f_Z^!\mathcal{O}_Z \simeq g_X^*f^!\mathcal{O}_Y$ where $g_X\colon Z \times_Y X \to X$ is the projection. In particular, the formation of $f^!\mathcal{O}_Y$ is compatible with open immersions $U \subseteq Y$. Finally, there is a natural map $\mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y \to f^!\mathcal{G}^\bullet$; in some cases, it is an isomorphism. One of those cases is when $\mathcal{G}^\bullet$ has finite homological dimension; to see this one has to prove that for any complex $\mathcal{F}^\bullet$ in $D(X)$ the induced map $\operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y) \to \operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$ is bijective, and for this it is enough to prove that $\mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{F}^\bullet, \mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y) \to \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$ is an isomorphism in $D(Y)$. The first member is isomorphic to $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{O}_Y) \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet$, by Equation (1.2), the projection formula and the duality isomorphism (1.3). Moreover, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{O}_Y) \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$ by Equation (1.2) and relative duality (1.3). Other case where $\mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y \to f^!\mathcal{G}^\bullet$ is an isomorphism is when f is a regular closed immersion; in this case one has to prove that the induced morphism $f_*(\mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y) \to f_*(f^!\mathcal{G}^\bullet)$ is an isomorphism. By the projection formula the first member is isomorphic to $\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f_*(f^!\mathcal{O}_Y)$. Since $f_*f^!\mathcal{O}_Y \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(f_*\mathcal{O}_X, \mathcal{O}_Y)$ and $f_*f^!\mathcal{G}^\bullet \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}^\bullet(f_*\mathcal{O}_X, \mathcal{G}^\bullet)$, the result follows again from Equation (1.2), which can be applied because $f_*\mathcal{O}_X$ is of finite homological dimension.

2. Relative Integral functors for singular schemes

2.1. **Boundedness conditions.** Let S be a scheme and let $X \to S$ and $Y \to S$ be proper morphisms. We denote by π_X and π_Y the projections of the fibre product $X \times_S Y$ onto its factors.

Let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times_S Y)$. The relative integral functor defined by $\mathcal{K}^\bullet$ is the functor $\Phi_{X\to Y}^{\mathcal{K}^\bullet}\colon D(X) \to D(Y)$ given by

$$\Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{F}^\bullet) = \mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet).$$

and it maps $D^-(X)$ to $D^-(Y)$.

By adjunction between the direct and inverse images and by relative Grothendieck duality, $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ has a right adjoint $H \colon D(Y) \to D(X)$ given by

$$(2.1) \qquad H(\mathcal{G}^\bullet) = \mathbf{R}\pi_{X,*}(\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{G}^\bullet)),$$

which maps $D^+(Y)$ to $D^+(X)$. We shall now study when either $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ or H take bounded complexes to bounded complexes.

Definition 2.1. Let $f \colon Z \to T$ be a morphism of schemes. An object $\mathcal{E}^\bullet$ in $D_c^b(Z)$ is said to be of *finite homological dimension* (resp. of *finite projective dimension*) over T, if $\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet$ (resp. $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}^\bullet(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)$) is bounded for any $\mathcal{G}^\bullet$ in $D_c^b(T)$. $\qquad \triangle$

Remark 2.2. The absolute notion of finite homological dimension corresponds to being "of finite homological dimension over Z with respect to the identity" [16, Lemma 1.2], rather than to being "of finite homological dimension over $\operatorname{Spec} k$"; actually, any object $\mathcal{E}^\bullet$ in $D_c^b(Z)$ is of finite homological dimension over $\operatorname{Spec} k$. The usual notion of morphism of finite homological dimension is equivalent to saying that $\mathcal{O}_Z$ is "of finite homological dimension over T". $\qquad \triangle$

For projective morphisms one has the following result (see [16]):

Proposition 2.3. *Let $f \colon Z \to T$ be a projective morphism and $\mathcal{O}(1)$ a relatively very ample line bundle. Let $\mathcal{E}^\bullet$ be an object of $D_c^b(Z)$ and let us set $\mathcal{E}^\bullet(r) = \mathcal{E}^\bullet \otimes \mathcal{O}(r)$. The following conditions are equivalent:*

(1) *$\mathcal{E}^\bullet$ is of finite projective dimension over T.*
(2) *$\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ is of finite homological dimension (i.e. a perfect complex) for every integer r.*
(3) *$\mathcal{E}^\bullet$ is of finite homological dimension over T.*

Thus, if f is locally projective, $\mathcal{E}^\bullet$ is of finite projective dimension over T if and only if it is of finite homological dimension over T. $\qquad \square$

Corollary 2.4. *Let $f \colon Z \to T$ be a locally projective morphism. The functor $f^!$ sends $D^b(T)$ to $D^b(Z)$ if and only if f is a morphism of finite homological dimension.* $\qquad \square$

We also state here the following lemma whose proof can be found in [16].

Lemma 2.5. *Let $f \colon Z \to T$ be a projective morphism.*

(1) *Let $\mathcal{E}^\bullet$ be an object of $D^-(Z)$. Then $\mathcal{E}^\bullet = 0$ (resp. is an object of $D^b(Z)$) if and only if $\mathbf{R}f_*(\mathcal{E}^\bullet(r)) = 0$ (resp. is an object of $D^b(T)$) for every integer r.*
(2) *Let $g \colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism in $D^-(Z)$. Then g is an isomorphism if and only if the induced morphism $\mathbf{R}f_*(\mathcal{E}^\bullet(r)) \to \mathbf{R}f_*(\mathcal{F}^\bullet(r))$ is an isomorphism in $D^-(T)$ for every integer r.*

$\qquad \square$

The arguments used in the proof of the above Lemma also show that an object $\mathcal{E}^\bullet$ of $D(Z)$ has coherent cohomology sheaves if and only if $\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ has coherent cohomology sheaves for every integer r. One then obtains the following result.

Proposition 2.6. *If $f \colon Z \to T$ is a locally projective morphism, the functor $f^!$ sends $D_c(T)$ to $D_c(Z)$.*

Proof. Let $\mathcal{G}^\bullet$ be an object of $D_c(T)$. Since the formation of $f^!\mathcal{G}^\bullet$ is compatible with open immersions, we can assume that f is projective. In this case, Grothendieck duality gives $\mathbf{R}f_*(f^!\mathcal{G}^\bullet(r)) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathbf{R}f_*\mathcal{O}_Z(-r), \mathcal{G}^\bullet)$, and the latter complex has coherent cohomology sheaves. We finish by Lemma 2.5. $\square$

Proposition 2.7. *Assume that $X \to S$ is locally projective and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times_S Y)$. The functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y}$ maps $D^b_c(X)$ to $D^b_c(Y)$ if and only if $\mathcal{K}^\bullet$ has finite homological dimension over X.*

Proof. Assume that $\Phi^{\mathcal{K}^\bullet}_{X \to Y}$ maps $D^b_c(X)$ to $D^b_c(Y)$. We have to prove that $\mathbf{L}\pi^*_X\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet$ is bounded for any bounded complex $\mathcal{F}^\bullet$. We may assume that $X \to S$ is projective, and then π_Y is also projective. By Lemma 2.5, it suffices to show that $\mathbf{R}\pi_{Y*}[\mathbf{L}\pi^*_X\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet \otimes \pi^*_X\mathcal{O}(r)]$ is bounded for any r. This is immediate from the equality $\mathbf{R}\pi_{Y*}[\mathbf{L}\pi^*_X\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet \otimes \pi^*_X\mathcal{O}(r)] = \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{F}^\bullet(r))$. The converse is clear. $\square$

Proposition 2.8. *Let $f\colon Z \to T$ be a locally projective morphism of schemes and $\mathcal{E}^\bullet$ an object of $D^b_c(Z)$ of finite homological dimension over T. One has*

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)$$

for $\mathcal{G}^\bullet$ in $D^b_c(T)$. In particular, $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T)$ is also of finite homological dimension over T.

Proof. One has natural morphisms
(2.2)
$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet \to \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, \mathbf{L}f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_T) \to \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet).$$

We have to prove that the composition is an isomorphism. This is a local question on T, so that we can assume that f is projective.

By Lemma 2.5 we have to prove that the induced morphism

(2.3) $\quad \mathbf{R}f_*(\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T) \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet \otimes \mathcal{O}(r)) \to \mathbf{R}f_*(\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet) \otimes \mathcal{O}(r))$

is an isomorphism for any integer r. By Grothendieck duality, this is equivalent to proving that the induced morphism

(2.4) $\qquad \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_T}(\mathbf{R}f_*\mathcal{E}^\bullet(-r), \mathcal{O}_T) \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet \to \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_T}(\mathbf{R}f_*\mathcal{E}^\bullet(-r), \mathcal{G}^\bullet),$

is an isomorphism. This follows from (1.2) because $\mathbf{R}f_*\mathcal{E}^\bullet(-r)$ is of finite homological dimension by Proposition 2.3. Finally, by Proposition 2.3, $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Z}(\mathcal{E}^\bullet, f^!\mathcal{O}_T)$ is also of finite homological dimension over T. $\square$

Proposition 2.9. *Assume that $X \to S$ is locally projective and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times_S Y)$ of finite homological dimension over both X and Y. The functor $\Phi^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S Y}}(\mathcal{K}^\bullet, \pi^!_Y\mathcal{O}_Y)}_{Y \to X}\colon D^b_c(Y) \to D^b_c(X)$ is a right adjoint to $\Phi^{\mathcal{K}^\bullet}_{X \to Y}\colon D^b_c(X) \to D^b_c(Y)$.*

Proof. Since π_Y is a locally projective morphism, Proposition 2.8 gives the result. $\square$

When the integral functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y}$ is an equivalence between the bounded categories, Proposition 2.9 can be refined as follows:

Proposition 2.10. *Let $X \to S$ and $Y \to S$ be projective morphisms and let $\mathcal{K}^\bullet$ be a kernel in $D_c^b(X \times_S Y)$ such that $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ induces an equivalence*

$$\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \simeq D_c^b(Y) \,.$$

Then one has:

(1) *$\mathcal{K}^\bullet$ is finite of homological dimension over both X and Y.*
(2) *The right adjoint to $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \simeq D_c^b(Y)$ is an integral functor, and its kernel $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$ is also of finite homological dimension over both X and Y.*

