

SIMPSON JACOBIANS OF REDUCIBLE CURVES

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ABSTRACT. For any projective curve X let $\overline{M}^d(X)$ be the Simpson moduli space of pure dimension one rank 1 degree d sheaves that are semistable with respect to a fixed polarization H on X . When X is a reduced curve the connected component of $\overline{M}^d(X)$ that contains semistable line bundles can be considered as the compactified Jacobian of X . In this paper we give explicitly the structure of this compactified Simpson Jacobian for the following projective curves: tree-like curves and all reduced and reducible curves that can appear as Kodaira singular fibers of an elliptic fibration, that is, the fibers of types *III*, *IV* and I_N with $N \geq 2$

1. INTRODUCTION

The problem of compactifying the (generalized) Jacobian of a singular curve has been studied since Igusa's work [16] around 1950. He constructed a compactification of the Jacobian of a nodal and irreducible curve X as the limit of the Jacobians of smooth curves approaching X . Igusa also showed that his compactification does not depend on the considered family of smooth curves. An intrinsic characterization of the boundary points of Igusa's compactification as the torsion free, rank 1 sheaves which are not line bundles is due to Mumford and Mayer. The complete construction for a family of integral curves over a noetherian Hensel local ring with residue field separably closed was carried out by D'Souza [9]. One year later, Altman and Kleiman [2] gave the construction for a general family of integral curves.

When the curve X is reducible and nodal, Oda and Seshadri [20] produced a family of compactified Jacobians Jac_ϕ parameterized by an

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element ϕ of a real vector space. Seshadri dealt in [22] with the general case of a reduced curve considering sheaves of higher rank as well.

In 1994, Caporaso showed [8] how to compactify the relative Jacobian over the moduli of stable curves and described the boundary points of the compactified Jacobian of a stable curve X as invertible sheaves on certain Deligne-Mumford semistable curves that have X as a stable model. Recently, Pandharipande [21] has given another construction with the boundary points now representing torsion free, rank 1 sheaves and he showed that Caporaso's compactification was equivalent to his.

On the other hand, Esteves [10] constructed a compactification of the relative Jacobian of a family of geometrically reduced and connected curves and compared it with Seshadri's construction [22] using theta functions and Alexeev [1] gave a description of the Jacobian of certain singular curves in terms of the orientations on complete subgraphs of the dual graph of the curve.

Most of the above papers are devoted to the construction of the compactified Jacobian of a curve, not to describe it. Moreover these constructions are only valid for certain projective curves. However Simpson's work [23] on the moduli of pure coherent sheaves on projective schemes allows us to define in a natural way the Jacobian of any polarized projective curve X as the space $\text{Jac}^d(X)_s$ of equivalence classes of stable invertible sheaves with degree d with respect to the fixed polarization. This is precisely the definition we adopt and we also denote by $\overline{\text{Jac}}^d(X)$ the space of equivalence classes of semistable pure dimension one sheaves with rank 1 on every irreducible component of X and degree d .

In some recent papers about the moduli spaces of stable vector bundles on elliptic fibrations, for instance [15], [4], [6], [7], the Jacobian in the sense of Simpson of spectral curves appears. Beauville [3] uses it as well in counting the number of rational curves on K3 surfaces. This suggests the necessity to determine the structure of these Simpson Jacobians.

In this paper we give an explicit description of the structure of these Simpson schemes, $\text{Jac}^d(X)_s$ and $\overline{\text{Jac}}^d(X)$ of any degree d , for X a polarized curve of the following types: tree-like curves and all reduced and reducible curves that can appear as singular fibers of an elliptic fibration. The paper is organized as follows. We will work over an algebraically closed field κ of characteristic zero.

In the second part we define the Simpson Jacobian $\text{Jac}^d(X)_s$ of any projective curve X over κ , but the definition is also functorial. By Simpson's work [23], the moduli space $\overline{M}^d(X)$ of equivalence classes of

semistable pure dimension one sheaves on X of (polarized) rank 1 and degree d is a projective scheme that contains $\text{Jac}^d(X)_s$ and then it is a compactification of $\text{Jac}^d(X)_s$. The subscheme $\overline{\text{Jac}}^d(X)$ is always a connected component of this compactification $\overline{M}^d(X)$. We show that $\overline{\text{Jac}}^d(X)$ is only one of the connected components that can appear in the moduli space $\overline{M}^d(X)$ (see Proposition 2.3) and it coincides with the compactifications constructed by other authors. The description of the connected components of $\overline{M}^d(X)$ different to $\overline{\text{Jac}}^d(X)$ is still an open problem and, as we show (see Example 2.4), it has to cover the study of moduli spaces of higher rational rank sheaves on reducible curves. This study is not included in this paper, however when the curve X is also reduced, as our curves, the connected component $\overline{\text{Jac}}^d(X)$ is projective so that it can be considered as a compactification of the Simpson Jacobian of X as well.

In the third part we collect some general properties about semistable pure dimension one rank 1 sheaves on reducible curves and some results relating the Picard groups of two reduced and projective curves if there is a birational and finite morphism between them.

In the fourth part we give the description (Theorem 4.5) of the Simpson Jacobian $\text{Jac}^d(X)_s$ and of the connected component $\overline{\text{Jac}}^d(X)$ of its compactification when X is a tree-like curve, that is, a projective reduced and connected curve such that the intersection points of its irreducible components are disconnecting ordinary double points, but the singularities lying only at one irreducible component can be arbitrary singularities. The Picard group of a tree-like curve is isomorphic to the direct product of the Picard groups of its irreducible components. Then we use a Teixidor's lemma [24] that allows us to order the irreducible components of X and to find subcurves X_i of X that are also tree-like curves and intersect their complements in X at just one point. With this lemma, the determination of the stable rank 1 sheaves on X which are locally free at all intersection points of the irreducible components of X is analogous to this given by Teixidor in [24] for curves of compact type. When the sheaf is not locally free at some intersection points, to study its stability we use a Seshadri's lemma describing the stalk of a pure dimension one sheaf at the intersection points. Finally for strictly semistable rank 1 sheaves, we give a recurrent algorithm that enables us to construct a Jordan-Hölder filtration and then to determine their S -equivalence classes. The descriptions given here, and in particular

the recurrent algorithm, will be essential in the next part for the analysis of the Simpson schemes of Kodaira reduced fibers of an elliptic fibration.

The aim of the fifth part is to describe the structure of the Simpson Jacobian and of the connected component $\overline{\text{Jac}}^d(X)$ of its compactification $\overline{M}^d(X)$ when X is a reducible and reduced Kodaira fiber of an elliptic fibration, that is, a fiber of type *III*, *IV* or I_N with $N \geq 2$. Using the results of the second section, we first determine the Picard group of these curves. We find then necessary and sufficient conditions for a line bundle to be (semi)stable. For the analysis of (semi)stable pure dimension one rank 1 sheaves which are not line bundles, the fundamental result is Lemma 5.3: if X is a projective reduced and connected curve, P a singular point at which X is Gorenstein and F is a pure dimension one rank 1 sheaf on X which is not invertible at P , then there is a projective curve X' , a finite birational morphism $\phi: X' \rightarrow X$ and a pure dimension one rank 1 sheaf G on X' such that $\phi_*(G) = F$. Hence we prove that there is an isomorphism

$$\overline{\text{Jac}}^d(X) - \text{Jac}^d(X) \simeq \overline{\text{Jac}}^{d-1}(X')$$

between the border of the connected component $\overline{\text{Jac}}^d(X)$ of X and the whole component $\overline{\text{Jac}}^{d-1}(X')$ of the curve X' . When X is a fiber of type *III* or I_N , the curve X' is a tree-like curve and then the structure of this border is determined by previous results. If X is a fiber of type *IV*, that is, three rational smooth curves meeting at one point P , the curve X' is also the union of three rational smooth curves meeting at one point but in such a way that the curve X' cannot be embedded (even locally) in a smooth surface (see Figure 3). The study of Simpson schemes for this curve X' is in the subsection 5.4. Lemma 5.11, whose proof is a generalization of the proof given by Seshadri for Lemma 3.1 in this paper, plays a fundamental role in the most difficult part of the analysis, the study of (semi)stable pure dimension one rank 1 sheaves on X' which are not line bundles. Subsection 5.4 completes then the description of the Simpson schemes for the curves of type *IV*.

We finish the paper with a special study for the case of degree zero. This case is very interesting because the polarization does not influence semistability conditions (Corollary 6.5). We also prove that if X is a fiber of type *III*, *IV* or I_N , all strictly semistable pure dimension one rank 1 sheaves of degree 0 on X are in the same S -equivalence class (Corollary 6.7) and that the moduli space of semistable pure dimension one sheaves of rank 1 and degree 0 on X is never a fine moduli space (Corollary 6.6). These results can be found in [6] for a curve X of type

I_2 where Căldăraru uses them to prove that the connected component $\overline{\text{Jac}}^0(X)$ is isomorphic to a rational curve with one node.

If $p: S \rightarrow B$ is an elliptic fibration with reduced fibers, the global structure of the compactified Simpson Jacobian $\overline{\text{Jac}}^0(S/B)$ is not totally known (when the fibers of p are geometrically integral some results can be found in [4]). We hope that the descriptions of the compactified Simpson Jacobians of degree zero given here (Propositions 6.2 and 6.3) will be very useful to study the singular fibers of $\overline{\text{Jac}}^0(S/B) \rightarrow B$, to know the singularities of the variety $\overline{\text{Jac}}^0(S/B)$ and to find, using the results of [5], examples of non isomorphic elliptic fibrations having isomorphic derived categories.

Two consequences of the given descriptions are the following: the first is that for reducible curves it is not true in general that the tensor product of semistable pure dimension one sheaves of rank 1 is semistable (see Example 4.11) and the second is that the pullback of a stable pure dimension one rank 1 sheaf by a finite morphism of reducible curves is not stable either (see Example 5.16). These two statements constitute the principal problem to generalize to reducible curves the study of Abel maps given by Esteves, Gagné and Kleiman in [12] for families of integral curves.

The results presented here are part of my Ph. D. thesis. I am very grateful to my advisor D. Hernández Ruipérez, who introduced me to the problem of compactifying the generalized Jacobian of a curve, for his invaluable help and his constant encouragement. I could not have done this work without his support.

2. THE SIMPSON JACOBIAN OF A PROJECTIVE CURVE

Let X be a projective curve over κ . Let \mathcal{L} be an ample invertible sheaf on X , let H be the associated polarization and let h be the degree of H .

Let F be a coherent sheaf on X . We say that F is *pure of dimension one* or *torsion free* if for all nonzero subsheaves $F' \hookrightarrow F$ the dimension of $\text{Supp}(F')$ is 1. The (polarized) *rank* and *degree* with respect to H of F are the rational numbers $r_H(F)$ and $d_H(F)$ determined by the Hilbert polynomial

$$P(F, n, H) = \chi(F \otimes \mathcal{O}_X(nH)) = h r_H(F)n + d_H(F) + r_H(F)\chi(\mathcal{O}_X).$$

The *slope* of F is defined by

$$\mu_H(F) = \frac{d_H(F)}{r_H(F)}$$

The sheaf F is *stable* (resp. *semistable*) with respect to H if F is pure of dimension one and for any proper subsheaf $F' \hookrightarrow F$ one has

$$\mu_H(F') < \mu_H(F) \text{ (resp. } \leq)$$

In [23] Simpson defined the multiplicity of F as the integer number $h r_H(F)$ and the slope as the quotient

$$\frac{d_H(F) + r_H(F)\chi(\mathcal{O}_X)}{h r_H(F)}$$

Stability and semistability considered in terms of Simpson's slope and in terms of μ_H are equivalent. We adopt these definitions of rank and degree of F because they coincide with the classical ones when the curve X is integral. Note however that if X is not integral, the rank and the degree of a pure dimension one sheaf are not in general integer numbers.

According to the general theory, for every semistable sheaf F with respect to H there is a *Jordan-Hölder filtration*

$$0 = F_0 \subset F_1 \subset \dots \subset F_n = F$$

with stable quotients F_i/F_{i-1} and $\mu_H(F_i/F_{i-1}) = \mu_H(F)$ for $i = 1, \dots, n$. This filtration need not be unique, but *the graded object* $Gr(F) = \bigoplus_i F_i/F_{i-1}$ does not depend on the choice of the Jordan-Hölder filtration. Two semistable sheaves F and F' on X are said to be *S-equivalent* if $Gr(F) \simeq Gr(F')$. Observe that two stable sheaves are *S-equivalent* only if they are isomorphic.

In the relative case, given a scheme S of finite type over κ , a projective morphism of schemes $f: X \rightarrow S$ whose geometric fibers are curves and a relative polarization H , we define the *relative rank* and *degree* of a coherent sheaf F on X , flat over S , as its rank and degree on fibers, and we say that F is *relatively pure of dimension one* (resp. *stable*, resp. *semistable*) if it is flat over S and if its restriction to every geometric fiber of f is pure of dimension one (resp. stable, resp. semistable).

Let $\mathbf{M}^d(X/S, r)_s$ (resp. $\overline{\mathbf{M}}^d(X/S, r)_s$) be the functor which to any S -scheme T associates the set of equivalence classes of stable locally free (resp. relatively pure dimension one) sheaves on $X_T = X \times_S T$ with relative rank r and degree d . Two such sheaves F and F' are said to be equivalent if $F' \simeq F \otimes f_T^* N$, where N is a line bundle on T and $f_T: X_T \rightarrow T$ is the natural projection. Similarly, we define the functor $\mathbf{M}^d(X/S, r)$ (resp. $\overline{\mathbf{M}}^d(X/S, r)$) of semistable locally free (resp. relatively pure dimension one) sheaves of relative rank r and degree d .

As a particular case of the Simpson's work [23], there exists a projective scheme $\overline{M}^d(X/S, r) \rightarrow S$ which coarsely represents the functor $\overline{\mathbf{M}}^d(X/S, r)$. Rational points of $\overline{M}^d(X/S, r)$ correspond to S -equivalence classes of semistable torsion free sheaves of rank r and degree d on X_s ($s \in S$). Moreover, $\mathbf{M}^d(X/S, r)$ is coarsely represented by a subscheme $M^d(X/S, r)$ and there are open subschemes $M^d(X/S, r)_s$ and $\overline{M}^d(X/S, r)_s$ which represent the other two functors.

Definition 2.1. The Simpson Jacobian of degree d of the curve X is $\text{Jac}^d(X)_s = M^d(X/\text{Spec } \kappa, 1)_s$. We denote $\overline{M}^d(X) = \overline{M}^d(X/\text{Spec } \kappa, 1)$. The projective scheme $\overline{M}^d(X)$ is the compactification of the Simpson Jacobian of X of degree d .