Proof. First, $\mathcal{K}^\bullet$ is of finite homological dimension over X by Proposition 2.7. We still don't know if $\mathcal{K}^\bullet$ is of finite homological dimension over Y, so we cannot apply Proposition 2.9. However, we can proceed as follows: The functor $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D(X) \to D(Y)$ over the whole derived category has a right adjoint H (cf. Equation 2.1). Since $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \simeq D_c^b(Y)$ is an equivalence, it has a right adjoint $\bar{H} \colon D_c^b(Y) \to D_c^b(X)$. Let us prove that H and $\bar{H}$ coincide over $D_c^b(X)$. For any $\mathcal{F}^\bullet \in D_c^b(X)$, $\mathcal{G}^\bullet \in D_c^b(Y)$ one has that

$$\mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \bar{H}(\mathcal{G}^\bullet)) \simeq \mathrm{Hom}_{D(Y)}(\Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, H(\mathcal{G}^\bullet))$$

Hence, there is a morphism $\eta \colon \bar{H}(\mathcal{G}^\bullet) \to H(\mathcal{G}^\bullet)$ in $D^+(X)$, such that the induced morphism

$$\mathrm{Hom}_{D(X)}^i(\mathcal{F}^\bullet, \bar{H}(\mathcal{G}^\bullet)) \to \mathrm{Hom}_{D(X)}^i(\mathcal{F}^\bullet, H(\mathcal{G}^\bullet))$$

is an isomorphism for any $\mathcal{F}^\bullet \in D_c^b(X)$ and every integer i. Let $\mathcal{C}^\bullet \in D^+(X)$ be the cone of η. Then $\mathrm{Hom}_{D(X)}^i(\mathcal{F}^\bullet, \mathcal{C}^\bullet) = 0$ for any $\mathcal{F}^\bullet \in D_c^b(X)$ and any i. Taking $\mathcal{F}^\bullet = \mathcal{O}_X(r)$ for r big enough, one concludes that $\mathcal{C}^\bullet = 0$, so that $\bar{H} \simeq H$.

Now, for any $\mathcal{G}^\bullet \in D_c^b(Y)$ one has that

$$\bar{H}(\mathcal{G}^\bullet \otimes \mathcal{O}(r)) \simeq \mathbf{R}\pi_{X*}[\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{G}^\bullet) \otimes \pi_Y^* \mathcal{O}(r)]$$

is bounded for any integer r. By Lemma 2.5, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{G}^\bullet)$ is bounded and then $\mathcal{K}^\bullet$ has finite homological dimension over Y. It follows from Proposition 2.9 that $\bar{H}$ is an integral functor of kernel $\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times_S Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$. Since $\bar{H}$ is an equivalence, its kernel is also of finite homological dimension over both X and Y. $\square$

Proposition 2.11. *Let $f \colon Z \to T$ be a locally projective morphism and $\mathcal{E}^\bullet$ an object of $D_c^b(Z)$. If $\mathcal{E}^\bullet$ has finite homological dimension over T then the natural morphism*

$$\mathcal{E}^\bullet \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathcal{E}^\bullet, f^! \mathcal{O}_T), f^! \mathcal{O}_T)$$

is an isomorphism.

Proof. The problem is local on T, so we can assume that f is projective. Since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathcal{E}^\bullet, f^! \mathcal{O}_T)$ is of finite homological dimension over T by Proposition 2.8, the object $\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathcal{E}^\bullet, f^! \mathcal{O}_T), f^! \mathcal{O}_T)$ has bounded cohomology. Thus, by Lemma 2.5, to prove that

$$\mathcal{E}^\bullet \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathcal{E}^\bullet, f^! \mathcal{O}_T), f^! \mathcal{O}_T)$$

is an isomorphism we have to prove that the induced morphism

$$\mathbf{R}f_*(\mathcal{E}^\bullet(r)) \to \mathbf{R}f_*(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}^\bullet(\mathcal{E}^\bullet, f^! \mathcal{O}_T), f^! \mathcal{O}_T)(r))$$

is an isomorphism for every integer r. Grothendieck duality implies that this morphism is the natural morphism $\mathbf{R}f_*(\mathcal{E}^\bullet(r)) \to (\mathbf{R}f_*(\mathcal{E}^\bullet(r)))^{\vee\vee}$ (where for a complex $\mathcal{F}^\bullet$ we

write $\mathcal{F}^{\bullet\vee} = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_T}(\mathcal{F}^\bullet, \mathcal{O}_T)$ for the derived dual), which is an isomorphism because $\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ is of finite homological dimension by Proposition 2.3. $\qquad\square$

If X is a proper scheme over a field, we denote by $\mathcal{E}^{\bullet\sharp}$ the "dual in the dualising complex", that is,

$$(2.5) \qquad \mathcal{E}^{\bullet\sharp} = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X).$$

If X is projective, we can apply Proposition 2.11 to the projection $f\colon X \to \operatorname{Spec} k$. Since any bounded complex $\mathcal{E}^\bullet$ has finite homological dimension over $\operatorname{Spec} k$ one has:

Corollary 2.12. *Let X be a projective scheme over a field. The contravariant functor $\sharp$ induces an anti-equivalence of categories $\sharp\colon D^b_c(X) \to D^b_c(X)$.* $\qquad\square$

We finish this subsection with the following property about commutation of integral functors with the functor $\sharp$ (cf. Equation 2.5), which will be used in Theorems 3.6 and 4.4.

Lemma 2.13. *Let X and Y be projective schemes over a field and $\mathcal{K}^\bullet \in D^b_c(X \times Y)$ a kernel of finite homological dimension over X. Then one has*

$$(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{E}^\bullet))^\sharp \simeq \Phi^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi^!_X\mathcal{O}_X)}_{X\to Y}(\mathcal{E}^{\bullet\sharp}).$$

for any object $\mathcal{E}^\bullet$ in $D^b_c(X)$.

Proof. On the one hand, we have

$$(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{E}^\bullet))^\sharp \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathbf{R}\pi_{Y*}(\pi^*_X\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet), \mathcal{D}^\bullet_Y) \simeq \mathbf{R}\pi_{Y*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\pi^*_X\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet, \pi^!_Y\mathcal{D}^\bullet_Y)$$

$$\simeq \mathbf{R}\pi_{Y*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\pi^*_X\mathcal{E}^\bullet, \pi^!_Y\mathcal{D}^\bullet_Y)).$$

On the other hand

$$\Phi^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi^!_X\mathcal{O}_X)}_{X\to Y}(\mathcal{E}^{\bullet\sharp}) \simeq \mathbf{R}\pi_{Y*}(\pi^*_X\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X) \overset{\mathbf{L}}{\otimes} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi^!_X\mathcal{O}_X))$$

$$\simeq \mathbf{R}\pi_{Y*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi^!_X\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X))$$

where the last isomorphism is by Proposition 2.8. Then, it is enough to see that $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\pi^*_X\mathcal{E}^\bullet, \pi^!_Y\mathcal{D}^\bullet_Y) \simeq \pi^!_X\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X)$. Since $\pi^!_Y\mathcal{D}^\bullet_Y \simeq \mathcal{D}^\bullet_{X\times Y} \simeq \pi^!_X\mathcal{D}^\bullet_X$, this follows from the isomorphisms

$$\operatorname{Hom}_{D^b(X\times Y)}(\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\pi^*_X\mathcal{E}^\bullet, \mathcal{D}^\bullet_{X\times Y})) \simeq \operatorname{Hom}_{D^b(X\times Y)}(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \pi^*_X\mathcal{E}^\bullet, \mathcal{D}^\bullet_{X\times Y})$$

$$\simeq \operatorname{Hom}_{D^b(X)}(\mathbf{R}\pi_{X*}(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \pi^*_X\mathcal{E}^\bullet), \mathcal{D}^\bullet_X)$$

$$\simeq \operatorname{Hom}_{D^b(X)}(\mathbf{R}\pi_{X*}\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet, \mathcal{D}^\bullet_X)$$

$$\simeq \operatorname{Hom}(\mathbf{R}\pi_{X*}\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X))$$

$$\simeq \operatorname{Hom}_{D^b(X\times Y)}(\mathcal{F}^\bullet, \pi^!_X\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{D}^\bullet_X))$$

which hold for any object $\mathcal{F}^\bullet$ in $D^b_c(X \times Y)$. $\qquad\square$

2.2. **Restriction to fibres: a criterion for equivalence.** In this subsection we prove that, in a very general situation, to see that a relative integral functor is fully faithful (or an equivalence) it is enough to prove that its restriction to each fibre is fully faithful (or an equivalence). Using this result and the theorem of Seidel and Thomas that proves that any twist functor along a spherical object is an equivalence of categories, we construct then a non-trivial auto-equivalence of the bounded derived category of an arbitrary genus one fibration.

The base field is here an algebraically closed field of arbitrary characteristic. Let $p\colon X \to S$ and $q\colon Y \to S$ be proper and *flat* morphisms. Let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times_S Y)$ and $\Phi = \Phi_{X \to Y}^{\mathcal{K}^\bullet}$. For any closed point $s \in S$ we write $X_s = p^{-1}(s)$, $Y_s = q^{-1}(s)$, and denote by $\Phi_s\colon D^-(X_s) \to D^-(Y_s)$ the integral functor defined by $\mathcal{K}^\bullet{}_s = \mathbf{L}j_s^*\mathcal{K}^\bullet$, with $j_s\colon X_s \times Y_s \hookrightarrow X \times_S Y$ the natural embedding.

When the kernel $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$ is of finite homological dimension over X, the functor Φ maps $D_c^b(X)$ into $D_c^b(Y)$ and $\mathcal{K}^\bullet{}_s \in D_c^b(X_s \times Y_s)$ for any $s \in S$. Morever, since q is flat, $\mathcal{K}^\bullet{}_s$ is of finite homological dimension over X_s.

From the base-change formula (Proposition 1.1, see also [2]) we obtain that

$$(2.6) \qquad \mathbf{L}j_s^*\Phi(\mathcal{F}^\bullet) \simeq \Phi_s(\mathbf{L}j_s^*\mathcal{F}^\bullet)$$

for every $\mathcal{F}^\bullet \in D(X)$, where $j_s\colon X_s \hookrightarrow X$ and $j_s\colon Y_s \hookrightarrow Y$ are the natural embeddings. In this situation, base change formula also gives that

$$(2.7) \qquad j_{s*}\Phi_s(\mathcal{G}^\bullet) \simeq \Phi(j_{s*}\mathcal{G}^\bullet)$$

for every $\mathcal{G}^\bullet \in D(X_s)$.