When X is an integral curve every torsion free rank 1 sheaf is stable, and then $\text{Jac}^d(X)_s$ is equal to the Picard scheme $\text{Pic}^d(X)$ and $\overline{M}^d(X)$ coincides with Altman-Kleiman's compactification [2].

In some papers about Jacobians of non irreducible curves, for instance [22], [24], [25], [21], torsion free rank 1 sheaves are considered as those pure dimension one sheaves having rank 1 on every irreducible component of the curve. If the notion of rank is given by the Hilbert polynomial, there can be, depending on the degree of the fixed polarization H on the curve X , pure dimension one sheaves of rank 1 whose restrictions to some irreducible components of X are not of rank 1.

Example 2.2. Let X be the nodal curve which is a union of two smooth curves C_1 and C_2 meeting transversally at one point P . Let H be a polarization on X such that $\deg(H|_{C_1}) = \deg(H|_{C_2}) = h$ and let F be a locally free sheaf on C_1 of rank 2 and degree d . Let us denote by $i: C_1 \hookrightarrow X$ the inclusion map. The sheaf $i_*(F)$ is pure of dimension one on X and, since

$$P(i_*(F), n, H) = P(F, n, H|_{C_1}) = 2hn + d + 2\chi(\mathcal{O}_{C_1}),$$

one has $r_H(i_*(F)) = 1$. However, the restriction of $i_*(F)$ to C_2 is a torsion sheaf supported at P .

In order to avoid the confusion, we will say that a sheaf F on X is of *polarized rank 1* if F has rank 1 with the Hilbert polynomial, that is, $r_H(F) = 1$, whereas by *rank 1* sheaves we mean those sheaves that have rank 1 on every irreducible component of X .

Semistable pure dimension one rank 1 sheaves of degree d have polarized rank 1 with respect to any polarization and they are a connected component of the compactification $\overline{M}^d(X)$ of the Simpson Jacobian of X . Let us denote by $\overline{\text{Jac}}^d(X)$ this connected component.

Depending on the degree of the polarization H on X , $\overline{\text{Jac}}^d(X)$ need not be the unique connected component of the moduli space $\overline{M}^d(X)$. The following proposition shows all connected components that can appear in $\overline{M}^d(X)$ when X is a union of two integral curves meeting transversally only at one point.

If X is a projective and reduced curve with irreducible components C_1, \dots, C_N and P_1, \dots, P_k are the intersection points of C_1, \dots, C_N , it is known (see [22]) that for every pure dimension one sheaf F on X there is an exact sequence

$$0 \rightarrow F \rightarrow F_{C_1} \oplus \dots \oplus F_{C_N} \rightarrow T \rightarrow 0$$

where we denote $F_{C_i} = F|_{C_i}/\text{torsion}$ and T is a torsion sheaf whose support is contained in the set $\{P_1, \dots, P_k\}$.

Proposition 2.3. *Let $X = C_1 \cup C_2$ be a projective curve with C_i integral curves for $i = 1, 2$ and $C_1 \cdot C_2 = P$. Let H be a polarization on X of degree h and let h_{C_i} be the degree of the induced polarization H_{C_i} on C_i for $i = 1, 2$. It holds that*

- (1) *If h is not a multiple of h_{C_i} for $i = 1, 2$, then the only connected component of $\overline{M}^d(X)$ is $\overline{\text{Jac}}^d(X)$.*
- (2) *If $h = rh_{C_i}$ only for one $i = 1$ or 2 , then*

$$\overline{M}^d(X) = \overline{\text{Jac}}^d(X) \sqcup \overline{M}^{d_i}(C_i, r)$$

where $d_i = d + \chi(\mathcal{O}_X) - r\chi(\mathcal{O}_{C_i})$.

- (3) *If $h = rh_{C_1} = r'h_{C_2}$, then*

$$\overline{M}^d(X) = \overline{\text{Jac}}^d(X) \sqcup \overline{M}^{d_1}(C_1, r) \sqcup \overline{M}^{d_2}(C_2, r')$$

where $d_1 = d + \chi(\mathcal{O}_X) - r\chi(\mathcal{O}_{C_1})$ and $d_2 = d + \chi(\mathcal{O}_X) - r'\chi(\mathcal{O}_{C_2})$.

Proof. Let F be a semistable pure dimension one sheaf on X of polarized rank 1 and degree d with respect to H . From the above exact sequence, we have

$$h = h_{C_1} r_{H_{C_1}}(F_{C_1}) + h_{C_2} r_{H_{C_2}}(F_{C_2}).$$

Thus, if $r_{H_{C_i}}(F_{C_i}) > 0$ for $i = 1, 2$, the sheaf F is of rank 1 and its S -equivalence class, denoted by $[F]$, belongs to $\overline{\text{Jac}}^d(X)$. Since $r_{H_{C_i}}(F_{C_i}) \in \mathbb{Z}$, if $r_{H_{C_1}}(F_{C_1})$ (resp. $r_{H_{C_2}}(F_{C_2})$) is zero, then h_{C_2} (resp. h_{C_1}) must divide to h , namely $h = r'h_{C_2}$ (resp. $h = rh_{C_1}$). In this case, one has $F \simeq F_{C_2}$ (resp. $F \simeq F_{C_1}$) so that F is a semistable pure dimension one sheaf on C_2 (resp. C_1) of rank r' (resp. r) and degree d_2 (resp. d_1) with respect to H_{C_2} (resp. H_{C_1}). Therefore, the

S -equivalence class of F belongs to $\overline{M}^{d_2}(C_2, r)$ (resp. $\overline{M}^{d_1}(C_1, r)$) and the result follows. \square

The problem of describing the connected components of $\overline{M}^d(X)$ given by the polarized rank 1 sheaves which are not of rank 1 have to cover then the analysis of Simpson's moduli spaces of sheaves of higher rank. Furthermore, when the number of irreducible components of the curve X is bigger than two, the compactification $\overline{M}^d(X)$ can even contain moduli spaces of rational rank sheaves on reducible curves as the following example shows.

Example 2.4. Let $X = C_1 \cup C_2 \cup C_3$ be a compact type curve. Let us consider a polarization H on X such that $h_{C_i} = 1$ for $i = 1, 2, 3$. Let L be an invertible sheaf on C_1 and let E be a vector bundle of rank 2 on C_2 . Let us consider the sheaf $F = i_*(L \oplus E)$ where $i: C_1 \cup C_2 \hookrightarrow X$ is the inclusion map. The sheaf F is pure of dimension one and it has polarized rank 1 with respect to H . Moreover, the (semi)stability of F with respect to H is equivalent to the (semi)stability of $L \oplus E$ with respect to the polarization $H_{C_1 \cup C_2}$ (see Lemma 3.2). However, $L \oplus E$ is a sheaf of polarized rank $3/2$.

Nevertheless, when the curve X is projective and reduced, the (semi)stability notion of a sheaf given by the slope μ_H is equivalent to the (semi)stability notion given by Seshadri [22] (it is enough to consider as weights the rational numbers $a_i = \frac{\deg(H_{C_i})}{\deg(H)}$). Therefore the connected component $\overline{\text{Jac}}^d(X)$ of $\overline{M}^d(X)$ coincides with Seshadri's compactification. Thus, in this case $\overline{\text{Jac}}^d(X)$ is a projective scheme that contains the Simpson Jacobian of X so that it can be considered as a compactification of $\text{Jac}^d(X)_s$.

On the other hand, if X is a stable curve and $\overline{P}_{d,X}$ denotes the compactification of the (generalized) Jacobian constructed by Caporaso in [8], by considering as polarization the canonical sheaf of X and by using essentially the Pandharipande's results of [21], one easily proves that there exists a bijective morphism

$$\Xi: \overline{P}_{d,X} \rightarrow \overline{\text{Jac}}^d(X)$$

from Caporaso's compactification to the connected component $\overline{\text{Jac}}^d(X)$.

Finally, the relation between Oda and Seshadri's compactifications $\text{Jac}_\phi(X)$ and the connected component $\overline{\text{Jac}}^d(X)$, when X is a nodal curve, can be found in [1].

3. TORSION FREE SHEAVES ON REDUCIBLE CURVES

3.1. General properties of semistable sheaves. Let X be a projective reduced and connected curve over κ . Let C_1, \dots, C_N denote the irreducible components of X and P_1, \dots, P_k the intersection points of C_1, \dots, C_N . Let H be a polarization on X of degree h . Henceforth we shall use the following notation.

Notation: If F is a pure dimension one sheaf on X , for every proper subcurve D of X , we will denote by F_D the restriction of F to D modulo torsion, that is, $F_D = (F \otimes \mathcal{O}_D)/\text{torsion}$, π_D will be the surjective morphism $F \rightarrow F_D$ and $F^D = \ker \pi_D$. We shall denote by h_D the degree of the induced polarization H_D on D . If $d = d_H(F)$ then we shall write $d_D = d_{H_D}(F_D)$. The complementary subcurve of D in X , that is, the closure of $X - D$, will be denoted by \overline{D} . If $g = g(X)$ denotes the arithmetic genus of X , that is, the dimension of $H^1(X, \mathcal{O}_X)$, for any pure dimension one sheaf F on X of polarized rank 1 and degree d with respect to H , let b , $0 \leq b < h$, be the residue class of $d - g$ modulo h so that

$$d - g = ht + b.$$

For every proper subcurve D of X , we shall write

$$(1) \quad k_D = \frac{h_D(b+1)}{h}.$$

If β is a real number, we use $[\beta]$ to denote the greatest integer less than or equal to β .

We collect here some general properties we will use later.

The following lemma, due to Seshadri, describes the stalk of a pure dimension one sheaf on X at the intersection points P_i that are ordinary double points.

Lemma 3.1. *Let F be a pure dimension one sheaf on X . If P_i is an ordinary double point lying in two irreducible components C_i^1 and C_i^2 , then*

$$F_{P_i} \simeq \mathcal{O}_{X, P_i}^{a_1} \oplus \mathcal{O}_{C_i^1, P_i}^{a_2} \oplus \mathcal{O}_{C_i^2, P_i}^{a_3}$$

where a_1, a_2, a_3 are the integer numbers determined by:

$$a_1 + a_2 = \text{rk}(F_{P_i} \otimes_{\mathcal{O}_{X, P_i}} \mathcal{O}_{C_i^1, P_i})$$

$$a_1 + a_3 = \text{rk}(F_{P_i} \otimes_{\mathcal{O}_{X, P_i}} \mathcal{O}_{C_i^2, P_i})$$

$$a_1 + a_2 + a_3 = \text{rk}(F_{P_i} \otimes \kappa)$$

Proof. See [22], Huitième Partie, Prop. 3. □

Lemma 3.2. *Let F be a pure dimension one sheaf on X supported on a subcurve D of X . Then F is stable (resp. semistable) with respect to H_D if and only if F is stable (resp. semistable) with respect to H .*

Proof. It follows from the equality

$$P(F, n, H) = \chi(i_* F \otimes \mathcal{O}_X(nH)) = \chi(F \otimes \mathcal{O}_D(nH_D)) = P(F, n, H_D)$$

where $i: D \hookrightarrow X$ is the inclusion map. \square

Lemma 3.3. *A torsion free rank 1 sheaf F on X is stable (resp. semistable) if and only if $\mu_H(F^D) < \mu_H(F)$ (resp. \leq) for every proper subcurve D of X .*

Proof. Given a subsheaf G of F such that $\text{Supp}(G) = D \subset X$, let us consider the complementary subcurve \bar{D} of D in X . Since $F_{\bar{D}}$ is torsion free, we have $G \subset F^{\bar{D}}$ with $r_H(G) = r_H(F^{\bar{D}})$ so that $\mu_H(G) \leq \mu_H(F^{\bar{D}})$ and the result follows. \square

Lemma 3.4. *Let L be an invertible sheaf on X of degree d . Then L is (semi)stable with respect to H if and only if for every proper connected subcurve D of X the following inequalities hold:*

$$-\chi(\mathcal{O}_D) + h_D t + k_D \underset{(\leq)}{<} d_D \underset{(\leq)}{<} -\chi(\mathcal{O}_D) + h_D t + k_D + \alpha_D$$

where $\alpha_D = D \cdot \bar{D}$ is the intersection multiplicity of D and \bar{D} .

Proof. Let us write $d = g + ht + b$. If L is (semi)stable with respect to H and D is a proper subcurve of X , the condition $\mu_H(L^D) \underset{(\leq)}{<} \mu_H(L)$ is

$$\frac{hd - hd_D + h_D \chi(\mathcal{O}_X) - \chi(\mathcal{O}_D)}{h - h_D} \underset{(\leq)}{<} d$$

which is equivalent to

$$(2) \quad -\chi(\mathcal{O}_D) + h_D t + k_D \underset{(\leq)}{<} d_D.$$

Considering the subsheaf $L^{\bar{D}}$ of L , yields

$$(3) \quad -\chi(\mathcal{O}_{\bar{D}}) + h_{\bar{D}} t + k_{\bar{D}} \underset{(\leq)}{<} d_{\bar{D}}.$$

Since $X = D \cup \bar{D}$ and $\alpha_D = D \cdot \bar{D}$, we have $d = d_D + d_{\bar{D}}$, $h = h_D + h_{\bar{D}}$ and $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_D) + \chi(\mathcal{O}_{\bar{D}}) - \alpha_D$. Then (2) and (3) give the desired inequalities.

Conversely, if D is a connected subcurve of X , by the left-hand side inequality of the statement, we have $\mu_H(L^D) \underset{(\leq)}{<} \mu_H(L)$. Otherwise, this holds for every connected component of D and it is easy to deduce it for D . The result follows then from the former lemma. \square

3.2. Picard groups and normalizations. We collect here some results relating the Picard groups of certain projective reduced curves. They are consequences of the following proposition due to Grothendieck ([11], Prop. 21.8.5).

Proposition 3.5. *Let X and X' be projective and reduced curves over κ . Let $\phi: X' \rightarrow X$ be a finite and birational¹ morphism. Let U be the open subset of X such that $\phi: \phi^{-1}(U) \rightarrow U$ is an isomorphism and let $S = X - U$. Let us denote $\mathcal{O}'_X = \phi_*(\mathcal{O}_{X'})$. Then, there is an exact sequence*

$$0 \rightarrow \left(\prod_{s \in S} \mathcal{O}'_{X,s} / \mathcal{O}_{X,s} \right) / \text{Im } H^0(X', \mathcal{O}'_{X'}) \rightarrow \text{Pic}(X) \xrightarrow{\phi^*} \text{Pic}(X') \rightarrow 0.$$

If the canonical morphism $H^0(X, \mathcal{O}_X) \rightarrow H^0(X', \mathcal{O}_{X'})$ is bijective, then the kernel of ϕ^ is isomorphic to $\prod_{s \in S} \mathcal{O}'_{X,s} / \mathcal{O}_{X,s}$. \square*

Corollary 3.6. *Let X and X' be two projective reduced and connected curves over κ . Let $\phi: X' \rightarrow X$ be a birational morphism which is an isomorphism outside $P \in X$. If $\phi^{-1}(P)$ is one point $Q \in X'$ and $\mathfrak{m}_{X',Q}^2 \subset \mathfrak{m}_{X,P}$, then the sequence*

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Pic}(X) \xrightarrow{\phi^*} \text{Pic}(X') \rightarrow 0$$

is exact. \square

Corollary 3.7. *Let $X = \cup_{i \in I} C_i$ be a projective reduced and connected curve over κ . Suppose that the intersection points $\{P_j\}_{j \in J}$ of its irreducible components are ordinary double points. Let $X' = \sqcup_{i \in I} C_i$ be the partial normalization of X at the nodes $\{P_j\}_{j \in J}$. Then, there is an exact sequence*

$$0 \rightarrow (\kappa^*)^m \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(X') \rightarrow 0$$

where $m = |J| - |I| + 1$. \square

4. TREE-LIKE CURVES

In this section we describe the structure of the Simpson Jacobian $\text{Jac}^d(X)_s$ and of the connected component $\overline{\text{Jac}}^d(X)$ of its compactification when X is a tree-like curve. We also give a recurrent algorithm to determine the S -equivalence class of a semistable sheaf on X which we will use to analyze these Simpson moduli spaces for reduced fibers of an elliptic fibration.

¹By a birational morphism $X' \rightarrow X$ of reducible curves we mean a morphism which is an isomorphism outside a discrete set of points of X

Definition 4.1. An ordinary double point P of a projective curve X is a disconnecting point if $X - P$ has two connected components.

Definition 4.2. A tree-like curve is a projective, reduced and connected curve $X = C_1 \cup \dots \cup C_N$ over κ such that the intersection points, P_1, \dots, P_k , of its irreducible components are disconnecting ordinary double points.

Note that the singularities lying only at one irreducible component of a tree-like curve can be arbitrary singularities.

In this section, we will assume that X is a tree-like curve of arithmetic genus g . The number of intersection points of the irreducible components of X is $k = N - 1$ and then, by Corollary 3.7, one has that

$$\mathrm{Pic}(X) \simeq \prod_{i=1}^N \mathrm{Pic}(C_i).$$

When the irreducible components of X are smooth, that is, X is a curve of compact type², Teixidor ([24], Lem.1) proves the following lemma. Her proof is also valid for any tree-like curve.

Lemma 4.3. *Let $X = C_1 \cup \dots \cup C_N$ a tree-like curve. Then, the following statements hold:*

- (1) *It is possible to order the irreducible components of X , so that for every $i \leq N - 1$ the subcurve $C_{i+1} \cup \dots \cup C_N$ is connected.*
- (2) *For $i \leq N - 1$ there are irreducible components, say C_{i_1}, \dots, C_{i_k} , with all subindices smaller than i , such that the subcurve $X_i = C_i \cup C_{i_1} \cup \dots \cup C_{i_k}$ is connected and intersects its complement \overline{X}_i in X in just one point P_i .*

□

Let H be a polarization on X of degree h and let us suppose from now on that an ordering of the irreducible components of X as in the lemma has been fixed. Fixing the degree d and using the notations of (1), this order allows us to define inductively integer numbers d_i^X as follows:

$$(4) \quad \begin{aligned} d_i^X &= -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + [k_{X_i}] + 1 - d_{i_1}^X - \dots - d_{i_k}^X, \quad \text{for } i = 1, \dots, N - 1 \\ d_N^X &= d - d_1^X - \dots - d_{N-1}^X. \end{aligned}$$

²Teixidor's tree-like curves are better known as compact type curves (see, for instance [14])

We are now going to modify the above numbers to obtain new numbers d_i associated with X . This is accomplished by a recurrent algorithm. In order to describe it we start by saying that a connected subcurve $D = C_{j_1} \cup \dots \cup C_{j_m}$, $m \geq 1$, of X ordered according to Lemma 4.3 is *final* either when the numbers $k_{D_{j_t}}$ are not integers for $t = 1, \dots, m - 1$ or D is irreducible.

If D is a final curve, we define d_{j_t} as follows:

1. if $m > 1$, $d_{j_t} = d_{j_t}^D$ for $t = 1, \dots, m$.
2. if $m = 1$, $d_{j_1} = h_{C_{j_1}} t + [k_{C_{j_1}}] - \chi(\mathcal{O}_{C_{j_1}})$.

Algorithm 4.4. *If the curve X is final, $d_i = d_i^X$ for all i . Otherwise, let i be the first index for which $k_{X_i} \in \mathbb{Z}$. We consider the two connected components, $Y = X_i$ and $\bar{Y} = \overline{X_i}$, of $X - P_i$ and we reorder them according to Lemma 4.3. This induces a new ordering P_r^Y (resp. $P_s^{\bar{Y}}$) of the points P_j ($j \neq i$) in Y (resp. \bar{Y}). Then,*

a) *If Y (resp. \bar{Y}) is a final curve, the process finishes for Y (resp. \bar{Y}).*

b) *If Y is not final, we take the first index r of Y for which $k_{Y_r} \in \mathbb{Z}$, consider the connected components, Z and \bar{Z} , of $Y - P_r^Y$ and reorder them according to Lemma 4.3. If Z and \bar{Z} are final, the process finishes for Y . Otherwise, we iterate the above argument for those components that are not final and so on. The process finishes for Y when all subcurves that we find are final.*

c) *If \bar{Y} is not final, we take the first index s of \bar{Y} such that $k_{\bar{Y}_s} \in \mathbb{Z}$, consider the connected components, W and \bar{W} , of $\bar{Y} - P_s^{\bar{Y}}$ and reorder them according to Lemma 4.3. If W and \bar{W} are final, the process finishes for \bar{Y} . Otherwise, we repeat the above argument for those components that are not final and so on. The process finishes for \bar{Y} when all subcurves that we obtain are final.*

The algorithm for X finishes when it finishes for both Y and \bar{Y} .

We can now state the theorem that determines the structure of the Simpson Jacobian of X and of the schemes $\overline{\text{Jac}}^d(X)_s$ and $\overline{\text{Jac}}^d(X)$.

Theorem 4.5. *Let $X = C_1 \cup \dots \cup C_N$, $N \geq 2$, be a tree-like curve.*

a) *If k_{X_i} is not an integer for every $i \leq N - 1$, then*

$$\text{Jac}^d(X)_s = \prod_{i=1}^N \text{Pic}^{d_i^X}(C_i) \quad \text{and}$$

$${}^3 \text{Jac}^d(X)_s \subseteq \overline{\text{Jac}}^d(X)_s = \overline{\text{Jac}}^d(X) \simeq \prod_{i=1}^N \overline{\text{Jac}}^{d_i^X}(C_i)$$

where d_i^X are the above integers.

b) If k_{X_i} is an integer for some $i \leq N - 1$, then

$$\text{Jac}^d(X)_s = \overline{\text{Jac}}^d(X)_s = \emptyset \quad \text{and}$$

$$\overline{\text{Jac}}^d(X) \simeq \prod_{i=1}^N \overline{\text{Jac}}^{d_i}(C_i)$$

where d_i are the integers constructed with the above algorithm.

We postpone the proof of this theorem to subsection 4.1 and we now give three examples to illustrate how Algorithm 4.4 works.

Example 4.6. Let $X = C_1 \cup \dots \cup C_N$, $N \geq 2$, be a tree-like curve with a polarization H whose degree h is a prime number. Suppose that the irreducible components of X are ordered according to Lemma 4.3. Then, since h_{X_i} is not divisible by h , $k_{X_i} = \frac{h_{X_i}(b+1)}{h}$ is an integer if and only if $b = h - 1$. Therefore, by Theorem 4.5, we have that if $b < h - 1$, $\text{Jac}^d(X)_s = \prod_{i=1}^N \text{Pic}^{d_i^X}(C_i)$ and

$$\text{Jac}^d(X)_s \subsetneq \overline{\text{Jac}}^d(X)_s = \overline{\text{Jac}}^d(X) \simeq \prod_{i=1}^N \overline{\text{Jac}}^{d_i^X}(C_i),$$

whereas for $b = h - 1$, $\text{Jac}^d(X)_s$ and $\overline{\text{Jac}}^d(X)_s$ are empty and

$$\overline{\text{Jac}}^d(X) \simeq \prod_{i=1}^N \overline{\text{Jac}}^{d_i}(C_i).$$

Moreover, in this case the number d_i is given by

$$d_i = h_{C_i}(t+1) - \chi(\mathcal{O}_{C_i}) \quad \text{for } i = 1, \dots, N.$$

Actually, if L is a strictly semistable line bundle of degree d on X , since k_D is integer for all $D \subset X$, the first index i such that k_{X_i} is integer is $i = 1$ and the connected components of $X - P_1$ are $Y = C_1$ and $\overline{Y} = C_2 \cup \dots \cup C_N$. Since C_1 is irreducible, it is a final subcurve so that $d_1 = h_{C_1}(t+1) - \chi(\mathcal{O}_{C_1})$. Since $k_{\overline{Y}_s}$ is integer for $s = 2, \dots, N - 1$, \overline{Y} is not a final curve so we have to apply the procedure to the curve \overline{Y} . Here, the first index s such that $k_{\overline{Y}_s}$ is integer is $s = 2$ and the connected components of $\overline{Y} - P_2$ are $W = C_2$ and $\overline{W} = C_3 \cup \dots \cup C_N$. Then $d_2 = h_{C_2}(t+1) - \chi(\mathcal{O}_{C_2})$. Since $k_{\overline{W}_s}$ is integer for $s = 3, \dots, N - 1$,

³The inclusion is an equality if all irreducible components of X are smooth

we apply the procedure to the curve \overline{W} , and so on. The iteration of this procedure will only finish when we obtain a sheaf supported on C_N that belongs to $\text{Pic}^{h_{C_N}(t+1)-\chi(\mathcal{O}_{C_N})}(C_N)$. Thus, $d_i = h_{C_i}(t+1) - \chi(\mathcal{O}_{C_i})$ for $i = 1, \dots, N$.

Example 4.7. We are now going to recover examples 1 and 2 of [1]. There, the irreducible components C_i are taken to be smooth and $d = g - 1$. Then, the residue class of $d - g$ modulo h is $b = h - 1$ and $t = -1$. It follows that k_{X_i} is integer for $i = 1, \dots, N$ and, arguing as in the former example, we have that $\text{Jac}^{g-1}(X)_s$ and $\overline{\text{Jac}}^{g-1}(X)_s$ are empty and

$$\overline{\text{Jac}}^{g-1}(X) \simeq \prod_{i=1}^N \overline{\text{Jac}}^{h_{C_i}(t+1)-\chi(\mathcal{O}_{C_i})}(C_i) \simeq \prod_{i=1}^N \text{Pic}^{g_i-1}(C_i)$$

as asserted in [1].

Example 4.8. Let X be the following tree-like curve

Fixing an ordering of the irreducible components of X as in Lemma 4.3, we obtain

so that $X_1 = C_1$, $X_2 = C_2$, $X_3 = C_1 \cup C_2 \cup C_3$. Assume that the first index i such that k_{X_i} is integer is $i = 3$ and let us compute the numbers d_i , $i = 1, \dots, 4$ in this case.

The connected components of $X - P_3$ are $Y = C_1 \cup C_2 \cup C_3$ and $\overline{Y} = C_4$. Since C_4 is a final subcurve, we have that $d_4 = h_{C_4}t + k_{C_4} - \chi(\mathcal{O}_{C_4})$. We now have to fix a new ordering for Y according to Lemma 4.3. We can take, for instance, $Y = C_{\sigma(1)} \cup C_{\sigma(2)} \cup C_{\sigma(3)}$ with $\sigma(1) = 1$, $\sigma(2) = 3$ and $\sigma(3) = 2$. Then, $Y_{\sigma(1)} = C_1$ and $Y_{\sigma(2)} = C_1 \cup C_3$. Therefore, since $k_{Y_{\sigma(1)}} = k_{X_1}$ and $k_{Y_{\sigma(2)}} = k_Y - k_{X_2}$ are not integers, Y is final and we

conclude that

$$\begin{aligned} d_1 &= d_{\sigma(1)}^Y = -\chi(\mathcal{O}_{Y_{\sigma(1)}}) + h_{Y_{\sigma(1)}}t + [k_{Y_{\sigma(1)}}] + 1 = d_1^X, \\ d_3 &= d_{\sigma(2)}^Y = -\chi(\mathcal{O}_{Y_{\sigma(2)}}) + h_{Y_{\sigma(2)}}t + [k_{Y_{\sigma(2)}}] + 1 - d_{\sigma(1)}^Y = \\ &= -\chi(\mathcal{O}_Y) + h_Y t + k_Y + 1 - d_1^X - d_2^X - 1 = d_3^X - 1, \\ d_2 &= d_{\sigma(3)}^Y = d_Y - d_{\sigma(1)}^Y - d_{\sigma(2)}^Y = d_2^X. \end{aligned}$$

4.1. Proof of Theorem 4.5. In order to prove that the Simpson Jacobian of X is not empty only when $k_{X_i} \notin \mathbb{Z}$ for all $i \leq N-1$, we need the next two lemmas that characterize stable invertible sheaves on X . These two lemmas are essentially steps 1 and 2 in [24].