Lemma 2.14. *Let $\Psi\colon D^-(X) \to D^-(Y)$ be an integral functor whose kernel is an object of $D^-(X \times Y)$. For any integer i there exists r_0 such that $\mathcal{H}^i(\Psi(\mathcal{G}^\bullet)) \simeq \mathcal{H}^i(\Psi(\sigma_{\geq r}\mathcal{G}^\bullet))$ for every $r \leq r_0$ and any $\mathcal{G}^\bullet \in D^-(X)$, where $\sigma_{\geq r}\mathcal{G}^\bullet$ is the truncation (cf. [14, I§7]).*

Proof. There exist an integer m such that if $\mathcal{F}^\bullet$ is an object of $D^-(X)$ and $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$ for $j \geq s$, then $\mathcal{H}^j(\Psi(\mathcal{F}^\bullet)) = 0$ for $j \geq s + m$. Let us take $r \leq i - 1 - m$ and let $\alpha_r\colon \mathcal{G}^\bullet \to \sigma_{\geq r}(\mathcal{G}^\bullet)$ be the natural morphism. Then $\mathcal{H}^j(\mathrm{Cone}(\alpha_r)) = 0$ for $j \geq r$, so that $\mathcal{H}^j(\Psi(\mathrm{Cone}(\alpha_r))) = 0$ for $j \geq r + m$. In particular $\mathcal{H}^{i-1}(\Psi(\mathrm{Cone}(\alpha_r))) = \mathcal{H}^i(\Phi^-(\mathrm{Cone}(\alpha_r))) = 0$, and we finish by taking cohomology on the exact triangle

$$\Psi(\mathcal{G}^\bullet) \to \Psi(\sigma_{\geq r}(\mathcal{G}^\bullet)) \to \Psi(\mathrm{Cone}(\alpha_r)) \to \Psi(\mathcal{G}^\bullet)[1]\,.$$

$\square$

Proposition 2.15. *Assume that $X \to S$ is locally projective and let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times_S Y)$ of finite homological dimension over both X and Y. The relative integral functor $\Phi = \Phi_{X \to Y}^{\mathcal{K}^\bullet}\colon D_c^b(X) \to D_c^b(Y)$ is fully faithful (resp. an equivalence) if and only if $\Phi_s\colon D_c^b(X_s) \to D_c^b(Y_s)$ is fully faithful (resp. an equivalence) for every closed point $s \in S$.*

Proof. By Proposition 2.9, the integral functor $H = \Phi_{Y \to X}^{\mathbf{R}\mathcal{H}om_{X \times Y}^\bullet(\mathcal{K}^\bullet, \pi_Y^!\mathcal{O}_Y)}\colon D_c^b(X) \to D_c^b(Y)$ is a right adjoint to Φ. We can now proceed as in the proof of [16, Thm. 2.4], which we explain here in some more detail.

If Φ is fully faithful the unit morphism $\mathrm{Id} \to H \circ \Phi$ is an isomorphism. Then, given a closed point $s \in S$ and $\mathcal{G}^\bullet \in D_c^b(X_s)$, one has an isomorphism $j_{s*}\mathcal{G}^\bullet \to (H \circ \Phi)(j_{s*}\mathcal{G}^\bullet)$. Since $(H \circ \Phi)(j_{s*}\mathcal{G}^\bullet) \simeq j_{s*}(H_s \circ \Phi_s)(\mathcal{G}^\bullet)$ by (2.7) and j_s is a closed immersion, the unit morphism $\mathcal{G}^\bullet \to (H_s \circ \Phi_s)(\mathcal{G}^\bullet)$ is an isomorphism; this proves that Φ_s is fully faithful.

Now assume that Φ_s is fully faithful for any closed point $s \in S$. To prove that Φ is fully faithful we have to see that the unit morphism $\eta \colon \mathrm{Id} \to H \circ \Phi$ is an isomorphism. For each $\mathcal{F}^\bullet \in D_c^b(X)$ we have an exact triangle

$$\mathcal{F}^\bullet \xrightarrow{\eta(\mathcal{F}^\bullet)} (H \circ \Phi)(\mathcal{F}^\bullet) \to \mathrm{Cone}(\eta(\mathcal{F}^\bullet)) \to \mathcal{F}^\bullet[1] \, .$$

Let us now fix a closed point $s \in S$. By Equation (2.6), we have an exact triangle

$$\mathbf{L}j_s^* \mathcal{F}^\bullet \to (H_s \circ \Phi_s)(\mathbf{L}j_s^* \mathcal{F}^\bullet) \to \mathbf{L}j_s^* \mathrm{Cone}(\eta(\mathcal{F}^\bullet)) \to \mathbf{L}j_s^* \mathcal{F}^\bullet[1] \, .$$

By Lemma 2.14, for every integer i there exists r small enough such that $\mathcal{H}^i(\mathbf{L}j_s^* \mathcal{F}^\bullet) \simeq \mathcal{H}^i(\sigma_{\geq r} \mathbf{L}j_s^* \mathcal{F}^\bullet)$ and $\mathcal{H}^i((H_s \circ \Phi_s)(\mathbf{L}j_s^* \mathcal{F}^\bullet)) \simeq \mathcal{H}^i((H_s \circ \Phi_s)(\sigma_{\geq r} \mathbf{L}j_s^* \mathcal{F}^\bullet))$. Since $\sigma_{\geq r} \mathbf{L}j_s^* \mathcal{F}^\bullet$ is a bounded complex, and $\eta_s \colon \mathrm{Id} \to H_s \circ \Phi_s$ is an isomorphism because Φ_s is fully faithful, one has that $\mathcal{H}^i(\sigma_{\geq r} \mathbf{L}j_s^* \mathcal{F}^\bullet) \simeq \mathcal{H}^i[(H_s \circ \Phi_s)(\sigma_{\geq r} \mathbf{L}j_s^* \mathcal{F}^\bullet)]$. Thus $\mathbf{L}j_s^* \mathcal{F}^\bullet \to (H_s \circ \Phi_s)(\mathbf{L}j_s^* \mathcal{F}^\bullet)$ induces isomorphisms between all the cohomology sheaves, so that it is an isomorphism, and then $\mathbf{L}j_s^* \mathrm{Cone}(\eta(\mathcal{F}^\bullet)) = 0$. Since this holds for every closed point $s \in S$, we finish by [16, Lemma 2.3].

A similar argument gives the statement about equivalence. $\qquad\square$

Related results concerning Azumaya smooth varieties have been proved by Kuznetsov in [22, Prop. 2.44 and Thm. 2.46]. To apply his results to our situation, X, Y and S have to be smooth though the flatness conditions on p and q can be removed.

2.2.1. *An auto-equivalence of the derived category of a genus one fibration.* Let $p \colon X \to S$ be a genus one fibration, that is, a projective Gorenstein morphism whose fibres are curves with arithmetic genus $\dim H^1(X_s, \mathcal{O}_{X_s}) = 1$ and have trivial dualising sheaf. No further assumptions on S or X are made here.

When the fibres are reduced of arithmetic genus one, then the condition on the dualising sheaf is always fulfilled. However, since nonreduced curves can also appear as degenerated fibres for such a genus one fibration, and for these curves the dualizing sheaf need not to be trivial (see for instance [12]), one needs to assume it.

There are some cases where the structure of the singular fibers is known: For smooth elliptic surfaces over the complex numbers, the classification was given by Kodaria [20] and for smooth elliptic threefolds over a base field of characteristic different from 2 and 3, they were classified by Miranda [23]. In both cases, the possible singular fibres are plane curves of the same type, the so-called Kodaira fibres. Nevertheless, in a genus one fibration non-plane curves could appear as degenerated fibres.

Since we are not putting any restriction on the characteristic of the base field or on the dimension of X, our genus one fibrations may have singular fibres other than the Kodaira fibres.

We consider the commutative diagram

$$
\begin{array}{ccc}
& X \times_S X & \\
{}^{\pi_1}\swarrow \quad \downarrow{}^{\rho} & & \searrow{}^{\pi_2} \\
X \quad\quad & & \quad\quad X \\
{}_p\searrow \quad & \downarrow & \quad \swarrow{}_p \\
& S &
\end{array}
$$

and the relative integral functor

$$\Phi = \Phi_{X \to X}^{\mathcal{I}_\Delta} \colon D_c^b(X) \to D_c^b(X) \, ,$$

with kernel the ideal sheaf $\mathcal{I}_\Delta$ of the relative diagonal immersion $\delta\colon X \hookrightarrow X \times_S X$.

Proposition 2.16. *The relative integral functor*

$$\Phi = \Phi^{\mathcal{I}_\Delta}_{X \to X}\colon D^b_c(X) \to D^b_c(X)$$

defined by the ideal sheaf of the relative diagonal is an equivalence of categories.

Proof. To prove that $\mathcal{I}_\Delta$ is of finite homological dimension over both factors, it is enough to see that it is of finite homological dimension over the first factor because of its symmetry. By the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X \times_S X} \to \delta_* \mathcal{O}_X \to 0\,,$$

it suffices to see that $\delta_* \mathcal{O}_X$ has finite homological dimension over the first factor. We have then to prove that for any $\mathcal{N}^\bullet \in D^b(X)$, the complex $\delta_* \mathcal{O}_X \otimes \pi_1^* \mathcal{N}^\bullet$ is bounded and this follows from the projection formula for δ.

For every closed point $s \in S$, the absolute functor $\Phi_s = \Phi^{\mathcal{I}_{\Delta_s}}_{X_s \to X_s}$ is equal to $T_{\mathcal{O}_{X_s}}[-1]$ where $T_{\mathcal{O}_{X_s}}$ denotes the twist functor along the object $\mathcal{O}_{X_s}$. Since X_s is a genus one projective curve with trivial dualizing sheaf, $\mathcal{O}_{X_s}$ is a spherical object. Thus, $T_{\mathcal{O}_{X_s}}$ is an equivalence of categories by [26] and we conclude by Proposition 2.15. $\square$

Similar results has been obtained by Burban and Kreussler in [11]. They proved a version of Proposition 2.16 in the case when the base field is of characteristic zero, S and X are reduced, X is connected, the fibration $p\colon X \to S$ has only integral fibres and it has a section taking values in the smooth locus of X. None of these assumptions have been made in this section, though we assume by technical reasons that S (and then X) is separated.

Notice that, since we are working in the relative setting, the integral functor $\Phi^{\mathcal{I}_\Delta}_{X \to X}$ whose kernel is the ideal sheaf of the relative diagonal is not easily described as a twist functor. Actually, the twist functor $T_{\mathcal{O}_X}[-1]$ is the integral functor whose kernel is the ideal sheaf of the absolute diagonal immersion $X \hookrightarrow X \times X$. Even the latter functor may fail to be an equivalence because in general $\mathcal{O}_X$ is not spherical.

We shall give an alternative proof of this result, without using the work of Seidel and Thomas, in Subsection 3.4.

3. Fully faithfulness criteria for integral functors

3.1. **Strongly simple objects and spanning classes.** Recall that if X is a Cohen-Macaulay scheme, for every point x there exist a zero cycle Z_x supported on x defined locally by a regular sequence (cf. [16, Lemma 1.9]); we refer to such cycles as to *locally complete intersection* or *l.c.i.* cycles. If $Z_x \hookrightarrow X$ is a l.c.i. cycle, by the Koszul complex theory, the structure sheaf $\mathcal{O}_{Z_x}$ has *finite homological dimension* as an $\mathcal{O}_X$-module.

In order to fix some notation, for any zero-cycle Z_x of X and any scheme S, we shall denote by j_{Z_x} the immersion $Z_x \times S \hookrightarrow X \times S$.

For further use, we gather here two equivalent characterisations of objects of the derived category of an equidimensional scheme (that is, a scheme with all its irreducible components of the same dimension) defined by single sheaves supported on a closed subscheme. The statements are a slight generalisation of [16, Props. 1.15, 1.16] and the proofs are completely analogous.

Proposition 3.1. *Let X, Y be equidimensional Cohen-Macaulay schemes, $j\colon Y \hookrightarrow X$ a closed immersion of codimension d, and $\mathcal{K}^\bullet$ an object of $D^b_c(X)$. Assume that*

(1) If $x \in X - Y$ is a closed point, then $\mathbf{L}j_{Z_x}^* \mathcal{K}^\bullet = 0$ for some l.c.i. zero cycle Z_x supported on x.
(2) If $x \in Y$ is a closed point, then $L_i j_{Z_x}^* \mathcal{K}^\bullet = 0$ for some l.c.i. zero cycle Z_x supported on x when either $i < 0$ or $i > d$.

Then there is a sheaf $\mathcal{K}$ on X whose topological support is contained in Y and such that $\mathcal{K}^\bullet \simeq \mathcal{K}$ in $D_c^b(X)$. Moreover, if $\mathcal{K}^\bullet$ is nonzero, then the topological support of $\mathcal{K}$ is a union of irreducible components of Y. □

Proposition 3.2. *Let* X, Y *be equidimensional Cohen-Macaulay schemes of dimensions* m *and* n *respectively,* $j\colon Y \hookrightarrow X$ *a closed immersion, and* $\mathcal{K}^\bullet$ *an object of* $D_c^b(X)$. *Assume that for any closed point* $x \in X$ *there is a l.c.i. zero cycle* Z_x *supported on* x *such that*

$$\mathrm{Hom}_{D(X)}^i(\mathcal{O}_{Z_x}, \mathcal{K}^\bullet) = 0\,,$$

unless $x \in Y$ *and* $m \leq i \leq n$. *Then there is a sheaf* $\mathcal{K}$ *on* X *whose topological support is contained in* Y *such that* $\mathcal{K}^\bullet \simeq \mathcal{K}$ *in* $D_c^b(X)$. *Moreover, if* $\mathcal{K}^\bullet$ *is nonzero, then the topological support of* $\mathcal{K}$ *is a union of irreducible components of* Y. □

In this paper we only use the first part of the statement, namely the fact that $\mathcal{K}^\bullet \simeq \mathcal{K}$ and its topological support is contained in Y.

Let X and Y be proper schemes. Assume that X is Cohen-Macaulay. In this situation, the notion of strong simplicity is the following.

Definition 3.3. An object $\mathcal{K}^\bullet$ in $D_c^b(X \times Y)$ is *strongly simple* over X if it satisfies the following conditions:

(1) For every closed point $x \in X$ there is a l.c.i. zero cycle Z_x supported on x such that
$$\mathrm{Hom}_{D(Y)}^i(\Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{Z_{x_1}}), \Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{O}_{x_2})) = 0$$
unless $x_1 = x_2$ and $0 \leq i \leq \dim X$.
(2) $\mathrm{Hom}_{D(Y)}^0(\Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x), \Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)) = k$ for every closed point $x \in X$.

△

The last condition can be written as $\mathrm{Hom}_{D(Y)}^0(\mathbf{L}j_x^* \mathcal{K}^\bullet, \mathbf{L}j_x^* \mathcal{K}^\bullet) = k$, because the restriction $\mathbf{L}j_x^* \mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ to the fibre $j_x\colon Y \simeq \{x\} \times Y \hookrightarrow X \times Y$ can also be computed as $\Phi_{X \to Y}^{\mathcal{K}^\bullet}(\mathcal{O}_x)$. Then, the objects $\mathbf{L}j_x^* \mathcal{K}^\bullet$ are simple and we may think of $\mathcal{K}^\bullet$ as a parametrisation of simple objects of $D_c^b(Y)$; this is the reason why one calls $\mathcal{K}^\bullet$ a strongly simple object over X.

Remark 3.4. When X and Y are smooth, our definition is weaker than the usual one given by Bondal and Orlov (see [2]). As a consequence of a result of Bondal and Orlov [6, Thm. 1.1], and of Theorem 3.6, our definition is equivalent to this one in the smooth case. In [16] we gave another notion of strongly simple objects for Gorenstein schemes. Again, Theorem 3.6 and the corresponding statement [16, Theorem 1.22] for the Gorenstein case, prove that the two notions are equivalent in that situation. △

3.1.1. *Spanning classes.* As in the Gorenstein case, the derived category $D_c^b(X)$ has a natural spanning class when X is a Cohen-Macaulay scheme.

Lemma 3.5. *If* X *is a Cohen-Macaulay scheme, then the set*

$\Omega = \{\mathcal{O}_{Z_x}$ *for all closed points* $x \in X$ *and all l.c.i. zero cycles* Z_x *supported on* $x\}$

is a spanning class for $D_c^b(X)$.

Proof. Take a non-zero object $\mathcal{E}^\bullet$ in $D_c^b(X)$. On the one hand, for every l.c.i. zero cycle as above there is a spectral sequence $E_2^{p,q} = \mathrm{Ext}_{\mathcal{O}_X}^p(\mathcal{H}^{-q}(\mathcal{E}^\bullet), \mathcal{O}_{Z_x})$ converging to $E_2^{p+q} = \mathrm{Hom}_{D(X)}^{p+q}(\mathcal{E}^\bullet, \mathcal{O}_{Z_x})$. If q_0 is the maximum of the q's such that $\mathcal{H}^q(\mathcal{E}^\bullet) \neq 0$ and x is a point of the support of $\mathcal{H}^q(\mathcal{E}^\bullet)$, then $E_2^{0,-q_0} \neq 0$ for every l.c.i. zero cycle Z_x and any non zero-element there survives to infinity. Then $\mathrm{Hom}_{D(X)}^{-q_0}(\mathcal{E}^\bullet, \mathcal{O}_{Z_x}) \neq 0$.

On the other hand, by Proposition 3.2 with $Y = \emptyset$, if $\mathrm{Hom}_{D(X)}^i(\mathcal{O}_{Z_x}, \mathcal{E}^\bullet) = 0$ for every i and every Z_x, then $\mathcal{E}^\bullet = 0$. $\square$

3.2. **A criterion in characteristic zero.** We now give the criterion for an integral functor between derived categories of Cohen-Macaulay proper schemes to be fully faithful.

Theorem 3.6. *Let X and Y be proper schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times Y)$ of finite homological dimension over both X and Y. Assume also that X is projective, Cohen-Macaulay and integral. Then the functor $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \to D_c^b(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is strongly simple over X.*

Proof. If the functor $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ is fully faithful, then $\mathcal{K}^\bullet$ is strongly simple over X.

Let us prove the converse. Before starting, we fix some notation: we denote by π_i the projections of $X \times X$ onto its factors and $\Phi = \Phi_{X \to Y}^{\mathcal{K}^\bullet}$.

By Proposition 2.9, the integral functor $H = \Phi_{Y \to X}^{\mathbf{R}\mathcal{H}om_{X \times Y}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)}$ is a right adjoint to Φ. By [16, Prop. 1.18] it suffices to show that $H \circ \Phi$ is fully faithful. We know that the composition of integral functors is again an integral functor (cf. [16]), and then $H \circ \Phi \simeq \Phi_{X \to X}^{\mathcal{M}^\bullet}$, with $\mathcal{M}^\bullet \in D_c^b(X \times X)$.

The strategy of the proof is similar to that of [16, Thm. 1.22]. We are going to prove that $\mathcal{M}^\bullet \simeq \delta_* \mathcal{N}$ where $\delta \colon X \hookrightarrow X \times X$ is the diagonal immersion and $\mathcal{N}$ is a line bundle on X; then $\Phi_{X \to X}^{\mathcal{M}^\bullet}$ consist of twisting by $\mathcal{N}$ which is an equivalence of categories, in particular fully faithful.

a) $\mathbf{L}j_x^* \mathcal{M}^\bullet$ *is a single sheaf topologically supported on x. Thus $\mathcal{M}^\bullet$ is a single sheaf $\mathcal{M}$ topologically supported on the diagonal and flat over X by the first projection by [7, Lemma 4.3].*

Let us fix a closed point $(x_1, x_2) \in X \times X$ and consider the l.c.i. zero cycle Z_{x_1} of the first condition of the definition of strongly simple object. One has

$$\mathrm{Hom}_{D(X)}^i(\mathcal{O}_{Z_{x_1}}, \Phi_{X \to X}^{\mathcal{M}^\bullet}(\mathcal{O}_{x_2})) \simeq \mathrm{Hom}_{D(Y)}^i(\Phi(\mathcal{O}_{Z_{x_1}}), \Phi(\mathcal{O}_{x_2})),$$

which is zero unless $x_1 = x_2$ and $0 \leq i \leq m$ because $\mathcal{K}^\bullet$ is strongly simple. Applying Proposition 3.2 to the immersion $\{x_2\} \hookrightarrow X$ we have that $\Phi_{X \to X}^{\mathcal{M}^\bullet}(\mathcal{O}_{x_2})$ reduces to a coherent sheaf topologically supported on x_2. Since $Lj_x^* \mathcal{M} \simeq \Phi_{X \to X}^{\mathcal{M}^\bullet}(\mathcal{O}_x)$, one has that $\mathcal{M}^\bullet$ is a sheaf $\mathcal{M}$ whose support is contained in the diagonal and $\pi_{1*}\mathcal{M}$ is locally free, where $\pi_1 \colon X \times X \to X$ is the projection onto the first factor. The rank of $\pi_{1*}\mathcal{M}$ can not be zero, by condition (2) of strongly simple. Hence, the topological support of $\mathcal{M}^\bullet$ is exactly the diagonal.

b) $\mathcal{M}$ *is schematically supported on the diagonal, that is, $\mathcal{M} = \delta_* \mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on X; moreover $\mathcal{N}$ is a line bundle.*

Let us denote by $\bar{\delta}\colon W \hookrightarrow X \times X$ the schematic support of $\mathcal{M}$ so that $\mathcal{M} = \bar{\delta}_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on W. Since $\mathcal{M}$ is topologically supported on the diagonal, the diagonal embedding δ factors through a closed immersion $\tau\colon X \hookrightarrow W$ which topologically is a homeomorphism.