Lemma 4.9. *Let L be a line bundle on X of degree d . If L is stable, then k_{X_i} is not an integer for every $i \leq N-1$ and L is obtained by gluing invertible sheaves L_i on C_i of degrees d_i^X , $i = 1, \dots, N$.*

Proof. Considering the subcurves X_i of X , $i = 1, \dots, N-1$, given by Lemma 4.3, from Lemma 3.4, we get

$$(5) \quad -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + k_{X_i} \leq d_{X_i} \leq -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + k_{X_i} + 1$$

because $\alpha_{X_i} = 1$. We have $r_{H_{X_i}}(L_{X_i}) = 1$ so that $d_{X_i} = d_{H_{X_i}}(L_{X_i})$ is an integer. Then, if $k_{X_i} \in \mathbb{Z}$ for some $i \leq N-1$, (5) becomes a contradiction. Thus $k_{X_i} \notin \mathbb{Z}$ for all $i \leq N-1$ and there is only one possibility for d_{X_i} , namely

$$d_{X_i} = -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + [k_{X_i}] + 1, \quad \text{for } i = 1, \dots, N-1.$$

From $d_{X_i} = d_{C_i} + d_{C_{i_1}} + \dots + d_{C_{i_k}}$ and the exact sequence

$$0 \rightarrow L \rightarrow L_{C_1} \oplus \dots \oplus L_{C_N} \rightarrow \bigoplus_{i=1}^{N-1} \kappa(P_i) \rightarrow 0$$

we deduce that $d_{C_i} = d_i^X$ for all i and the proof is complete. \square

The following lemma proves the converse of Lemma 4.9.

Lemma 4.10. *Let L be the invertible sheaf on X of degree $d = g + ht + b$ obtained by gluing line bundles L_i on C_i of degrees d_i^X , $i = 1, \dots, N$. Suppose that k_{X_i} is not integer for every $i \leq N-1$. Then L is stable.*

Proof. Taking as weights $\frac{\deg(H_{C_i})}{\deg(H)}$ and proceeding as in Lemma 2 of [24], we obtain that for every proper connected subcurve D of X the following inequalities are satisfied:

$$-\chi(\mathcal{O}_D) + h_D t + k_D < d_D < -\chi(\mathcal{O}_D) + h_D t + k_D + \alpha_D$$

where $\alpha_D = D \cdot \bar{D}$. The stability of L follows then from Lemma 3.4. \square

By Lemmas 4.9 and 4.10, $\text{Jac}^d(X)_s$ is not empty only if k_{X_i} is not integer for every $i \leq N-1$, and in this case it is equal to $\prod_{i=1}^N \text{Pic}^{d_i^X}(C_i)$.

We prove now the remaining statements of the theorem. If L is a strictly semistable line bundle on X of degree d then, d_{X_i} is equal to one of the two extremal values of the inequality (5). In particular, k_{X_i} is an integer for some $i \leq N-1$.

Let i be the first index such that k_{X_i} is integer. Then, there are two possibilities for d_{X_i} :

- a) $d_{X_i} = -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + k_{X_i}$
- b) $d_{X_i} = -\chi(\mathcal{O}_{X_i}) + h_{X_i}t + k_{X_i} + 1$

Let us construct a Jordan-Hölder filtration for L in both cases. Since case a) and case b) are the same but with the roles of X_i and \overline{X}_i intertwined, we give the construction in the case a).

We have that $\mu_H(L_{X_i}) = \mu_H(L^{\overline{X}_i}) = \mu_H(L)$. Then, L_{X_i} and $L^{\overline{X}_i} \simeq L_{\overline{X}_i}(-P_i)$ are semistable with respect to H and, by Lemma 3.2, they are semistable with respect to H_{X_i} and $H_{\overline{X}_i}$ respectively.

For simplicity, we shall write $Y = X_i = C_{i_0} \cup C_{i_1} \cup \dots \cup C_{i_k}$ with $i_1, \dots, i_k < i_0 = i$ and $Z = \overline{X}_i$, which are again tree-like curves.

Let us see when the sheaves L_Y and $L_Z(-P_i)$ are stable. We can fix an ordering for Y as in Lemma 4.3, so that $Y = C_{\sigma(i_0)} \cup \dots \cup C_{\sigma(i_k)}$ and we obtain subcurves Y_r of Y for $r = \sigma(i_0), \dots, \sigma(i_{k-1})$.

Claim 1. The sheaf L_Y is stable if and only if k_{Y_r} is not an integer for $r = \sigma(i_0), \dots, \sigma(i_{k-1})$.

Proof. Since the residue class of $d_Y - g(Y)$ modulo h_Y is $b_Y = k_Y - 1$, the numbers $\frac{h_{Y_r}(b_Y+1)}{h_Y} = k_{Y_r}$ are not integers for $r = \sigma(i_0), \dots, \sigma(i_{k-1})$.

Then, from Lemma 4.10, we have only to prove that L_Y is in $\prod_r \text{Pic}^{d_r^Y}(C_r)$, where d_r^Y are the integer numbers defined as d_i^X but with the new ordering of Y and r runs through the irreducible components of Y . This is equivalent to proving that

$$d_{Y_r} = -\chi(\mathcal{O}_{Y_r}) + h_{Y_r}t + [k_{Y_r}] + 1 \quad \text{for } r = \sigma(i_0), \dots, \sigma(i_{k-1}).$$

Actually, since L is semistable and Y_r is a proper subcurve of X , by Lemma 3.4, we obtain

$$(7) \quad -\chi(\mathcal{O}_{Y_r}) + h_{Y_r}t + k_{Y_r} \leq d_{Y_r} \leq -\chi(\mathcal{O}_{Y_r}) + h_{Y_r}t + k_{Y_r} + \alpha$$

where α is the number of intersection points of Y_r and its complement in X . We have that $\alpha \leq 2$ and d_{Y_r} is not equal to the extremal values of (7) because $k_{Y_r} \notin \mathbb{Z}$. Moreover, if it were

$$d_{Y_r} = -\chi(\mathcal{O}_{Y_r}) + h_{Y_r}t + [k_{Y_r}] + 2,$$

since $d_Y = d_{Y_r} + d_{\overline{Y_r}^Y}$, $h_Y = h_{Y_r} + h_{\overline{Y_r}^Y}$ and $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{Y_r}) + \chi(\mathcal{O}_{\overline{Y_r}^Y}) - 1$, $\overline{Y_r}^Y$ being the complement of Y_r in Y , then

$$d_{\overline{Y_r}^Y} = -\chi(\mathcal{O}_{\overline{Y_r}^Y}) + h_{\overline{Y_r}^Y}t + [k_{\overline{Y_r}^Y}]$$

which contradicts the semistability of L . Thus,

$$d_{Y_r} = -\chi(\mathcal{O}_{Y_r}) + h_{Y_r}t + [k_{Y_r}] + 1$$

and the proof of the claim 1 is complete.

On the other hand, the irreducible components of Z are ordered according the instructions in Lemma 4.3 and the subcurves Z_s , where s runs through the irreducible components of Z and $s \leq N - 1$, are equal to either X_s or $X_s - Y$.

Claim 2. The sheaf $L_Z(-P_i)$ is stable if and only if k_{X_s} is not an integer for every $s > i$.

Proof. Since the residue class of $d_Z - 1 - g(Z)$ modulo h_Z is $b_Z = k_Z - 1$ and $k_Y \in \mathbb{Z}$, by the hypothesis, the numbers $\frac{h_{Z_s}(b_Z+1)}{h_Z}$ are not integers for $s > i$ and, by the choice of i , they aren't for $s < i$ either. Then, by Lemma 4.10, it is enough to prove that

$$d_{H_{Z_s}}(L_Z(-P_i)|_{Z_s}) = -\chi(\mathcal{O}_{Z_s}) + h_{Z_s}t + [k_{Z_s}] + 1 \quad \text{for } s \leq N - 1.$$

Since L is semistable, we have that

$$d_{X_s} = -\chi(\mathcal{O}_{X_s}) + h_{X_s}t + [k_{X_s}] + 1.$$

Moreover, if $Z_s = X_s$ then, $d_{H_{Z_s}}(L_Z(-P_i)|_{Z_s}) = d_{X_s}$ and if $Z_s = X_s - Y$ then, $d_{H_{Z_s}}(L_Z(-P_i)|_{Z_s}) = d_{X_s} - d_Y - 1$. We obtain the desired result in both cases and the proof of the claim 2 is complete.

We return now to the proof of the theorem. If k_{Y_r} and k_{X_s} are not integers for $r = \sigma(i_0), \dots, \sigma(i_{k-1})$ and $s > i$ (i.e. Y and Z are final curves), then $0 \subset L_Z(-P_i) \subset L$ is a Jordan-Hölder filtration for L and the S -equivalence class of L belongs to $\prod_r \text{Pic}^{d_r^Y}(C_r) \times \prod_s \text{Pic}^{d_s^X}(C_s)$.

On the other hand, if k_{Y_r} is integer for some $r = \sigma(i_0), \dots, \sigma(i_{k-1})$, the sheaf L_Y is strictly semistable and we have to repeat the above procedure with L_Y in the place of L and the curve Y in the place of X . Similarly, if k_{X_s} is integer for some $s > i$, the sheaf $L_Z(-P_i)$ is strictly semistable. Then, we have to repeat the above procedure for $L_Z(-P_i)$. By iterating this procedure, we get a Jordan-Hölder filtration for L_Y :

$$0 = F_0 \subset F_1 \subset \dots \subset F_m = L_Y$$

and another for $L_Z(-P_i)$:

$$0 = G_0 \subset G_1 \subset \dots \subset G_n = L_Z(-P_i).$$

Therefore, a filtration for L is given by

$$0 = G_0 \subset G_1 \subset \dots \subset L_Z(-P_i) \subset \pi_Y^{-1}(F_1) \subset \dots \subset \pi_Y^{-1}(L_Y) = L.$$

Thus, the S -equivalence class of L belongs to $\prod_{i=1}^N \text{Pic}^{d_i}(C_i)$, where d_i are the integer numbers constructed with the algorithm.

Finally, let us consider a pure dimension one sheaf F on X of rank 1 and degree d which is not a line bundle. When F is locally free at the intersection points P_i for all $i = 1, \dots, N - 1$, calculations and results are analogous to the former ones. If F is not locally free at P_i for some $i = 1, \dots, N - 1$, then there is a natural morphism

$$F \rightarrow F_Y \oplus F_Z$$

where Y, Z are the connected components of $X - P_i$, that is clearly an isomorphism outside P_i . But this is an isomorphism at P_i as well because by Lemma 3.1, $F_{P_i} \simeq \mathcal{O}_{C_i^1, P_i} \oplus \mathcal{O}_{C_i^2, P_i}$ and this is precisely the stalk of $F_Y \oplus F_Z$ at P_i . We conclude that F is not stable and if it is semistable, then k_Y and k_Z are integers, d_Y and d_Z are given by

$$d_Y = -\chi(\mathcal{O}_Y) + h_Y t + k_Y, \quad d_Z = -\chi(\mathcal{O}_Z) + h_Z t + k_Z,$$

and F_Y and F_Z are semistable as well. Then, the construction of a Jordan-Hölder filtration for F can be done as above and thus the S -equivalence class of F belongs to $\prod_{i=1}^N \overline{\text{Jac}}^{d_i}(C_i)$. \square

We give here an example that shows that for reducible curves the tensor product of two semistable torsion free sheaves of rank 1 is not in general a semistable sheaf, even when the considered sheaves are line bundles.

Example 4.11. Let $X = C_1 \cup C_2$ be a tree-like curve with $C_1 \cdot C_2 = P$. Suppose that C_i are rational curves and fix on X a polarization H such that $P \notin \text{Supp}(H)$ and $h_{C_i} = 1$ for $i = 1, 2$. Let Q_1 and Q_2 be two smooth points of C_1 . From Lemma 3.4, it is easy to prove that the sheaves $\mathcal{O}_X(Q_1)$ and $\mathcal{O}_X(Q_2)$ are semistable with respect to H . The same lemma proves that $\mathcal{O}_X(Q_1) \otimes \mathcal{O}_X(Q_2)$ is not a semistable sheaf.

5. KODAIRA REDUCED FIBERS

Let B be a projective smooth curve over κ and let $p: S \rightarrow B$ be an elliptic fibration. By this we mean a proper flat morphism of schemes whose fibers are Gorenstein curves of arithmetic genus 1. By a Kodaira's result (Thm. 6.2 [17]) the singular reduced fibers of p can be classified as follows:

$I_1: X = C_1$ a rational curve with one node.

$I_2 : X = C_1 \cup C_2$, where C_1 and C_2 are rational smooth curves with $C_1 \cdot C_2 = P + Q$.

$I_N : X = C_1 \cup C_2 \cup \dots \cup C_N$, $N = 3, 4, \dots$, where C_i , $i = 1, \dots, N$, are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = \dots = C_{N-1} \cdot C_N = C_N \cdot C_1 = 1$.

$II : X = C_1$ a rational curve with one cusp.

$III : X = C_1 \cup C_2$ where C_1 and C_2 are rational smooth curves with $C_1 \cdot C_2 = 2P$.

$IV : X = C_1 \cup C_2 \cup C_3$, where C_1, C_2, C_3 are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = P$.

In this section we give the description of the Simpson Jacobian and of the connected component $\overline{\text{Jac}}^d(X)$ of its compactification $\overline{M}^d(X)$ for all reduced singular fibers of an elliptic fibration. Since when the fiber X is an irreducible curve (i.e. a rational curve with one node or one cusp) every pure dimension one rank 1 sheaf on X is stable, we mean the fibers of types III , IV and I_N for $N \geq 2$.

5.1. Preliminary results. Lemmas 5.1 and 5.2 can be found in [12] for integral curves and the proofs given are valid for finite morphisms of reducible curves.

Lemma 5.1. *Let $\phi: X' \rightarrow X$ be a finite birational morphism of projective reduced and connected curves over κ . Let G_1 and G_2 be two pure dimension one rank 1 sheaves on X' and let $u: \phi_*(G_1) \rightarrow \phi_*(G_2)$ be a morphism. Then there is a unique morphism $v: G_1 \rightarrow G_2$ such that $\phi_*(v) = u$. \square*

Lemma 5.2. *Let X be a projective reduced and connected curve over κ and P a singular point of X . Let \mathfrak{m} denote the maximal ideal of P . Set $\mathfrak{B} := \text{End}_{\mathcal{O}_X}(\mathfrak{m})$ and $X^* := \text{Spec}(\mathfrak{B})$. Let $\psi: X^* \rightarrow X$ denote the natural map. Then the following assertions hold:*

- (1) $\mathfrak{B} = \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{m}, \mathcal{O}_X)$.
- (2) *The curve X is Gorenstein at P if and only if $\psi: X^* \rightarrow X$ is a finite birational morphism and $g(X^*) = g(X) - 1$.*
- (3) *If X is Gorenstein at P and $\phi: X' \rightarrow X$ is a birational morphism nontrivial at P , then ϕ factors through ψ .*

\square

Moreover, we can adapt the proof of Lemma 3.8 in [12] to show the following:

Lemma 5.3. *Let X be a projective reduced and connected curve over κ and P a singular point at which X is Gorenstein. Let \mathfrak{m} denote the maximal ideal of P . Set $\mathfrak{B} := \text{End}_{\mathcal{O}_X}(\mathfrak{m})$ and $X^* := \text{Spec}(\mathfrak{B})$. Let*

$\psi: X^* \rightarrow X$ denote the natural map. Let F be a pure dimension one rank 1 sheaf on X . Then, F is not locally free at P if and only if there is a pure dimension one rank 1 sheaf G on X^* such that $\psi_*(G) = F$. If G exists, then it is unique.