Since $\mathcal{M}$ is flat over X by π_1, $\mathcal{N}$ is flat over X by the composition $\bar{\pi}_1 = \pi_1 \circ \bar{\delta}\colon W \to X$. Moreover $\bar{\pi}_1$ is a finite morphism, so that $\pi_{1*}\mathcal{M} \simeq \bar{\pi}_{1*}\mathcal{N}$ is locally free.

Now, as in the proof of [16, Thm. 1.22], to conclude is enough to see that the rank of the locally free sheaf $\pi_{1*}\mathcal{M}$ is one.

One has that $\mathrm{Hom}^0(\mathcal{O}_x, \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)) \simeq \mathrm{Hom}^0(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \simeq k$. Hence there is an injective morphism $\mathcal{O}_x \to \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x) \simeq j_x^*\mathcal{M}$. It suffices to show that this is an isomorphism for at least one closed point x. If C_x is the cokernel, we have to see that $\mathrm{Hom}^0(\mathcal{O}_x, C_x) = 0$. Arguing like in [7] it is enough to show that $\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Hom}^1(\Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x), \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x))$ is injective.

Let us denote $\widetilde{\mathcal{M}} = \mathbf{R}\mathcal{H}om_{X \times X}^\bullet(\mathcal{M}, \pi_1^!\mathcal{O}_X)$, which is again a sheaf supported topologically on the diagonal and flat over X by the first projection, because

$$\pi_{1*}\mathbf{R}\mathcal{H}om_{X \times Y}^\bullet(\mathcal{M}, \pi_1^!\mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om_{X \times Y}^\bullet(\pi_{1*}\mathcal{M}, \mathcal{O}_X)$$

and $\pi_{1*}\mathcal{M}$ is locally free.

One has that $\mathcal{O}_x^\sharp \simeq \mathcal{O}_x$ by relative duality for the closed immersion $\{x\} \hookrightarrow X$, so that $\mathbf{L}j_x^*\widetilde{\mathcal{M}} \simeq (\mathbf{L}j_x^*\mathcal{M})^\sharp$ by Lemma 2.13. Then,

$$\mathrm{Hom}^0(\mathbf{L}j_x^*\widetilde{\mathcal{M}}, \mathcal{O}_x) \simeq \mathrm{Hom}^0((\mathbf{L}j_x^*\mathcal{M})^\sharp, \mathcal{O}_x^\sharp) \simeq \mathrm{Hom}^0(\mathcal{O}_x, \mathbf{L}j_x^*\mathcal{M}) \simeq k$$

because the functor $\sharp$ is an anti-equivalence of categories by Corollary 2.12. Hence, $\mathbf{L}j_x^*\widetilde{\mathcal{M}}$ is the sheaf of a zero cycle supported on x. We can then apply [7, Lemmas 5.2, 5.3] to $\widetilde{\mathcal{M}}$ to conclude that there exists a point x such that
(3.1)
$$\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Hom}^1(\Phi_{X \to X}^{\widetilde{\mathcal{M}}}(\mathcal{O}_x), \Phi_{X \to X}^{\widetilde{\mathcal{M}}}(\mathcal{O}_x)) \simeq \mathrm{Hom}^1(\Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)^\sharp, \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)^\sharp)$$

is injective.

Now, again by Corollary 2.12, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{O}_x) & \longrightarrow & \mathrm{Hom}^1(\Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x), \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)) \\
\simeq \downarrow \sharp & & \simeq \downarrow \sharp \\
\mathrm{Hom}^1(\mathcal{O}_x, \mathcal{O}_x) & \longrightarrow & \mathrm{Hom}^1(\Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)^\sharp, \Phi_{X \to X}^{\mathcal{M}}(\mathcal{O}_x)^\sharp).
\end{array}
$$

Since the bottom arrow is injective, the top arrow is injective as well and we conclude. $\square$

The condition that the characteristics of the base field k is zero is only used to prove that there exists a point x such that the morphism (3.1) is injective [7, Lemmas 5.2, 5.3]. This is proving by showing that (3.1) is the Kodaira-Spencer map for the family $\widetilde{\mathcal{M}}$. Moreover, the sheaves $\widetilde{\mathcal{M}}_x$ define points of a Hilbert scheme and the above Kodaira-Spencer map is the composition of the tangent map to the map $x \mapsto \widetilde{\mathcal{M}}_x$, and the Kodaira-Spencer map for the universal family. The latter is an isomorphism and since k has characteristics zero and $x \mapsto \widetilde{\mathcal{M}}_x$ is injective on closed points, its tangent map is injective at some point. It is this last statement what fails to be true in positive characteristics.

3.3. **A criterion in arbitrary characteristic.** As we showed in [16], Theorem 3.6 is no longer true in positive characteristic even in the smooth case. We reproduce here the counterexample given there. Let X be a smooth projective scheme of dimension m over a field k of characteristic $p > 0$, and $F \colon X \to X^{(p)}$ the *relative Frobenius morphism* [17, 3.1], which is topologically a homeomorphism. Let $\Gamma \hookrightarrow X \times X^{(p)}$ be the graph of F, whose associated integral functor is the direct image $F_* \colon D_c^b(X) \to D_c^b(X^{(p)})$. Since $F_*(\mathcal{O}_x) \simeq \mathcal{O}_{F(x)}$, one easily sees that Γ is strongly simple over X. However, $F_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_{X^{(p)}}$-module of rank p^m [17, 3.2], so that $\operatorname{Hom}^0_{D(X^{(p)})}(F_*(\mathcal{O}_X), \mathcal{O}_{F(x)}) \simeq k^{p^m}$ whereas $\operatorname{Hom}^0_{D(X)}(\mathcal{O}_X, \mathcal{O}_x) \simeq k$; thus F_* is not fully faithful.

Then, in arbitrary characteristic we need another characterisation of those kernels which give rise to fully faithful integral functors. The right notion is the following

Definition 3.7. An object $\mathcal{K}^\bullet$ of $D_c^b(X \times Y)$ satisfies the Cohen-Macaulay orthonormality conditions over X if it has the following properties:

(1) For every closed point $x \in X$ there is a l.c.i. zero cycle Z_x supported on x such that
$$\operatorname{Hom}^i_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_{x_1}}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{x_2})) = 0$$
unless $x_1 = x_2$ and $0 \le i \le \dim X$.

(2) There exists a closed point x such that at least one of the following conditions is fulfilled:

(2.1) $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_X), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k$.

(2.2) $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k$ for any l.c.i. zero cycle Z_x supported on x.

(2.3) $1 \le \dim_k \operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x})) \le l(\mathcal{O}_{Z_x})$ for any l.c.i. zero cycle Z_x supported on x, where $l(\mathcal{O}_{Z_x})$ is the length of $\mathcal{O}_{Z_x}$.

$\triangle$

Notice that now the objects $\mathbf{L}j_x^* \mathcal{K}^\bullet$ are not required to be simple; hence we avoid the word simple in the denomination of the objects satisfying the above conditions and turn back to something closer to the original Bondal and Orlov way to describe them.

We prove now a variant of Theorem 3.6 which is valid in arbitrary characteristic. Here, the requirement that X is integral can be relaxed.

Theorem 3.8. *Let X and Y be proper schemes over an algebraically closed field of arbitrary characteristic, and let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times Y)$ of finite homological dimension over both X and Y. Assume also that X is projective, Cohen-Macaulay, equidimensional and connected. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y} \colon D_c^b(X) \to D_c^b(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ satisfy the Cohen-Macaulay orthonormality conditions over X (Definition 3.7).*

Proof. The direct is immediate. For the converse we proceed as in the proof of 3.6. As there, using condition (1) of Definition 3.7, one sees that Φ has a right adjoint H and that $H \circ \Phi \simeq \Phi^{\mathcal{M}}_{X \to X}$, where $\mathcal{M}$ is a sheaf whose support is contained in the diagonal and $\pi_{1*}\mathcal{M}$ is locally free. Since X is connected, we can consider the rank r of $\pi_{1*}\mathcal{M}$, which is nonzero by condition (2) of Definition 3.7; thus the support of $\mathcal{M}$ is the diagonal. To conclude, we have only to prove that $r = 1$.

If $\mathcal{K}^\bullet$ satisfies (2.1) of Definition 3.7, then
$$\operatorname{Hom}^0_{D(X)}(\mathcal{O}_X, \Phi^{\mathcal{M}}_{X \to X}(\mathcal{O}_x)) \simeq \operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_X), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k.$$

Hence $\Phi^{\mathcal{M}}_{X \to X}(\mathcal{O}_x) \simeq \mathcal{O}_x$ and $r = 1$.

If $\mathcal{K}^\bullet$ satisfies (2.2) of Definition 3.7, then

$$\operatorname{Hom}^0_{D(X)}(\mathcal{O}_{Z_x}, \Phi^{\mathcal{M}}_{X \to X}(\mathcal{O}_x)) \simeq \operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_{Z_x}), \Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{O}_x)) \simeq k$$

for any l.c.i. zero cycle Z_x. Hence $\Phi^{\mathcal{M}}_{X \to X}(\mathcal{O}_x) \simeq \mathcal{O}_x$ and $r = 1$.