Proof. If G exists, it is unique by Lemma 5.1. The sheaf G exists if and only if F is a \mathfrak{B} -module, so if and only if $\mathcal{E}nd_{\mathcal{O}_X}(F)$ contains \mathfrak{B} .

To reduce the notation, we set $\mathcal{O} := \mathcal{O}_{X,P}$, $M := \mathfrak{m}_P$, $B := \mathfrak{B}_P$ and $F := F_P$.

Hence G does not exist if F is invertible at P because then $\text{End}_{\mathcal{O}}(F) = \mathcal{O}$, whereas the cokernel of $\mathcal{O} \hookrightarrow B$ has, by (2) of Lemma 5.2, length 1.

Suppose now that F is not invertible. We have to prove that $BF \subset F$. Set $F^* := \text{Hom}_{\mathcal{O}}(F, \mathcal{O})$. Since the curve X is Gorenstein at P , then $F^{**} = F$ (see, for example, [13]). Let \tilde{X} be the total normalization of X , that is, if $X = \cup_i C_i$, then $\tilde{X} = \sqcup_i \tilde{C}_i$ where \tilde{C}_i is the normalization of the integral curve C_i . Let us denote $\overline{\mathcal{O}} := \mathcal{O}_{\tilde{X},P}$ and $K := \overline{\mathcal{O}}^*$.

Since the $\overline{\mathcal{O}}$ -module

$$F^* \overline{\mathcal{O}} := (F^* \otimes_{\mathcal{O}} \overline{\mathcal{O}}) / \text{torsion}$$

is free of rank 1, there is an element $g \in F^*$ such that $F^* \overline{\mathcal{O}} = g \overline{\mathcal{O}}$. Then we have $g \mathcal{O} \subset F^* \subset g \overline{\mathcal{O}}$. By applying $\text{Hom}_{\mathcal{O}}(-, g \mathcal{O})$, we get

$$(6) \quad K \subset gF \subset \mathcal{O}.$$

Since F is not invertible, $gF \neq \mathcal{O}$. Hence $gF \subset M$.

Let \overline{M} be the Jacobson radical of $\overline{\mathcal{O}}$ which is invertible as $\overline{\mathcal{O}}$ -module. Set $G := (\overline{M}^{-1} \otimes_{\overline{\mathcal{O}}} K) / \text{torsion}$. The natural morphism

$$\overline{M}^{-1} \otimes_{\overline{\mathcal{O}}} K \otimes_{\overline{\mathcal{O}}} gF \rightarrow K$$

induces an inclusion

$$gGF = (G \otimes_{\overline{\mathcal{O}}} gF) / \text{torsion} \hookrightarrow K.$$

So (6) yields $gGF \subset gF$. Therefore,

$$(7) \quad GF \subset F.$$

Note that $G \not\subset \mathcal{O}$. Indeed, otherwise $G \subset K$ because K is the largest $\overline{\mathcal{O}}$ -submodule of \mathcal{O} . However, if $G \hookrightarrow K$, there is a morphism

$$\overline{M}^{-1} \otimes_{\overline{\mathcal{O}}} K \rightarrow K$$

and then

$$K \hookrightarrow \overline{M}K := (\overline{M} \otimes_{\overline{\mathcal{O}}} K).$$

Hence, by Nakayama's Lemma, $K = 0$ which is not true as Lemma 5.4, that we will see later, proves.

The natural morphism

$$\overline{M}^{-1} \otimes_{\overline{\mathcal{O}}} K \rightarrow \mathrm{Hom}_{\overline{\mathcal{O}}}(\overline{M}, \mathrm{Hom}_{\mathcal{O}}(\overline{\mathcal{O}}, \mathcal{O})) \simeq \mathrm{Hom}_{\mathcal{O}}(\overline{M}, \mathcal{O}),$$

gives an inclusion

$$G \hookrightarrow \mathrm{Hom}_{\mathcal{O}}(\overline{M}, \mathcal{O})/\mathrm{torsion}.$$

Since $\mathcal{O} \hookrightarrow \mathrm{Hom}_{\mathcal{O}}(\overline{M}, \mathcal{O})/\mathrm{torsion}$, it is possible to consider $D := G + \mathcal{O}$. Then $\mathcal{O} \subset D$, but $\mathcal{O} \neq D$. Set $D^* := \mathrm{Hom}_{\mathcal{O}}(D, \mathcal{O})$. Since X is Gorenstein, $D^{**} = D$ and we have $D^* \subset \mathcal{O}$ and $D^* \neq \mathcal{O}$. Hence $D^* \subset M$. Therefore $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}) \subset D$. Since, by (1) of Lemma 5.2, it is $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O}) = B$, we have

$$BF \subset DF = F + GF.$$

Hence (7) implies $BF \subset F$ and the proof is complete. \square

Lemma 5.4. *Using the above notations, one has $K \neq 0$.*

Proof. Since $\tilde{X} = \sqcup_i \tilde{C}_i$, we have $\overline{\mathcal{O}} = \bigoplus_i \mathcal{O}_i$ with $\mathcal{O}_i = \mathcal{O}_{\tilde{C}_i, P}$. Then $K = \overline{\mathcal{O}}^* = \bigoplus_i \mathcal{O}_i^*$ and it is enough to prove that $\mathcal{O}_i^* \neq 0$.

Using the local duality, one has

$$\mathrm{Ext}_{\mathcal{O}}^1(\mathcal{O}_i, \mathcal{O})^* \simeq \mathrm{Hom}(\mathrm{h}_P^0(\mathcal{O}_i), \mathrm{h}_P^1(\mathcal{O})).$$

Since \mathcal{O}_i has depth one, $\mathrm{h}_P^0(\mathcal{O}_i) = 0$ and then $\mathrm{Ext}_{\mathcal{O}}^1(\mathcal{O}_i, \mathcal{O}) = 0$.

The exact sequence

$$0 \rightarrow \mathcal{O} \rightarrow \bigoplus_i \mathcal{O}_i \rightarrow T \rightarrow 0$$

yields

$$0 \rightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_i, \mathcal{O}) \rightarrow \mathcal{O}_i \rightarrow \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_i, T) \rightarrow 0$$

because $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_i, \mathcal{O}_j) = 0$ for $i \neq j$.

Since T is a torsion sheaf, we conclude that $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_i, \mathcal{O}) \neq 0$ and the result follows. \square

Lemma 5.5. *Let $\phi: X' \rightarrow X$ be a finite birational and surjective morphism of projective reduced and connected curves with $g(X') = g(X) - s$. Let H be a polarization on X of degree h such that $H' := \phi^*(H)$ has degree h . Let G be a pure dimension one sheaf on X' of rank 1 and degree d with respect to H' . Then the following statements hold:*

- (1) *The sheaf $\phi_*(G)$ is pure of dimension one of rank 1 and degree $d + s$ with respect to H .*
- (2) *G is (semi)stable with respect to H' if and only if $\phi_*(G)$ is (semi)stable with respect to H .*

Proof. The sheaf $\phi_*(G)$ is pure of dimension one because it is the direct image of a torsion free sheaf by a finite morphism. Moreover, for any sheaf G' on X' , we have $P(\phi_*(G'), n, H) = P(G', n, H')$, so that

$$\begin{aligned} r_H(\phi_*(G')) &= r_{H'}(G'), \\ d_H(\phi_*(G')) &= d_{H'}(G') + r_{H'}(G')(\chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_X)). \end{aligned}$$

Then

$$\mu_H(\phi_*(G')) = \mu_{H'}(G') + s$$

and the result is now straightforward. \square

5.2. The description for the fibers of type III. Here X will denote a fiber of type III of an elliptic fibration, that is, $X = C_1 \cup C_2$ with C_1 and C_2 two rational smooth curves and $C_1 \cdot C_2 = 2P$ (figure 1).

Figure 1.

If $X' = C_1 \cup C_2$ is the blow-up of X at the point $2P$, then X' is a tree-like curve with $C_1 \cdot C_2 = P$ and there is a finite birational morphism $\phi: X' \rightarrow X$ which is an isomorphism outside $2P$ such that $\phi^{-1}(2P) = P$. Since it is possible to write the completions of the local rings as

$$\widehat{\mathcal{O}}_{X,2P} = \kappa[[x, y]]/x(x - y^2) \quad \text{and} \quad \widehat{\mathcal{O}}_{X',P} = \kappa[[y, \lambda]]/\lambda(\lambda - y)$$

with $x = \lambda y$, we have that $\mathfrak{m}_{X',P}^2 \subset \mathfrak{m}_{X,2P}$. Hence, by Corollary 3.6 and taking into account that X' is a tree-like curve, there is an exact sequence

$$(8) \quad 0 \rightarrow \mathbb{G}_a \rightarrow \text{Pic}(X) \rightarrow \prod_{i=1}^2 \text{Pic}(C_i) \rightarrow 0$$

where the last morphism is given by $L \rightarrow (L_{C_1}, L_{C_2})$.

Let H be a polarization on X of degree h . Fixing the degree d and using the notations of (1) (in this case $g = 1$), the proposition describing the structure of the Simpson Jacobian of X and of the scheme $\text{Jac}^d(X)$ of S -equivalence classes of semistable line bundles is the following:

Proposition 5.6. *Let $X = C_1 \cup C_2$ be a curve of type III.*

a) If $k_{C_1} \in \mathbb{Z}$, then there is an exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow \prod_{i=1}^2 \text{Pic}^{h_{C_i}t+k_{C_i}}(C_i) \rightarrow 0, \text{ and}$$

$$\text{Jac}^d(X) - \text{Jac}^d(X)_s = \prod_{i=1}^2 \text{Pic}^{h_{C_i}t+k_{C_i}-1}(C_i).$$

b) If $k_{C_1} \notin \mathbb{Z}$, then there is an exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow \bigsqcup_{i,j=1}^2 \text{Pic}^{h_{C_i}t+[k_{C_i}]}(C_i) \times \text{Pic}^{h_{C_j}t+[k_{C_j}]+1}(C_j) \rightarrow 0.$$

In this case,

$$\text{Jac}^d(X) - \text{Jac}^d(X)_s = \emptyset,$$

that is, there are not strictly semistable line bundles.

Proof. Since $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2}) - 2$, if L is a line bundle on X of degree d , one has the exact sequence

$$0 \rightarrow L \rightarrow L_{C_1} \oplus L_{C_2} \rightarrow T \rightarrow 0$$

where T is a torsion sheaf with support at $2P$ and $\chi(T) = 2$.

By Lemma 3.4, the sheaf L is (semi)stable if and only if

$$(9) \quad h_{C_1}t + k_{C_1} - 1 \underset{(\leq)}{<} d_{C_1} \underset{(\leq)}{<} h_{C_1}t + k_{C_1} + 1.$$

Since $d_{C_1} \in \mathbb{Z}$ and $d = d_{C_1} + d_{C_2}$, we get the following:

a) if $k_{C_1} \in \mathbb{Z}$, then L is stable if and only if $d_{C_i} = h_{C_i}t + k_{C_i}$ for $i = 1, 2$.

b) if $k_{C_1} \notin \mathbb{Z}$, then L is stable if and only if $d_{C_1} = h_{C_1}t + [k_{C_1}]$ and $d_{C_2} = h_{C_2}t + [k_{C_2}] + 1$ or the same but with the roles of C_1 and C_2 intertwined.

From (8), we obtain then the exact sequences in a) and b).

Suppose now that the sheaf L is strictly semistable. In this case, d_{C_1} is equal to one of the two extremal values of (9). Assume that $d_{C_1} = h_{C_1}t + k_{C_1} - 1$ (the other case is the same but with the roles of C_1 and C_2 intertwined). Then $k_{C_1} \in \mathbb{Z}$, $\mu_H(L_{C_1}) = \mu_H(L^{C_1}) = d$ and L_{C_1} and L^{C_1} are stable sheaves with respect to H_{C_1} and H_{C_2} respectively. Thus, $L^{C_1} \subset L$ is a Jordan-Hölder filtration for L and then its S -equivalence class belongs to $\prod_{i=1}^2 \text{Pic}^{h_{C_i}t+k_{C_i}-1}(C_i)$. Hence, there is only one S -equivalence class of strictly semistable line bundles so that $\text{Jac}^d(X) - \text{Jac}^d(X)_s = \prod_{i=1}^2 \text{Pic}^{h_{C_i}t+k_{C_i}-1}(C_i)$. \square

As before, let $X' = C_1 \cup C_2$ be the tree-like curve obtained by gluing transversally at P the irreducible components of the curve X and $\phi: X' \rightarrow X$ the natural morphism. Let H be a polarization on X of degree h such that $H' := \phi^*(H)$ is of degree h as well.

The following proposition proves that there is an isomorphism between the set of boundary points of the connected component $\overline{\text{Jac}}^d(X)$ and the component $\overline{\text{Jac}}^{d-1}(X')$ when we consider the polarization H on X and the polarization H' on X' . Since X' is a tree-like curve, the structure of the border $\overline{\text{Jac}}^d(X) - \text{Jac}^d(X)$ is determined by Theorem 4.5.