Finally, assume that $\mathcal{K}^\bullet$ satisfies (2.3) of Definition 3.7, and let us prove that then condition (2.2) of Definition 3.7 holds as well.

If $\widetilde{\mathcal{M}^\bullet} = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times X}}(\mathcal{M}, \pi_1^! \mathcal{O}_X)$, proceeding as in the proof of Theorem 3.6, one has that $\widetilde{\mathcal{M}^\bullet}$ is a sheaf $\widetilde{\mathcal{M}}$ supported topologically on the diagonal and that $\pi_{1*}\widetilde{\mathcal{M}}$ is locally free. It follows that the functor $\Phi^{\mathcal{M}}_{X \to X}$ has a left adjoint G, defined as $G(\mathcal{F}^\bullet) = \mathbf{R}\pi_{1*}(\pi_2^* \mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \widetilde{\mathcal{M}})$. This can be seen as follows: one has that $\mathcal{M} \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times X}}(\widetilde{\mathcal{M}}, \pi_1^! \mathcal{O}_X)$ by Proposition 2.11, and then $\pi_1^* \mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{M} \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times X}}(\widetilde{\mathcal{M}}, \pi_1^! \mathcal{G}^\bullet)$ by Proposition 2.8; thus

$$\operatorname{Hom}(\mathcal{F}^\bullet, \Phi^{\mathcal{M}}_{X \to X}(\mathcal{G}^\bullet)) \simeq \operatorname{Hom}(\pi_2^* \mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times X}}(\widetilde{\mathcal{M}}, \pi_1^! \mathcal{G}^\bullet)) \simeq \operatorname{Hom}(G(\mathcal{F}^\bullet), \mathcal{G}^\bullet),$$

which proves that G is a left adjoint to $\Phi^{\mathcal{M}}_{X \to X}$. Thus, condition (2.2) is equivalent to $\operatorname{Hom}^0_{D(X)}(G(\mathcal{O}_{Z_x}), \mathcal{O}_x) \simeq k$.

We know that $\mathcal{M}$ is a sheaf topologically supported on the diagonal and $\pi_{1*}\mathcal{M}$ is locally free. Then, if $\mathcal{F}$ is a sheaf, $\Phi^{\mathcal{M}}_{X \to X}(\mathcal{F})$ is also a sheaf and the functor $\mathcal{F} \mapsto \phi(\mathcal{F}) = \Phi^{\mathcal{M}}_{X \to X}(\mathcal{F})$ is exact. One has that $\mathcal{H}^i(G(\mathcal{F})) = 0$ for $i > 0$ because $\widetilde{\mathcal{M}}$ is topologically supported on the diagonal. Hence, $\operatorname{Hom}_{D(X)}(G(\mathcal{F}_1), \mathcal{F}_2) \simeq \operatorname{Hom}_{D(X)}(G^0(\mathcal{F}_1), \mathcal{F}_2)$ for whatever sheaves $\mathcal{F}_1$, $\mathcal{F}_2$, where $G^0 = \mathcal{H}^0 \circ G$. This has two consequences: first, G^0 is a left adjoint to ϕ; second, there are isomorphisms

$$\operatorname{Hom}^0_{D(X)}(G(\mathcal{O}_{Z_x}), \mathcal{O}_x) \simeq \operatorname{Hom}^0_{D(X)}(G^0(\mathcal{O}_{Z_x}), \mathcal{O}_x) \simeq \operatorname{Hom}_{\mathcal{O}_{Z_x}}(j_{Z_x}^* G^0(\mathcal{O}_{Z_x}), \mathcal{O}_x),$$

so that we are reduced to prove that $\operatorname{Hom}_{\mathcal{O}_{Z_x}}(j_{Z_x}^* G^0(\mathcal{O}_{Z_x}), \mathcal{O}_x) \simeq k$. Then, it is enough to see that $j_{Z_x}^* G^0(\mathcal{O}_{Z_x}) \simeq \mathcal{O}_{Z_x}$.

Using the exactness of ϕ, one proves by induction on the length $\ell(\mathcal{F})$ that the unit map $\mathcal{F} \to \phi(\mathcal{F})$ is injective for any sheaf $\mathcal{F}$ supported on x. It follows that the morphism $G^0(\mathcal{F}) \to \mathcal{F}$ is an epimorphism. To see that this is indeed the case, we have only to prove that for every sheaf $\mathcal{F}'$ supported on the point x, the morphism $\operatorname{Hom}(\mathcal{F}, \mathcal{F}') \to \operatorname{Hom}(G^0(\mathcal{F}), \mathcal{F}')$ is injective. By the adjunction formula, this is identified with the morphism $\operatorname{Hom}(\mathcal{F}, \mathcal{F}') \to \operatorname{Hom}(\mathcal{F}, \phi(\mathcal{F}'))$, which is injective because $\mathcal{F}' \to \phi(\mathcal{F}')$ is so.

In particular, $\eta \colon G^0(\mathcal{O}_{Z_x}) \to \mathcal{O}_{Z_x}$ is surjective, and $\dim \operatorname{Hom}^0_{D(X)}(G^0(\mathcal{O}_{Z_x}), \mathcal{O}_{Z_x}) \geq \ell(\mathcal{O}_{Z_x})$. Thus, by condition (2.3) of Definition 3.7, $\dim \operatorname{Hom}^0_{D(X)}(G^0(\mathcal{O}_{Z_x}), \mathcal{O}_{Z_x}) = \ell(\mathcal{O}_{Z_x})$. Since $\mathcal{O}_{Z_x}$ is free, the exact sequence of $\mathcal{O}_{Z_x}$-modules

$$0 \to \mathcal{N} \to j_{Z_x}^* G^0(\mathcal{O}_{Z_x}) \xrightarrow{j_{Z_x}^*(\eta)} \mathcal{O}_{Z_x} \to 0$$

splits, so that

$$0 \to \operatorname{Hom}_{\mathcal{O}_{Z_x}}(\mathcal{O}_{Z_x}, \mathcal{O}_{Z_x}) \to \operatorname{Hom}_{\mathcal{O}_{Z_x}}(j_{Z_x}^* G^0(\mathcal{O}_{Z_x}), \mathcal{O}_{Z_x}) \to \operatorname{Hom}_{\mathcal{O}_{Z_x}}(\mathcal{N}, \mathcal{O}_{Z_x}) \to 0$$

is an exact sequence. Moreover, $\operatorname{Hom}_{\mathcal{O}_{Z_x}}(\mathcal{N}, \mathcal{O}_{Z_x}) = 0$ because the two first terms have the same dimension. Let us see that this implies $\mathcal{N} = 0$. If $\mathcal{O}_x \to \mathcal{O}_{Z_x}$ is a nonzero, and then injective, morphism, we have $\operatorname{Hom}_{\mathcal{O}_{Z_x}}(\mathcal{N}, \mathcal{O}_x) = 0$ so that $\mathcal{N} = 0$ by Nakayama's lemma. $\qquad \square$

Remark 3.9. Even if the base field has characteristic zero, there is no obvious direct relationship between strongly simplicity and the Cohen-Macaulay orthonormality conditions for a kernel $\mathcal{K}^\bullet$ in $D_c^b(X \times Y)$, despite the fact that they are equivalent due to Theorems 3.6 and 3.8. Then, Theorem 3.8 is actually a new characterisation of which kernels induce a fully faithful integral functor.

If we wish to compare directly strongly simplicity with the Cohen-Macaulay orthonormality conditions, we see that the first condition in both definitions is the same; however, the second condition of strongly simplicity is a property that has to be satisfied at *every* closed point for *one* cycle supported on the point, whereas two of the forms of the second condition of the Cohen-Macaulay orthonormality conditions refer to a property which has to be satisfied for *all* zero cycles supported on only *one* closed point.   △

3.4. **A criterion in the relative setting.** In the relative situation the notions of strongly simple object and of an object satisfying the Cohen-Maculay orthonormality conditions, are the following.

Definition 3.10. Assume that $X \to S$ is Cohen-Macaulay. An object $\mathcal{K}^\bullet \in D_c^b(X \times_S Y)$ is *relatively strongly simple (resp. satisfies the relative Cohen-Macaulay orthonormality conditions)* over X if $\mathcal{K}^\bullet_s$ is strongly simple (resp. satisfies the Cohen-Maculay orthonormality conditions) over X_s for every closed point $s \in S$.   △

As a corollary of Proposition 2.15 and Theorems 3.6 and 3.8, we obtain the following result.

Theorem 3.11. *Let $X \to S$ and $Y \to S$ be proper and flat morphisms. Assume also that $X \to S$ is locally projective with Cohen-Macaulay, equidimensional, and connected fibers. Let $\mathcal{K}^\bullet$ be an object in $D_c^b(X \times_S Y)$ of finite homological dimension over both X and Y.*

 (1) *Assume that we are in characteristic 0 and $X \to S$ has integral fibers. The relative integral functor $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \to D_c^b(Y)$ is fully faithful if and only if $\mathcal{K}^\bullet$ is relatively strongly simple over X.*
 (2) *In arbitrary characteristic, $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \to D_c^b(Y)$ is fully faithful if and only if $\mathcal{K}^\bullet$ satisfies the relative Cohen-Macaulay orthonormality conditions over X.*

3.4.1. *Application to genus one fibrations in arbitrary characteristic.* We now apply Theorem 3.11 to give an alternative proof for Proposition 2.16 without using spherical objects.

With the same notation than in the Subsection 2.2.1 and taking into account the symmetry of $\mathcal{I}_\Delta$, to conclude that the functor $\Phi = \Phi_{X \to X}^{\mathcal{I}_\Delta}$ is an auto-equivalence of $D_c^b(X)$, it is enough to prove that $\mathcal{I}_\Delta$ satisfies the relative Cohen-Macaulay orthonormality conditions over the first factor. We fix a closed point $s \in S$ and consider two points x and $\bar{x}$ in the fiber X_s.