Proposition 5.7. *Let $X = C_1 \cup C_2$ be a curve of type III with $C_1 \cdot C_2 = 2P$. If $X' = C_1 \cup C_2$ with $C_1 \cdot C_2 = P$ and we consider on X (resp. X') a polarization H (resp. H') as above, then there are isomorphisms*

$$\overline{\text{Jac}}^d(X)_s - \text{Jac}^d(X)_s \simeq \overline{\text{Jac}}^{d-1}(X')_s \quad \text{and}$$

$$\overline{\text{Jac}}^d(X) - \text{Jac}^d(X) \simeq \overline{\text{Jac}}^{d-1}(X').$$

Proof. Let \mathfrak{m} be the maximal ideal of $2P$ in X and let us denote $\mathfrak{B} = \mathcal{E}nd_{\mathcal{O}_X}(\mathfrak{m})$ and $X^* = \text{Spec}(\mathfrak{B})$. Taking into account that X is Gorenstein and that the natural morphism $\phi: X' \rightarrow X$ is finite, birational and non trivial at $2P$, by (2) and (3) of Lemma 5.2, we get a morphism $X' \rightarrow X^*$. Since X' and X^* are both of arithmetic genus zero, this morphism is an isomorphism. Let F be a pure dimension one sheaf on X of rank 1 and degree d which is not a line bundle, that is, it is not locally free at the point $2P$. By Lemma 5.3, there is a unique pure dimension one rank 1 sheaf G on X' such that $\phi_*(G) \simeq F$. Since ϕ is finite, birational and surjective and $g(X') = g(X) - 1$, by Lemma 5.5, G has degree $d - 1$ and it is (semi)stable with respect to H' if and only if F is (semi)stable with respect to H . Therefore, the direct image ϕ_* produces the desired isomorphisms. \square

5.3. The description for the fibers of type IV. In this part, X will denote a fiber of an elliptic fibration of type IV, that is, $X = C_1 \cup C_2 \cup C_3$ with C_i rational smooth curves and $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = P$ (figure 2).

Figure 2.

Let $X' = C_1 \cup C_2 \cup C_3$ be the curve obtained by gluing the irreducible components of X at the point P in such a way that this curve X' cannot be embedded, even locally, in a smooth surface (see figure 3).

Figure 3.

There exists a finite birational morphism $\phi: X' \rightarrow X$ which is an isomorphism outside P and $\phi^{-1}(P)$ is the point P of X' . Since the completions of the local rings of X and X' at P can be written as

$$\widehat{\mathcal{O}}_{X,P} = \kappa[[x, y]]/xy(y-x) \quad \text{and} \quad \widehat{\mathcal{O}}_{X',P} = \kappa[[\bar{x}, \bar{y}, \bar{z}]]/(\bar{x}\bar{y}, \bar{x}\bar{z}, \bar{y}\bar{z})$$

with $x = \bar{x} + \bar{y}$, $y = \bar{y} + \bar{z}$ by ϕ , it is easy to prove that $\mathfrak{m}_{X',P}^2 \subset \mathfrak{m}_{X,P}$. Then, from Corollary 3.6, there is an exact sequence

$$(10) \quad 0 \rightarrow \mathbb{G}_a \rightarrow \text{Pic}(X) \xrightarrow{\phi^*} \text{Pic}(X') \rightarrow 0.$$

Moreover, the Picard group of X' is isomorphic to $\prod_{i=1}^3 \text{Pic}(C_i)$. Indeed, if $\tilde{X} = \sqcup_{i=1}^3 C_i$ is the total normalization of X' at P and $\pi: \tilde{X} \rightarrow X'$ is the projection map, by Proposition 3.5, we have the following exact sequence:

$$(11) \quad 0 \rightarrow (\pi_* \mathcal{O}_{\tilde{X},P}^* / \mathcal{O}_{X',P}^*) / \text{Im } H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) \rightarrow \text{Pic}(X') \xrightarrow{\pi^*} \text{Pic}(\tilde{X}) \rightarrow 0.$$

Considering the exact sequence

$$0 \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \oplus \mathcal{O}_{C_3} \xrightarrow{\beta} \kappa \oplus \kappa \rightarrow 0$$

where the morphism β is defined by:

$$\beta(s_1, s_2, s_3) = (s_1(P) - s_2(P), s_1(P) - s_3(P)),$$

we get

$$\pi_* \mathcal{O}_{\tilde{X},P}^* / \mathcal{O}_{X',P}^* \simeq \kappa^* \oplus \kappa^*.$$

Since the morphism $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) = \bigoplus_{i=1}^3 H^0(C_i, \mathcal{O}_{C_i}^*) \xrightarrow{\beta} \kappa^* \oplus \kappa^*$, which is given by

$$\beta(u_1, u_2, u_3) = \left(\frac{u_1(P)}{u_2(P)}, \frac{u_1(P)}{u_3(P)} \right),$$

is surjective, by (11), we obtain that

$$\text{Pic}(X') \simeq \text{Pic}(\tilde{X}) \simeq \prod_{i=1}^3 \text{Pic}(C_i).$$

From (10), we conclude that the Picard group of the curve X is given by the following exact sequence:

$$(12) \quad 0 \rightarrow \mathbb{G}_a \rightarrow \text{Pic}(X) \rightarrow \prod_{i=1}^3 \text{Pic}(C_i) \rightarrow 0.$$

We start now with the (semi)stability analysis. Let H be a polarization on X of degree h . As in (1), if d is the degree of the Jacobian we are considering, b will be the residue class of $d - 1$ modulo h . When the numbers $k_{C_i} = \frac{h_{C_i}(b+1)}{h}$ are not integers, we write $k_{C_i} = [k_{C_i}] + a_i$ with $0 < a_i < 1$. Since $\sum_{i=1}^3 k_{C_i} \in \mathbb{Z}$, then $\sum_{i=1}^3 a_i$ is equal to 1 or 2. Thus the proposition describing the schemes $\text{Jac}^d(X)_s$ and $\text{Jac}^d(X)$ is the following:

Proposition 5.8. *Let $X = C_1 \cup C_2 \cup C_3$ be a curve of type IV. We have the following three cases:*

a) *If $k_{C_i} \in \mathbb{Z}$ for $i = 1, 2, 3$, then there is an exact sequence*

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow \prod_{i=1}^3 \text{Pic}^{h_{C_i}t+k_{C_i}}(C_i) \rightarrow 0, \text{ and}$$

$$\text{Jac}^d(X) - \text{Jac}^d(X)_s = \prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$$

where $d_i = h_{C_i}t + k_{C_i} - 1$ for all i .

b) *Since it is not possible to have $k_{C_i} \notin \mathbb{Z}$ only for one index i because the sum of these three numbers is an integer, the following case to consider is k_{C_i} integer only for one i . Set $k_{C_1} \in \mathbb{Z}$ and $k_{C_i} \notin \mathbb{Z}$ for $i = 2, 3$. Then, there is an exact sequence*

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow K \rightarrow 0$$

where $K = \bigsqcup_{j,l=2}^3 \text{Pic}^{h_{C_1}t+k_{C_1}}(C_1) \times \text{Pic}^{h_{C_j}t+[k_{C_j}]}(C_j) \times \text{Pic}^{h_{C_l}t+[k_{C_l}]+1}(C_l)$. In this case, we have

$$\text{Jac}^d(X) - \text{Jac}^d(X)_s = \prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$$

where $d_1 = h_{C_1}t + k_{C_1} - 1$ and d_2, d_3 are the integer numbers obtained by applying Algorithm 4.4 to the tree-like curve $C_2 \cup C_3$ for a sheaf of degree $d - d_1 - 2$.

c) *Suppose that $k_{C_i} \notin \mathbb{Z}$ for $i = 1, 2, 3$.*

1. *If $\sum_i a_i = 1$, there is an exact sequence*

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow K \rightarrow 0$$

with $K = \bigsqcup_{i,j,l=1}^3 \text{Pic}^{h_{C_i}t+[k_{C_i}]+1}(C_i) \times \text{Pic}^{h_{C_j}t+[k_{C_j}]}(C_j) \times \text{Pic}^{h_{C_l}t+[k_{C_l}]}(C_l)$.

2. If $\sum_i a_i = 2$, there is an exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \text{Jac}^d(X)_s \rightarrow K \rightarrow 0$$

with $K = \bigsqcup_{i,j,l=1}^3 \text{Pic}^{h_{C_i}t+[k_{C_i}]+1}(C_i) \times \text{Pic}^{h_{C_j}t+[k_{C_j}]+1}(C_j) \times \text{Pic}^{h_{C_l}t+[k_{C_l}]}(C_l)$.

In both cases, $\text{Jac}^d(X) - \text{Jac}^d(X)_s$ is empty, that is, every semistable line bundle is stable.

Proof. Since $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{C_1}) + \chi(\mathcal{O}_{C_2}) + \chi(\mathcal{O}_{C_3}) - 3$, for every line bundle L of degree d on X , one has an exact sequence

$$0 \rightarrow L \rightarrow L_{C_1} \oplus L_{C_2} \oplus L_{C_3} \rightarrow T \rightarrow 0$$

where T is a torsion sheaf supported at the point P and $\chi(T) = 3$.

The only connected subcurves of X are C_i , $i = 1, 2, 3$, and their complements $\overline{C_i} = C_j \cup C_l$ and $C_i \cdot \overline{C_i} = 2$ so that, by Lemma 3.4, the sheaf L is (semi)stable if and only if

$$(13) \quad h_{C_i}t + k_{C_i} - 1 \underset{(\leq)}{<} d_{C_i} \underset{(\leq)}{<} h_{C_i}t + k_{C_i} + 1, \text{ for } i = 1, 2, 3.$$

Taking into account that $d_{C_i} \in \mathbb{Z}$ and $d = d_{C_1} + d_{C_2} + d_{C_3}$, in case a) L is stable if and only if $d_{C_i} = h_{C_i}t + k_{C_i}$ for $i = 1, 2, 3$. Since the following case is $k_{C_i} \notin \mathbb{Z}$ for two indices, reordering the irreducible components of X , we can assume that we are in case b). In this case, L is stable if and only if $d_{C_1} = h_{C_1}t + k_{C_1}$, $d_{C_2} = h_{C_2}t + [k_{C_2}]$ and $d_{C_3} = h_{C_3}t + [k_{C_3}] + 1$ or the same with the roles of C_2 and C_3 intertwined. Finally, in case c), L is stable if and only if $d_{C_i} = h_{C_i}t + [k_{C_i}] + \epsilon_i$ where $\epsilon_i = 0$ or 1 . Therefore, reordering the irreducible components of X if it were necessary, the possibilities are:

1. if $\sum_i a_i = 1$, then $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon_3 = 0$.
2. if $\sum_i a_i = 2$, then $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = 0$.

This together with (12) proves the exact sequences of the statement.

When L is a strictly semistable line bundle on X of degree d , d_{C_i} is equal to one of the two extremal values of (13), and then $k_{C_i} \in \mathbb{Z}$, for some $i = 1, 2, 3$. Reordering the irreducible components of X , we can assume that $k_{C_1} \in \mathbb{Z}$ and $d_{C_1} = h_{C_1}t + k_{C_1} - 1$. Hence, $\mu_H(L_{C_1}) = \mu_H(L^{C_1}) = d$ so that, by Lemma 3.2, L_{C_1} and L^{C_1} are semistable with respect to H_{C_1} and $H_{C_2 \cup C_3}$ respectively. Since C_1 is an integral curve $L_{C_1} \in \text{Pic}^{d_{C_1}}(C_1)$ is stable. On the other hand, since $C_2 \cup C_3$ is a tree-like curve, by Theorem 4.5, we have that $[L^{C_1}] \in \prod_{i=2}^3 \text{Pic}^{d_i}(C_i)$ where d_2 and d_3 are the integer numbers obtained by applying Algorithm 4.4 to $C_2 \cup C_3$ for the sheaf L^{C_1} which has degree $d - d_{C_1} - 2$. Bearing in mind that if $k_{C_i} \in \mathbb{Z}$ for $i = 2, 3$, the only final subcurves of $C_2 \cup C_3$ are

its irreducible components, we conclude that the S -equivalence class of L belongs to $\prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$ with d_i the integers of the statement. \square

We study now the set of boundary points of the connected component $\overline{\text{Jac}}^d(X)$.

Proposition 5.9. *Let $X = C_1 \cup C_2 \cup C_3$ be a curve of type IV with $C_1 \cdot C_2 = C_1 \cdot C_3 = C_2 \cdot C_3 = P$ and let $X' = C_1 \cup C_2 \cup C_3$ be the curve of figure 3. Let $\phi: X' \rightarrow X$ denote the natural morphism. Let H be a polarization on X of degree h such that $H' = \phi^*(H)$ is also of degree h . Then, considering on X' the polarization H' , there are isomorphisms*

$$\overline{\text{Jac}}^d(X)_s - \text{Jac}^d(X)_s \simeq \overline{\text{Jac}}^{d-1}(X')_s \quad \text{and}$$

$$\overline{\text{Jac}}^d(X) - \text{Jac}^d(X) \simeq \overline{\text{Jac}}^{d-1}(X').$$

Proof. Since the curve X is Gorenstein, $g(X') = g(X) - 1$ and ϕ is a finite birational morphism non trivial at the point P , arguing as in the proof of Proposition 5.7, the morphism ϕ_* produces the desired isomorphisms. \square

Thus the description of the connected component $\overline{\text{Jac}}^d(X)$ when X is a fiber of type IV implies the analysis of Simpson schemes $\text{Jac}^d(X')$ and $\overline{\text{Jac}}^d(X')$ for the curve X' of figure 3. This analysis is given in the following subsection which completes then the description for the curves of type IV.

5.4. The description for the curve X' . Let $X' = C_1 \cup C_2 \cup C_3$ be the curve of figure 3. Let H' be a polarization on X' of degree h . With the notations we come using, we have the following

Proposition 5.10. *If $X' = C_1 \cup C_2 \cup C_3$ is the curve of the figure 3, we have the following two cases:*

a) *If $k_{C_i} \in \mathbb{Z}$ for some $i = 1, 2, 3$, then*

$$\text{Jac}^d(X')_s = \emptyset \quad \text{and}$$

$$\text{Jac}^d(X') - \text{Jac}^d(X')_s = \prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$$

where $d_i = h_{C_i}t + k_{C_i} - 1$ if $k_{C_i} \in \mathbb{Z}$ and $d_j, d_k, j, k \neq i$ are the integer numbers obtained by applying Algorithm 4.4 to the tree-like curve \overline{C}_i for a sheaf of degree $d - d_i - 1$.

b) *Suppose that $k_{C_i} \notin \mathbb{Z}$ for $i = 1, 2, 3$.*

1. If $\sum_i a_i = 1$, then

$$\mathrm{Jac}^d(X')_s = \prod_{i=1}^3 \mathrm{Pic}^{h_{C_i}t + [k_{C_i}]}(C_i).$$

2. If $\sum_i a_i = 2$, then

$$\mathrm{Jac}^d(X')_s = \emptyset.$$

In these two cases,

$$\mathrm{Jac}^d(X') - \mathrm{Jac}^d(X')_s = \emptyset.$$

Proof. We have seen that the Picard group of this curve X' is isomorphic to the direct product of the Picard groups of its irreducible components. Moreover, if L is a line bundle on X' of degree d , by Lemma 3.4, L is (semi)stable if and only if

$$h_{C_i}t + k_{C_i} - 1 \underset{(\leq)}{<} d_{C_i} \underset{(\leq)}{<} h_{C_i}t + k_{C_i}, \text{ for } i = 1, 2, 3.$$

Then, the result follows arguing as in the proof of Proposition 5.8. \square

In order to determine boundary points of $\overline{\mathrm{Jac}}^d(X')$, we need the following lemma that describes the stalk of a pure dimension one sheaf on the curve X' at its only singular point P .