Let $Z_x \hookrightarrow X_s$ be a l.c.i. zero cycle supported on x of length ℓ defined by an ideal $\mathcal{I}_{Z_x}$; then $\mathcal{I}_{Z_x}$ is an invertible sheaf of $\mathcal{O}_{X_s}$-modules. We denote by $\pi \colon Z_x \times X_s \to X_s$ the second projection and by $\mathcal{J}_{Z_x}$ the push-forward by π of the ideal of the graph of

$Z_x \hookrightarrow X_s$. We have a commutative diagram of exact rows

$$(3.2) \qquad \begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_{Z_x} & \longrightarrow & \mathcal{O}_{X_s} & \longrightarrow & \mathcal{O}_{Z_x} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{J}_{Z_x} & \longrightarrow & \pi_*\mathcal{O}_{Z_x \times X_s} & \longrightarrow & \mathcal{O}_{Z_x} & \longrightarrow & 0
\end{array}$$

Since $\pi_*\mathcal{O}_{Z_x \times X_s}$ is a free $\mathcal{O}_{X_s}$-module of rank ℓ, and π has a section, one easily sees that the quotient sheaf $\pi_*\mathcal{O}_{Z_x \times X_s}/\mathcal{O}_{X_s}$ is free of rank $\ell - 1$ and we have an exact sequence

$$0 \to \mathcal{I}_{Z_x} \to \mathcal{J}_{Z_x} \to \mathcal{O}_{X_s}^{\oplus(\ell-1)} \to 0 \,,$$

which proves that $\mathcal{J}_{Z_x}$ is locally free of rank ℓ.

One has that $\Phi_s(\mathcal{O}_{Z_x}) \simeq \mathcal{J}_{Z_x}$ and $\Phi_s(\mathcal{O}_x) \simeq \mathcal{I}_x$, where $\mathcal{I}_x$ is the ideal of the point x; we have then to compute the groups $\mathrm{Hom}^i_{D(X_s)}(\mathcal{J}_{Z_x}, \mathcal{I}_{\bar{x}})$. Since $\mathcal{J}_{Z_x}$ is locally free, we know that $\mathrm{Hom}^i_{D(X_s)}(\mathcal{J}_{Z_x}, \mathcal{I}_{\bar{x}}) = 0$ unless $i = 0, 1$, so we have to worry only about the cases $i = 0$ and $i = 1$.

We have that

$$\mathrm{Hom}^0(\pi_*\mathcal{O}_{Z_x \times X_s}, \mathcal{I}_{\bar{x}}) \simeq \mathrm{Hom}^0(\mathcal{O}_{X_s}^{\oplus \ell}, \mathcal{I}_{\bar{x}}) \simeq H^0(X_s, \mathcal{I}_{\bar{x}})^{\oplus \ell} = 0 \,.$$

Then, from the bottom row of diagram (3.2) we obtain the following diagram
(3.3)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}^0(\mathcal{J}_{Z_x}, \mathcal{I}_{\bar{x}}) & \longrightarrow & \mathrm{Hom}^1(\mathcal{O}_{Z_x}, \mathcal{I}_{\bar{x}}) & \longrightarrow & \mathrm{Hom}^1(\pi_*\mathcal{O}_{Z_x \times X_s}, \mathcal{I}_{\bar{x}}) & \longrightarrow & \mathrm{Hom}^1(\mathcal{J}_{Z_x}, \mathcal{I}_{\bar{x}}) \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \downarrow \\
\cdots & \to & \mathrm{Hom}^0(\mathcal{J}_{Z_x}, \mathcal{O}_{X_s}) & \to & \mathrm{Hom}^1(\mathcal{O}_{Z_x}, \mathcal{O}_{X_s}) & \xrightarrow{\,g\,} & \mathrm{Hom}^1(\pi_*\mathcal{O}_{Z_x \times X_s}, \mathcal{O}_{X_s}) & \to & \mathrm{Hom}^1(\mathcal{J}_{Z_x}, \mathcal{O}_{X_s}) \to \cdots
\end{array}$$

a) *The morphisms h and g are isomorphisms.*

The fact that h is an isomorphism follows from the formula $\pi_*\mathcal{O}_{Z_x \times X_s} \simeq \mathcal{O}_{X_s}^{\oplus \ell}$. To prove that g is also an isomorphism, we first notice that g is the morphism obtained by applying the functor $\mathrm{Hom}^1(-, \mathcal{O}_{X_s})$ to the projection $\varpi \colon \pi_*\mathcal{O}_{Z_x \times X_s} \to \mathcal{O}_{Z_x}$. By duality, $\mathrm{Hom}^1(-, \mathcal{O}_{X_s}) \simeq H^0(X_s, -)^*$ and one finishes because $\varpi \colon \pi_*\mathcal{O}_{Z_x \times X_s} \to \mathcal{O}_{Z_x}$ induces a isomorphism between the corresponding spaces of global sections.

b) *Condition (1) of Definition 3.7*

It suffices to see that $\mathrm{Hom}^1(\mathcal{O}_{Z_x}, \mathcal{I}_{\bar{x}}) \to \mathrm{Hom}^1(\pi_*\mathcal{O}_{Z_x \times X_s}, \mathcal{I}_{\bar{x}})$ is an isomorphism when $x \neq \bar{x}$. In this case, the second vertical arrow of diagram (3.3) is an isomorphism, so that the above morphism is identified with g, which is an isomorphism.

c) *Condition (2.2) of Definition 3.7*

We have to prove that $\mathrm{Hom}^0(\mathcal{J}_{Z_x}, \mathcal{I}_x) \simeq k$. From diagram (3.3) $\mathrm{Hom}^0(\mathcal{J}_{Z_x}, \mathcal{I}_x)$ is isomorphic to the kernel of the second vertical arrow $\mathrm{Hom}^1(\mathcal{O}_{Z_x}, \mathcal{I}_x) \to \mathrm{Hom}^1(\mathcal{O}_{Z_x}, \mathcal{O}_{X_s})$, which is isomorphic to $\mathrm{Hom}^0(\mathcal{O}_{Z_x}, \mathcal{O}_x) \simeq k$.

Notice that the proof of this result does not use spanning classes and is valid in any characteristic.

4. FOURIER-MUKAI PARTNERS

When two proper schemes X and Y have equivalent derived categories $D_c^b(X) \simeq D_c^b(Y)$, they are called *D-equivalent*. When the equivalence is given by an integral functor (such functors are called Fourier-Mukai functors), we have the following more restrictive notion.

Definition 4.1. Two proper schemes X and Y are *Fourier-Mukai partners* if there is a Fourier-Mukai functor
$$\Phi_{X \to Y}^{\mathcal{K}^\bullet} \colon D_c^b(X) \simeq D_c^b(Y) \,.$$

$\triangle$

Due to Orlov's representation theorem [25], if X and Y are smooth and projective then they are D-equivalent if and only if they are Fourier-Mukai partners. Since the validity of Orlov's theorem for singular varieties is still unknown, we shall adopt the (hopefully provisional) notion of Fourier-Mukai partners.

In the rest of the section X and Y are projective and $\mathcal{K}^\bullet$ is a kernel in $D_c^b(X \times Y)$ such that $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ is an equivalence $D_c^b(X) \simeq D_c^b(Y)$.

Proposition 4.2. *There is a natural isomorphism*
$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_X^! \mathcal{O}_X) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y) \,.$$

Proof. By Proposition 2.10 the right adjoint to $\Phi_{X \to Y}^{\mathcal{K}^\bullet}$ is $\Phi_{Y \to X}^{\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)}$ and the right adjoint to the latter functor is $\Phi_{X \to Y}^{\mathcal{L}^\bullet}$ where
$$\mathcal{L}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y), \pi_X^! \mathcal{O}_X) \,.$$
Since the left and right adjoint of an equivalence are naturally isomorphic, one has that $\Phi_{X \to Y}^{\mathcal{K}^\bullet} \simeq \Phi_{X \to Y}^{\mathcal{L}^\bullet}$.

Moreover, for any scheme T we can consider $X_T = X \times T$ and $Y_T = Y \times T$. The kernel $\mathcal{K}^\bullet$ gives rise to a relative kernel $\mathcal{K}^\bullet{}_T = \pi_{X \times Y}^* \mathcal{K}^\bullet \in D_c^b(X_T \times_T Y_T)$, where $\pi_{X \times Y}$ is the projection $X_T \times_T Y_T \simeq X \times Y \times T \to X \times Y$. This relative kernel is of finite homological dimension over both X_T and Y_T, and the relative integral functor $\Phi_{X_T \to Y_T}^{\mathcal{K}^\bullet{}_T} \colon D_c^b(X_T) \to D_c^b(Y_T)$ is an equivalence by Proposition 2.15. Arguing as above, we get $\Phi_{X_T \to Y_T}^{\mathcal{K}^\bullet{}_T} \simeq \Phi_{X_T \to Y_T}^{\mathcal{L}^\bullet{}_T}$. If we take $T = X$ and apply the above isomorphism to the sheaf $\mathcal{O}_\Delta$ of the diagonal, we obtain an isomorphism
$$\mathcal{K}^\bullet \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathbf{R}\mathcal{H}om_{\mathcal{O}_{X \times Y}}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y), \pi_X^! \mathcal{O}_X) \,.$$
Since $\mathbf{R}\mathcal{H}om_{X \times Y}^\bullet(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)$ is of finite homological dimension over X, we conclude by Proposition 2.11. $\qquad\square$

Remark 4.3. When X and Y are smooth, Proposition 4.2 is equivalent to the fact that any equivalence commutes with the Serre functors. Then, Proposition 4.2 can be considered as a generalisation to that property for arbitrary singular schemes. $\quad\triangle$

Our next aim is to prove that any integral Fourier-Mukai partner of a projective Cohen-Macaulay (resp. Gorenstein) scheme X is also Cohen-Macaulay (resp. Gorenstein).

Theorem 4.4. *Let X be a projective equidimensional Cohen-Macaulay scheme and Y a projective Fourier-Mukai partner of X. Then one has*

(1) *If Y is reduced, then Y is equidimensional of dimension $m = \dim X$.*

(2) *If Y is equidimensional and $\dim Y = \dim X$, then Y is Cohen-Macaulay. More-over, if X is Gorenstein, then Y is Gorenstein as well.*

Proof. By Proposition 4.2 the integral functors

$$\Psi_1 = \Phi_{X \to Y}^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times Y}}(\mathcal{K}^\bullet, \pi_X^! \mathcal{O}_X)} : D_c^b(X) \to D_c^b(Y)$$

$$\Psi_2 = \Phi_{X \to Y}^{\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times Y}}(\mathcal{K}^\bullet, \pi_Y^! \mathcal{O}_Y)} : D_c^b(X) \to D_c^b(Y) \,,$$

are naturally isomorphic. Hence, for any l.c.i zero cycle Z_x supported on a closed point $x \in X$ and any closed point $y \in Y$ we have

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_1(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_2(\mathcal{O}_{Z_x})) \,.$$

This gives rise to the formula:

$$(4.1) \qquad \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}), \mathcal{O}_y) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}), \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \mathcal{O}_Y))[m] \,,$$

where $\Phi_{Z_x} : D_c^b(Z_x) \to D_c^b(Y)$ is the integral functor of kernel $\mathcal{K}^\bullet_{Z_x} = \mathbf{L}j_{Z_x}^* \mathcal{K}^\bullet \in D_c^b(Z_x \times Y)$ (see Lemmas 4.5 and 4.6 for details).