Lemma 5.11. *If F is a pure dimension one sheaf on the curve X' , then*

$$F_P \simeq \mathcal{O}_{X',P}^a \oplus \mathcal{O}_{C_{12},P}^{a_{12}} \oplus \mathcal{O}_{C_{13},P}^{a_{13}} \oplus \mathcal{O}_{C_{23},P}^{a_{23}} \oplus \mathcal{O}_{C_1,P}^{a_1} \oplus \mathcal{O}_{C_2,P}^{a_2} \oplus \mathcal{O}_{C_3,P}^{a_3}$$

where $C_{ij} = C_i \cup C_j$ and a, a_{ij} and a_i are integer numbers determined by the following equalities:

$$a + a_{12} + a_{13} + a_1 = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_1,P})$$

$$a + a_{12} + a_{23} + a_2 = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_2,P})$$

$$a + a_{13} + a_{23} + a_3 = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_3,P})$$

$$a + a_{12} + \frac{a_{13}}{2} + \frac{a_{23}}{2} + \frac{a_1}{2} + \frac{a_2}{2} = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_{12},P})$$

$$a + \frac{a_{12}}{2} + a_{13} + \frac{a_{23}}{2} + \frac{a_1}{2} + \frac{a_3}{2} = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_{13},P})$$

$$a + \frac{a_{12}}{2} + \frac{a_{13}}{2} + a_{23} + \frac{a_2}{2} + \frac{a_3}{2} = \mathrm{rk}(F_P \otimes_{\mathcal{O}_{X',P}} \mathcal{O}_{C_{23},P})$$

$$a + a_{12} + a_{13} + a_{23} + a_1 + a_2 + a_3 = \mathrm{rk}(F_P \otimes \kappa).$$

Proof. Since F is a pure dimension one sheaf, the $\mathcal{O}_{X',P}$ -module $M := F_P$ has depth 1. For $i = 1, 2, 3$, let t_i be a local parameter of C_i at P . If \mathfrak{m} denotes the maximal ideal of P in X' and \mathfrak{m}_i is the maximal ideal of P in C_i , we have $\mathfrak{m} = \bigoplus_{i=1}^3 \mathfrak{m}_i$ so that $t_i t_j = 0$ for $i \neq j$. Since $t_i M$ is a torsion free $\mathcal{O}_{C_i,P}$ -module, it is free:

$$(14) \quad t_i M \simeq \mathcal{O}_{C_i,P}^{r_i}, \quad i = 1, 2, 3.$$

The map

$$\begin{aligned} \Psi: M &\rightarrow t_1 M \oplus t_2 M \oplus t_3 M \\ m &\mapsto (t_1 m, t_2 m, t_3 m) \end{aligned}$$

is injective because M has depth 1. Consider the inclusion map

$$\begin{aligned} i: t_1 M \oplus t_2 M \oplus t_3 M &\hookrightarrow M \\ (t_1 m, t_2 m', t_3 m'') &\mapsto t_1 m + t_2 m' + t_3 m''. \end{aligned}$$

From (14), one has that

$$\Psi(\text{Im}(i)) = \mathfrak{m}_1^{r_1} \oplus \mathfrak{m}_2^{r_2} \oplus \mathfrak{m}_3^{r_3}$$

and then

$$(t_1 M \oplus t_2 M \oplus t_3 M) / \Psi(\text{Im}(i)) = \kappa^{r_1} \oplus \kappa^{r_2} \oplus \kappa^{r_3}.$$

Hence, there is a map

$$\chi: M \rightarrow \kappa^{r_1} \oplus \kappa^{r_2} \oplus \kappa^{r_3}$$

whose kernel is $t_1 M \oplus t_2 M \oplus t_3 M$.

For $i = 1, 2, 3$, let M_i be the following submodule of M :

$$M_i = \{f \in M \text{ such that } f \notin t_i M \text{ and } t_j f = t_k f = 0 \text{ for } j, k \neq i\}.$$

If $N_i := \chi(M_i) \subseteq \kappa^{r_i}$, then $N_i = \kappa^{r_i} \cap \text{Im}(\chi)$. Indeed, let u be a nonzero element of $\kappa^{r_i} \cap \text{Im}(\chi)$ with $u = \chi(f)$ for $f \in M$. Since $t_j f \in t_j^2 M$ for $j \neq i$, we can write $t_j f = t_j^2 g$ for some $g \in M$. If $f' = f - t_j g$, then $\chi(f) = \chi(f') = u$ and $f' \in M_i$ which proves the claim.

Let f_1, \dots, f_n be elements of M_1 such that $\chi(f_1), \dots, \chi(f_n)$ are free and let us check that f_1, \dots, f_n are free over $\mathcal{O}_{C_1,P}$. Suppose that

$$\alpha_1 f_1 + \dots + \alpha_n f_n = 0$$

with $\alpha_i \in \mathcal{O}_{C_1,P}$ and $\alpha_i \neq 0$ for some i . Let t_1^m be the maximal power of t_1 dividing all α_i and write $\alpha_i = t_1^m \beta_i$ for $i = 1, \dots, n$. Since $t_i t_j = 0$ for $i \neq j$, we have that

$$t_i (t_1^{m-1} \beta_1 f_1 + \dots + t_1^{m-1} \beta_n f_n) = 0$$

for $i = 1, 2, 3$, which implies that

$$t_1^{m-1} \beta_1 f_1 + \dots + t_1^{m-1} \beta_n f_n = 0$$

because M has depth 1.

Recurrently, we get

$$\beta_1 f_1 + \dots + \beta_n f_n = 0.$$

Therefore,

$$\beta_1(P)\chi(f_1) + \dots + \beta_n(P)\chi(f_n) = 0$$

which is absurd because $\beta_i(P) \neq 0$ for some $i = 1, \dots, n$ and $\chi(f_1), \dots, \chi(f_n)$ are free. Thus M_1 is a free $\mathcal{O}_{C_1, P}$ -module.

The same argument proves that M_2 (resp. M_3) is a free $\mathcal{O}_{C_2, P}$ (resp. $\mathcal{O}_{C_3, P}$) module.

One has $M_i \cap M_j = \{0\}$ for $i \neq j$ as $t_k(M_i \cap M_j) = 0$ for $k = 1, 2, 3$ and M has depth 1.

Let K_1 be a vector subspace of $\text{Im}(\chi)$ which is supplementary of $N_1 \oplus N_2 \oplus N_3$. For $i \neq j$ with $i, j = 1, 2, 3$, let M_{ij} the following submodule of M :

$$M_{ij} = \{f \in M \text{ such that } f \notin t_i M \oplus t_j M, t_k f = 0 \text{ for } k \neq i, j, \chi(f) \in K_1\}.$$

Arguing as above, if $N_{ij} := \chi(M_{ij})$, then $N_{ij} = (\kappa^{r_i} \oplus \kappa^{r_j}) \cap \text{Im}(\chi)$.

Let f_1, \dots, f_n be elements of M_{12} such that $\chi(f_1), \dots, \chi(f_n)$ are free and let us prove that f_1, \dots, f_n are also free over $\mathcal{O}_{C_{12}, P}$. Suppose that

$$(15) \quad \alpha_1 f_1 + \dots + \alpha_n f_n = 0$$

where $\alpha_i \in \mathcal{O}_{C_{12}, P}$ are not all nulls. Since $\chi(f_1), \dots, \chi(f_n)$ are free, $\alpha_i(P) = 0$ for $i = 1, \dots, n$. Let t_1^m and t_2^s be the maximal powers of t_1 and t_2 dividing all α_i and write $\alpha_i = t_1^m u_i + t_2^s v_i$, $i = 1, \dots, n$. Since $t_1 t_2 = 0$, we have that either $u_i(P)$ are not all nulls or $v_i(P)$ are not all nulls. Suppose that $u_i(P)$ are not all nulls. By multiplying by t_1 the equality (15), we get

$$t_1(t_1^m u_1 f_1 + \dots + t_1^m u_n f_n) = 0.$$

Since $t_i(t_1^m u_1 f_1 + \dots + t_1^m u_n f_n) = 0$ for $i = 2, 3$ as well and M has depth 1, we have that

$$t_1^m u_1 f_1 + \dots + t_1^m u_n f_n = 0.$$

Recurrently, we get

$$t_1(u_1 f_1 + \dots + u_n f_n) = 0.$$

Set $w := u_1 f_1 + \dots + u_n f_n$. Since $w \notin M_2$, then $w \in t_2 M$ so that $\chi(w)$ is zero, but this is absurd because $u_i(P)$ are not all nulls and $\chi(f_1), \dots, \chi(f_n)$ are free. Thus the $\mathcal{O}_{C_{12}, P}$ -module M_{12} is free.

Analogously, M_{13} and M_{23} are free modules and it is not difficult to prove that all intersections of M_{ij} and M_i are zero.

Let K_2 a vector subspace of $\text{Im}(\chi)$ supplementary of $N_1 \oplus N_2 \oplus N_3 \oplus N_{12} \oplus N_{13} \oplus N_{23}$.

Let g_1, \dots, g_d be elements of M such that $\chi(g_1), \dots, \chi(g_d)$ are a basis of K_2 . Let M_0 be the submodule of M generated by g_1, \dots, g_d . Let us prove that g_1, \dots, g_d are free over $\mathcal{O}_{X',P}$. Suppose that

$$(16) \quad \alpha_1 g_1 + \dots + \alpha_d g_d = 0$$

where $\alpha_i \in \mathcal{O}_{X',P}$ are not all nulls. Since $\chi(g_1), \dots, \chi(g_d)$ are free, $\alpha_i(P) = 0$ for $i = 1, \dots, d$. Let t_1^m, t_2^s and t_3^r be the maximal powers of t_1, t_2 and t_3 dividing α_i for $i = 1, \dots, d$ and write $\alpha_i = t_1^m u_i + t_2^s v_i + t_3^r w_i$, $i = 1, \dots, d$. Since $t_i t_j = 0$ for $i \neq j$, one of the following conditions holds: $u_i(P) \neq 0$ for some i , $v_i(P) \neq 0$ for some i or $w_i(P) \neq 0$ for some i . Suppose that $u_i(P) \neq 0$ for some i . By multiplying by t_1 equality (16) and arguing as in the former case, we get

$$t_1(u_1 g_1 + \dots + u_d g_d) = 0.$$

If $w' := u_1 g_1 + \dots + u_d g_d$, then $\chi(w')$ is a nonzero element of K_2 . But this is impossible because $w' \in t_2 M \oplus t_3 M$ and $t_i w' \neq 0$ for $i = 1, 2, 3$. Hence M_0 is a free $\mathcal{O}_{X',P}$ -module.

One has also $M_0 \cap M_i = M_0 \cap M_{ij} = \{0\}$ for all $i, j = 1, 2, 3$.

From Nakayama's Lemma, we conclude then that

$$M = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \oplus M_{12} \oplus M_{13} \oplus M_{23}$$

which proves the result. \square

Proposition 5.12. *If $X' = C_1 \cup C_2 \cup C_3$ is the curve of figure 3, then the only stable pure dimension one rank 1 sheaves on X' are stable line bundles, that is,*

$$\overline{\text{Jac}}^d(X')_s - \text{Jac}^d(X')_s = \emptyset.$$

Moreover, we have that

a) if $k_{C_i} \notin \mathbb{Z}$ for all $i = 1, 2, 3$, then the set of boundary points is empty, that is,

$$\overline{\text{Jac}}^d(X') - \text{Jac}^d(X') = \emptyset,$$

b) if $k_{C_i} \in \mathbb{Z}$ for some $i = 1, 2, 3$, then

$$\overline{\text{Jac}}^d(X') - \text{Jac}^d(X') = \prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$$

where $d_i = h_{C_i} t + k_{C_i} - 1$ and d_j $j \neq i$ are the integer numbers obtained by applying Algorithm 4.4 to the tree-like curve \overline{C}_i for a sheaf of degree $d - d_i - 1$.

Proof. Let F be a pure dimension one sheaf on X' of rank 1 and degree d with respect to the polarization H' . If F is not a line bundle, from Lemma 5.11, we get that, reordering the irreducible components of X'

if it were necessary, the sheaf F is isomorphic either to $F_{C_1} \oplus F_{C_2} \oplus F_{C_3}$ or to $F_{C_1 \cup C_2} \oplus F_{C_3}$. So F is not stable.

Suppose that F is semistable and let us determine its S -equivalence class. If $F \simeq F_{C_1} \oplus F_{C_2} \oplus F_{C_3}$, the condition $\mu_H(F_{C_i}) = d$ for $i = 1, 2, 3$ implies that $d_{C_i} = h_{C_i}t + k_{C_i} - 1$. Then $k_{C_i} \in \mathbb{Z}$ for $i = 1, 2, 3$ and the S -equivalence class of F belongs to $\prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$ where d_i are the integers of the statement because in this case the only final subcurves are the irreducible components. If $F \simeq F_{C_1 \cup C_2} \oplus F_{C_3}$, since $\mu_H(F_{C_1 \cup C_2}) = \mu_H(F_{C_3}) = d$, then $d_{C_3} = h_{C_3}t + k_{C_3} - 1$ so that $k_{C_3} \in \mathbb{Z}$ and the sheaf $F_{C_1 \cup C_2}$ is semistable of degree $d - d_3 - 1$ with respect to $H_{C_1 \cup C_2}$ on the tree-like curve $C_1 \cup C_2$. By Theorem 4.5, we conclude that $[F] \in \prod_{i=1}^3 \text{Pic}^{d_i}(C_i)$ with d_i the integers of the statement as well. \square

5.5. The description for the fibers of type I_N . In all this subsection X will be a fiber of an elliptic fibration of type I_N (figure 4), that is, if $N > 2$, then $X = C_1 \cup C_2 \cup \dots \cup C_N$ with $C_1 \cdot C_2 = C_2 \cdot C_3 = \dots = C_{N-1} \cdot C_N = C_N \cdot C_1 = 1$, and if $N = 2$, then $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = P + Q$. In both cases, the irreducible components of X are rational smooth curves.

Figure 4.

Since the number of intersection points of the irreducible components of X is equal to the number of irreducible components, by Corollary 3.7, there is an exact sequence

$$(17) \quad 0 \rightarrow \kappa^* \rightarrow \text{Pic}(X) \rightarrow \prod_{i=1}^N \text{Pic}(C_i) \rightarrow 0.$$

Let H be a polarization on X of degree h . Fixing the degree d and using the above notations, that is, b is the residue class of $d - 1$ module h and $k_{C_i} = \frac{h_{C_i}(b+1)}{h}$, let us write $k_{C_i} = [k_{C_i}] + a_i$ with $0 < a_i < 1$ when these numbers are not integers and set us $\epsilon_i = 0$ or 1 . Then the proposition describing the structure of the Simpson Jacobian of X of degree d is the following:

Proposition 5.13. *Let $X = C_1 \cup \dots \cup C_N$ be a curve of type I_N , $N \geq 2$.*

a) If $k_{C_i} \in \mathbb{Z}$ for $i = 1, \dots, N$, there is an exact sequence

$$0 \rightarrow \kappa^* \rightarrow \text{Jac}^d(X)_s \rightarrow \prod_{i=1}^N \text{Pic}^{h_{C_i}t + k_{C_i}}(C_i) \rightarrow 0.$$

b) If there are exactly $r \geq 2$ indices, say i_1, \dots, i_r , such that the numbers $k_{C_{i_1}}, \dots, k_{C_{i_r}}$ are not integers, then $\text{Jac}^d(X)_s$ is not empty if and only if for any subset $J \subseteq \{i_1, \dots, i_r\}$ such that either $\cup_{j \in J} C_j$ is a connected subcurve or it is connected by adding irreducible components C_i with $i \neq i_1, \dots, i_r$ the following inequalities hold:

$$\sum_{j \in J} \epsilon_{i_j} - 1 < \sum_{j \in J} a_{i_j} < \sum_{j \in J} \epsilon_{i_j} + 1.$$

In this case, there is an exact sequence

$$0 \rightarrow \kappa^* \rightarrow \text{Jac}^d(X)_s \rightarrow \prod_{s \neq i_j} \text{Pic}^{h_{C_s}t + k_{C_s}}(C_s) \times \prod_{j=1}^r \text{Pic}^{h_{C_{i_j}}t + [k_{C_{i_j}}] + \epsilon_{i_j}}(C_{i_j}) \rightarrow 0.$$

Proof. Since $\chi(\mathcal{O}_X) = \sum_{i=1}^N \chi(\mathcal{O}_{C_i}) - N$, for every line bundle L on X of degree d there is an exact sequence

$$0 \rightarrow L \rightarrow L_{C_1} \oplus \dots \oplus L_{C_N} \rightarrow T \rightarrow 0$$

where T is a torsion sheaf with $\chi(T) = N$. By the exact sequence (17), it is enough then to find the values of the degrees d_{C_i} so that L is stable. Note that every connected subcurve D of X is a tree-like curve of arithmetic genus 0 and $D \cdot \bar{D} = 2$. Hence, by Lemma 3.4, L is (semi)stable if and only if

$$(18) \quad -h_D t + k_D - 1 < d_D < h_D t + k_D + 1$$

(\leq) (\leq)

for every connected subcurve D of X . Since the degrees d_{C_i} are integers, we have that

a) if $k_{C_i} \in \mathbb{Z}$ for $i = 1, \dots, N$, then L is stable if and only if $d_{C_i} = h_{C_i}t + k_{C_i}$ for $i = 1, \dots, N$, because $d_D = \sum_j d_{C_j}$ if $D = \cup_j C_j$.

b) Suppose that only $k_{C_{i_1}}, \dots, k_{C_{i_r}}$, $r \geq 2$, are not integers. If L is stable, by (18), $d_{C_s} = h_{C_s}t + k_{C_s}$ for $s \neq i_1, \dots, i_r$ and $d_{C_{i_j}} = h_{C_{i_j}}t + [k_{C_{i_j}}] + \epsilon_{i_j}$ with $\epsilon_{i_j} = 0$ or 1 for $j = 1, \dots, r$. Conversely, suppose that L is a line bundle on X obtained by gluing line bundles L_i on C_i of degrees $d_{C_s} = h_{C_s}t + k_{C_s}$ for $s \neq i_1, \dots, i_r$ and $d_{C_{i_j}} = h_{C_{i_j}}t + [k_{C_{i_j}}] + \epsilon_{i_j}$ with $\epsilon_{i_j} = 0$ or 1 for $j = 1, \dots, r$. Let D be a connected subcurve of X . If D contains no component C_{i_j} , one easily check that d_D holds (18). Otherwise, consider $J = \{i_j, j = 1, \dots, r \text{ such that } C_{i_j} \subset D\}$. Since J satisfies the second condition of the statement, by the hypothesis, $d_D = h_D t + k_D - \sum_{j \in J} a_{i_j} + \sum_{j \in J} \epsilon_{i_j}$ holds (18) as well. Then L is stable and the proof is complete. \square

For the subscheme of strictly semistable line bundles, we have:

Proposition 5.14. *Let $X = C_1 \cup \dots \cup C_N$ be a curve of type I_N , $N \geq 2$. Then,*

$$\mathrm{Jac}^d(X) - \mathrm{Jac}^d(X)_s = \prod_i \mathrm{Pic}^{d_i}(C_i) \times \prod_j \mathrm{Pic}^{d_j}(C_j)$$

where i (resp. j) runs through the irreducible components of a connected subcurve $D \subset X$ (resp. \overline{D}) such that $k_D \in \mathbb{Z}$ and d_i (resp. d_j) are the integers obtained by applying Algorithm 4.4 to D (resp. \overline{D}) for a sheaf of degree $h_D t + k_D - 1$ (resp. $h_{\overline{D}} t + k_{\overline{D}} - 1$). If there is not so connected subcurve D of X , then $\mathrm{Jac}^d(X) - \mathrm{Jac}^d(X)_s = \emptyset$.

Proof. If L is a strictly semistable line bundle on X of degree d , there is a connected subcurve D of X such that d_D is equal to one of the two extremal values of (18). In particular, $k_D \in \mathbb{Z}$. Suppose that $d_D = h_D t + k_D - 1$ (the other case is similar). Then $\mu_H(L^D) = \mu_H(L_D) = \mu_H(L)$ and, by Lemma 3.2, the sheaves L^D and L_D are semistable with respect to $H_{\overline{D}}$ and H_D respectively. Since D and \overline{D} are tree-like curves, we conclude thanks to Theorem 4.5. \square

Arguing as in the proof of Proposition 5.7, we get the following proposition which together with Theorem 4.5 gives us the structure of the border of $\overline{\mathrm{Jac}}^d(X)$ when X is a curve of type I_N .

Proposition 5.15. *Let $X = C_1 \cup \dots \cup C_N$ be a curve of type I_N , $N \geq 2$ and let $X' = C_1 \cup \dots \cup C_N$ be the tree-like curve obtained by blowing up X at one of its singular points. Let $\phi: X' \rightarrow X$ denote the natural projection. Let H be a polarization on X of degree h such that $H' := \phi^* H$ is also of degree h . Then, considering on X' the polarization H' , there are isomorphisms*

$$\begin{aligned} \overline{\mathrm{Jac}}^d(X)_s - \mathrm{Jac}^d(X)_s &\simeq \overline{\mathrm{Jac}}^{d-1}(X')_s, \\ \overline{\mathrm{Jac}}^d(X) - \mathrm{Jac}^d(X) &\simeq \overline{\mathrm{Jac}}^{d-1}(X'). \end{aligned}$$

It is known that the pull-back of a stable torsion free sheaf by a finite morphism of integral curves is stable. Using the above descriptions, the following example shows that this result is not longer true for reducible curves.

Example 5.16. Let $X = C_1 \cup C_2$ be a curve of type I_2 with $C_1 \cdot C_2 = P + Q$. Let $X' = C_1 \cup C_2$ be the blow-up of X at one of its singular points, say P , and let $\phi: X' \rightarrow X$ be the natural morphism.

Let H be a polarization on X of degree h such that $H' = \phi^*(H)$ is also of degree h . Let L_1 (resp. L_2) be a line bundle on C_1 (resp. C_2) of degree $d_1 = h_{C_1}(t+1)$ (resp. $d_2 = h_{C_2}(t+1) - 1$) for some $t \in \mathbb{Z}$. Let L

be a line bundle on X such that $L_{C_i} \simeq L_i$ for $i = 1, 2$. With the above notations, since L has degree $d = ht + (h - 1)$, we have that the residue class of $d - g(X)$ module h is equal to $h - 2$, $k_{C_i} = h_{C_i} - \frac{h_{C_i}}{h} \notin \mathbb{Z}$ for $i = 1, 2$ and $a_1 = \frac{h_{C_2}}{h}$, $a_2 = \frac{h_{C_1}}{h}$, $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Since the conditions of the Proposition 5.13 hold for all $J \subseteq \{1, 2\}$, L is stable with respect to H . The line bundle $\phi^*(L)$ on X' has degree d with respect to H' . The curve X' is a tree-like curve of arithmetic genus zero, its irreducible components are ordered according to Lemma 4.3 and $X'_1 = C_1$. Since now the residue class of $d - g(X')$ is equal to $h - 1$ and the corresponding number $k'_{X'_1} = h_{C_1}$ is integer, by Theorem 4.5, the sheaf $\phi^*(L)$ is not stable.

6. THE CASE OF DEGREE ZERO.

In this section we want to make a special emphasis in the case of degree zero. This case is particularly significative because, as Corollary 6.5 proves, if X is a fiber of type *III*, *IV* or I_N of an elliptic fibration, for degree zero the conditions of semistability are independent of the polarization on X . Moreover, the results of the previous section now take a simpler and more explicit form.

We need previously the following

Lemma 6.1. *If $X = C_1 \cup \dots \cup C_N$ is a polarized tree-like curve whose irreducible components are rational, then there is no stable pure dimension one sheaf on X of rank 1 and degree -1.*

Proof. Suppose that the irreducible components of X are ordered according to Lemma 4.3. Since the arithmetic genus of X is 0 and $d = -1$, using the notations of (1), we have that $b = h - 1$. Then, $k_{X_i} = h_{X_i} \in \mathbb{Z}$ for all i and the result follows from Theorem 4.5. \square

Proposition 6.2. *Let $X = \cup_i C_i$ be a polarized curve of type *III*, *IV* or I_N . Then the following statements hold:*

- (1) *If L is a line bundle on X of degree 0, then L is stable if and only if $L_{C_i} = \mathcal{O}_{\mathbb{P}^1}$ for all i .*
- (2) *Every stable pure dimension one sheaf on X of rank 1 and degree 0 is a line bundle.*

Proof. 1. Since $d = 0$ and $g(X) = 1$, then $b = h - 1$, $t = -1$ and $k_{C_i} = h_{C_i}$ is integer for all i . Thus the result follows from Propositions 5.6, 5.8 and 5.13.

2. If F is a stable pure dimension one sheaf on X of rank 1 and degree 0 which is not a line bundle, by Propositions 5.7, 5.9 and 5.15, $F = \phi_*(G)$ where G is a stable pure dimension one sheaf of rank 1 and

degree -1 on a curve X' we have determined. When X is of type III or I_N , X' is a tree-like curve with rational components. Then this is impossible by Lemma 6.1. When X is of type IV , X' is the curve of figure 3 and this is also absurd by the description given in 5.3. \square

Proposition 6.3. *Let X be a polarized curve of type III , IV or I_N and let L be a line bundle on X of degree 0.*

- (1) *If X is of type III , L is strictly semistable if and only if $L_{C_1} = \mathcal{O}_{\mathbb{P}^1}(-1)$ and $L_{C_2} = \mathcal{O}_{\mathbb{P}^1}(1)$.*
- (2) *If X is of type IV , L is strictly semistable if and only if $L_{C_1} = \mathcal{O}_{\mathbb{P}^1}(-1)$, $L_{C_2} = \mathcal{O}_{\mathbb{P}^1}$ and $L_{C_3} = \mathcal{O}_{\mathbb{P}^1}(1)$.*
- (3) *If X is of type I_N , L is strictly semistable if and only if $L_{C_i} = \mathcal{O}_{\mathbb{P}^1}(r)$ where $r = -1, 0$ or 1 in such a way that when we remove the components C_i for which $r = 0$ there are neither two consecutive $r = 1$ nor two consecutive $r = -1$.*

Proof. If h is the degree of the polarization on X , then the residue class of $d_H(L) - g(X) = -1$ module h is $b = h - 1$ and $t = -1$. Then, for every connected subcurve D of X , it is $k_D = \frac{h_D(b+1)}{h} = h_D \in \mathbb{Z}$. Since the arithmetic genus of D is 0 and $D \cdot \bar{D} = 2$, by Lemma 3.4, L is strictly (semi)stable if and only if for every connected $D \subset X$ we have

$$-1 \underset{(\leq)}{<} d_D \underset{(\leq)}{<} 1$$

and d_D is equal to one of the two extremal values for some D . The result is now straightforward. \square

Example 6.4. The only possibilities for a strictly semistable line bundle of degree 0 on a curve of type I_3 and I_4 are:

where the numbers denote the degrees of the line bundles we have to consider on each irreducible component. The following examples in I_4 and I_6 are not possible for a strictly semistable line bundle of degree 0:

Corollary 6.5. *If X is a polarized curve of type III , IV or I_N , the (semi)stability of a pure dimension one sheaf of rank 1 and degree 0 on X does not depend on the polarization.*

Proof. For line bundles, the result follows from Propositions 6.2 and 6.3. If F is a pure dimension one sheaf on X of rank 1 and degree 0 which is not a line bundle, using Lemma 3.3, it is semistable if and only if $-\chi(\mathcal{O}_D) \leq d_D$ for any $D \subset X$ which does not depend on the polarization because $d_D = d_{H_D}(F_D) = \chi(F_D) - \chi(\mathcal{O}_D)$. \square

Since we have proved that for these curves there are always strictly semistable sheaves, we can ensure that

Corollary 6.6. *Let X be a curve of type III, IV or I_N . The moduli space of semistable pure dimension one sheaves of rank 1 and degree 0 on X is not a fine moduli space, that is, there is no universal sheaf.*

Corollary 6.7. *Let X be a polarized curve of type III, IV or I_N . If F is a strictly semistable pure dimension one sheaf of rank 1 and degree 0 on X , then its graded object is $Gr(F) = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Bearing in mind that for these sheaves it is $b = h - 1$ and $t = -1$, the result follows from the descriptions given in the preceding section (see [18] for details). \square

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