Since X is Cohen-Macaulay, every l.c.i. cycle Z_x is Cohen-Macaulay as well, and its dualizing complex $\mathcal{D}^\bullet_{Z_x}$ is a single sheaf ω_{Z_x}.

As Φ is an equivalence of categories, it follows from Lemma 3.5 that the objects $\Phi_{Z_x}(\mathcal{O}_{Z_x}) \simeq \Phi(\mathcal{O}_{Z_x})$ for all l.c.i. cycles Z_x form a spanning class for $D_c^b(Y)$. Then, if we fix a closed point $y \in Y$, there is a l.c.i. cycle Z_x such that $y \in \mathrm{supp}(\Phi_{Z_x}(\mathcal{O}_{Z_x}))$.

Let us denote by $\phi \colon Y' = \mathrm{Spec}\,\mathcal{O}_{Y,y} \to Y$ the natural flat morphism. The composition $\Phi'_{Z_x} = \phi^* \circ \Phi_{Z_x}$ is an integral functor with kernel $\mathcal{K}'^\bullet_{Z_x} = (1 \times \phi)^* \mathcal{K}^\bullet_{Z_x} \in D_c^b(Z_x \times Y')$. If we apply ϕ^* to (4.1) we get the analogous formula

$$(4.2) \qquad \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\omega_{Z_x}), \mathcal{O}_y) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}))[m] \,,$$

After shifting degrees if necessary, we can assume that $\mathcal{H}^0(\mathcal{K}'^\bullet_{Z_x}) \neq 0$ and $\mathcal{H}^i(\mathcal{K}'^\bullet_{Z_x}) = 0$ for $i > 0$. Then $\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})) \neq 0$ and $\mathcal{H}^i(\Phi'_{Z_x}(\mathcal{O}_{Z_x})) = 0$ for $i > 0$. For every nonzero finite $\mathcal{O}_{Z_x}$-module $\mathcal{F}$, one has that $\mathcal{H}^0(\pi_{Z_x}^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{K}'^\bullet_{Z_x}) \simeq \pi_{Z_x}^* \mathcal{F} \otimes \mathcal{H}^0(\mathcal{K}'^\bullet_{Z_x}) \neq 0$ and that $\mathcal{H}^i(\pi_{Z_x}^* \mathcal{F} \overset{\mathbf{L}}{\otimes} \mathcal{K}'^\bullet_{Z_x}) = 0$ for $i > 0$, where $\pi_{Z_x} \colon Z_x \times Y' \to Z_x$ is the projection. Thus, $\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{F})) \neq 0$ and $\mathcal{H}^i(\Phi'_{Z_x}(\mathcal{F})) = 0$ for $i > 0$.

It follows that $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\omega_{Z_x}), \mathcal{O}_y)$ has no negative cohomology sheaves and that its 0-th cohomology sheaf is nonzero. By Equation (4.2), the same happens for the object $\mathcal{Q}^\bullet = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}))[m]$, that is,

$$(4.3) \qquad\qquad \mathcal{H}^i(\mathcal{Q}^\bullet) = 0 \text{ for } i < 0, \text{ and } \mathcal{H}^0(\mathcal{Q}^\bullet) \neq 0 \,.$$

(1) Assume that Y is reduced and we choose y to be a smooth point of an irreducible component Y_0 of Y. Then $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}) \simeq \mathcal{O}_y[-n]$, where $n = \dim Y_0$, and we have $\mathcal{Q}^\bullet \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathcal{O}_y)[m-n]$. Since both $\mathcal{Q}^\bullet$ and $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathcal{O}_y)$ have no negative cohomology sheaves and nonzero 0-th cohomology sheaf, one must have $m - n = 0$. Thus all the irreducible components of Y have dimension m.

(2) Let $y \in Y$ be a closed point and let j_0 be the first index j with $\mathcal{E}xt^j_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}) \neq 0$, that is, j_0 is the depth of the local ring $\mathcal{O}_{Y,y}$. Then

$$\mathcal{H}^{j_0}[\mathcal{Q}^\bullet[-m]] \simeq \mathcal{H}om_{\mathcal{O}_{Y'}}(\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})), \mathcal{E}xt^{j_0}_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'})) \,,$$

which is not zero because $\mathcal{E}xt^{j_0}_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'})$ is a nonzero $\mathcal{O}_y$-vector space. Hence $j_0 \geq m = \dim Y$ (cf. Equation (4.3)), and Y is Cohen-Macaulay.

Assume now that X is Gorenstein. Then $\omega_{Z_x} \simeq \mathcal{O}_{Z_x}$ and we have

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathcal{O}_y) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\Phi'_{Z_x}(\mathcal{O}_{Z_x}), \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}))[m],$$

by Equation (4.2). Since $\mathcal{E}xt^j_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}) = 0$ for $j < m$, we deduce that

$$\mathrm{Hom}_{Y'}(\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})), \mathcal{O}_y) \simeq \mathrm{Hom}_{Y'}(\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})), \mathcal{E}xt^m_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'})),$$

and then,

$$\mathrm{Hom}_{\mathcal{O}_y}(\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})) \otimes \mathcal{O}_y, \mathcal{O}_y) \simeq \mathrm{Hom}_{\mathcal{O}_y}(\mathcal{H}^0(\Phi'_{Z_x}(\mathcal{O}_{Z_x})) \otimes \mathcal{O}_y, \mathcal{E}xt^m_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'})).$$

Thus, $\dim \mathcal{E}xt^m_{\mathcal{O}_{Y'}}(\mathcal{O}_y, \mathcal{O}_{Y'}) = 1$ and Y is Gorenstein. $\qquad\square$

We now prove the auxiliary Lemmas used in the proof of Theorem 4.4.

Lemma 4.5. $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_1(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}), \mathcal{O}_y).$

Proof. We first compute $\Psi_1(\mathcal{O}_{Z_x})$. By Corollary 2.12 and Lemma 2.13

$$\Psi_1(\mathcal{O}_{Z_x}) \simeq \Psi_1(\mathcal{O}^{\sharp\sharp}_{Z_x}) \simeq (\Phi(\mathcal{O}^\sharp_{Z_x}))^\sharp \simeq (\Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}))^\sharp.$$

where the last isomorphism is due to the isomorphism $\mathcal{O}^\sharp_{Z_x} \simeq j_{Z_x*}\mathcal{D}^\bullet_{Z_x}$, which follows by relative duality for the closed immersion $j_{Z_x} : Z_x \hookrightarrow X$. Considering Equation (1.1), we obtain:

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_1(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, (\Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}))^\sharp)$$

$$\simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y \overset{\mathbf{L}}{\otimes} \Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}), \mathcal{D}^\bullet_Y) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{D}^\bullet_{Z_x}), \mathcal{O}^\sharp_y)$$

Relative duality for the closed immersion $\{y\} \hookrightarrow Y$ gives $\mathcal{O}^\sharp_y \simeq \mathcal{O}_y$, which finishes the proof. $\qquad\square$

Lemma 4.6. *One has*

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_2(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}), \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \mathcal{O}_Y))[m].$$

Proof. We first compute $\Psi_2(\mathcal{O}_{Z_x})$. Let us denote π_{Z_x} and $\bar{\pi}_Y$ the projections of $Z_x \times Y$ onto Z_x and Y respectively. Using that $\mathcal{O}_{Z_x \times Y}$ is of finite homological dimension as a module over $\mathcal{O}_{X \times Y}$ and taking into account Equation (1.2), we have that

$$\Psi_2(\mathcal{O}_{Z_x}) \simeq \bar{\pi}_{Y*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Z_x \times Y}}(\mathcal{K}^\bullet_{Z_x}, \mathbf{L}j^*_{Z_x}\pi^!_Y\mathcal{O}_Y).$$

The term $\mathbf{L}j^*_{Z_x}\pi^!_Y\mathcal{O}_Y$ is computed as $\mathbf{L}j^*_{Z_x}\pi^*_X\mathcal{D}^\bullet_X \simeq \pi^*_{Z_x}\mathbf{L}j^*_{Z_x}\mathcal{D}^\bullet_X$. Furthermore, relative duality for the regular immersion $j_{Z_x} : Z_x \hookrightarrow X$ gives $j^!_{Z_x}\mathcal{O}_X \simeq \mathcal{O}_{Z_x}[-m]$, where $m = \dim X$ and $\mathcal{D}^\bullet_{Z_x} \simeq j^!_{Z_x}\mathcal{O}_X \overset{\mathbf{L}}{\otimes} \mathbf{L}j^*_{Z_x}\mathcal{D}^\bullet_X$. Thus, $\pi^*_{Z_x}\mathbf{L}j^*_{Z_x}\mathcal{D}^\bullet_X \simeq \pi^*_{Z_x}\mathcal{D}^\bullet_{Z_x}[m] \simeq \bar{\pi}^!_Y\mathcal{O}_Y[m]$, so that

$$\Psi_2(\mathcal{O}_{Z_x}) \simeq \bar{\pi}_{Y*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{Z_x \times Y}}(\mathcal{K}^\bullet_{Z_x}, \bar{\pi}^!_Y\mathcal{O}_Y[m])$$

$$\simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}), \mathcal{O}_Y[m]).$$

Then

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \Psi_2(\mathcal{O}_{Z_x})) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathcal{O}_y, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\Phi_{Z_x}(\mathcal{O}_{Z_x}), \mathcal{O}_Y[m])$$

and one concludes by Equation (1.1).

$\qquad\square$

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E-mail address: `ruiperez@usal.es`

E-mail address: `anacris@usal.es`

E-mail address: `fsancho@usal.es`

Departamento de Matemáticas and Instituto Universitario de Física Fundamental y Matemáticas (IUFFYM), Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain