FULLY FAITHFULNESS CRITERIA FOR QUASI-PROJECTIVE SCHEMES

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Abstract. We extend to quasi-projective schemes that kind of Bondal-Orlov's criterion for a Fourier-Mukai functor to be fully faithful and we prove a variant of it valid in arbitrary characteristic. As a first application of these criteria we prove the fully faithfulness of the Fourier-Mukai functor used to show the autoduality of the compactified Jacobian for reduced projective curves with locally planar singularities.

1. Introduction

Let X and Y be two proper schemes over a field k, let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times Y)$ and let $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X)\to D^b_c(Y)$ be the corresponding integral functor. In the last 10 years, several authors have considered the problem of characterizing those objects $\mathcal{K}^\bullet$ such that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is a fully faithful functor. The classical criterion of fully faithfulness by Bondal and Orlov [5] is valid in characteristic zero and when X and Y are smooth and projective varieties. In [12, 13] we started to study the question for singular projective varieties. In this direction, in [12] we generalized Bondal-Orlov's result to projective and Gorenstein schemes and in [13] we give fully faithfulness criteria in arbitrary characteristic assuming that X is a projective and Cohen-Macaulay scheme. Finally, F. Sancho [18] was able to provide characterizations in any characteristic and when X is a projective scheme with arbitrary singularities. Nevertheless, the case when X is non-projective has not been yet considered in the literature.

Here we start to extend to non-projective schemes that kind of criteria. In this article, we prove two fully faithful criteria for integral functors, one in characteristic zero (Theorem 2.5) and a variant of it valid in arbitrary characteristic (Theorem 2.6). In both criteria the scheme X is just assumed to be quasi-projective. The

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generalization could be useful for many practical situations. As an example, in the last section we give an application to the autoduality of the compactified Jacobian for reduced, projective curves with locally planar singularities that has been proved very recently in [16]. Namely, using our Theorem 2.6, we give an alternative proof of Theorem A in [16]: if X is a reduced curve with locally planar singularities, the integral functor defined by a Poincaré sheaf from the derived category of the generalized Jacobian of X (in general, a non-projective scheme) to the derived category of any fine compactified Jacobian of X is fully faithful .

2. Fully faithfulness criteria

Let X be a quasi-projective scheme over an algebraically closed field k. Denote by $D(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves and by $D^b(X) \subset D(X)$ its bounded derived subcategory. We will use the subscript c to refer to the corresponding subcategory of complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves.

Let Y be another quasi-projective scheme over k and consider the following commutative diagram

$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X & & & & Y \\ & \searrow_{p} & & \swarrow_{q} & \\ & & \operatorname{Spec} k & & \end{array}$$

If $\mathcal{K}^\bullet$ is an object in $D(X \times Y)$, the *integral functor* with kernel $\mathcal{K}^\bullet$ is the functor

$$\begin{aligned} \Phi^{\mathcal{K}^\bullet}_{X\to Y} \colon D(X) &\to D(Y) \\ \mathcal{E}^\bullet &\mapsto \mathbf{R}p_{2_*}(p_1^*\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)\,. \end{aligned}$$

In the case that the kernel $\mathcal{K}^\bullet$ is a line bundle $\mathcal{P}$ on $X \times Y$, the functor $\Phi^{\mathcal{P}}_{X\to Y}$ maps $D^b(X)$ to $D^b(Y)$. Observe that, if p_2 is not a proper morphism (as in the case we are interested in), an integral functor might not preserve coherence so that it is important to work with D (or D^b) instead of D_c (or D^b_c).

Before stating the criteria, let us collect some lemmas that will be used in the proof.

Lemma 2.1. *Let X and Y be quasi-projective k-schemes. Suppose that X is smooth and connected and Y is projective, connected and Gorenstein of dimension n and denote by ω_Y its dualizing sheaf. Let $\mathcal{P}$ be a line bundle on $X \times Y$. Then, the integral functor*

$$\Phi^{\mathcal{P}^{-1}\otimes p_2^*\omega_Y[n]}_{Y\to X} \colon D^b(Y) \to D^b(X)$$

is a left adjoint to the functor $\Phi^{\mathcal{P}}_{X\to Y}\colon D^b(X)\to D^b(Y)$.

Proof. There is a chain of isomorphisms

$$\begin{aligned}\operatorname{Hom}_{D^b(Y)}(\mathcal{G}^\bullet,\Phi^{\mathcal{P}}_{X\to Y}(\mathcal{F}^\bullet)) &\simeq \operatorname{Hom}_{D^b(X\times Y)}(\mathbf{L}p_2^*\mathcal{G}^\bullet, p_1^*\mathcal{F}^\bullet\otimes\mathcal{P})\simeq\\ &\simeq \operatorname{Hom}_{D^b(X\times Y)}(\mathbf{L}p_2^*\mathcal{G}^\bullet\otimes\mathcal{P}^*, p_1^*\mathcal{F}^\bullet)\simeq\\ &\simeq \operatorname{Hom}_{D^b(X\times Y)}(\mathbf{L}p_2^*\mathcal{G}^\bullet\otimes\mathcal{P}^*\otimes p_2^*\omega_Y[n], p_1^!\mathcal{F}^\bullet)\simeq\\ &\simeq \operatorname{Hom}_{D^b(X)}(\Phi^{\mathcal{P}^{-1}\otimes p_2^*\omega_Y[n]}_{Y\to X}(\mathcal{G}^\bullet),\mathcal{F}^\bullet).\end{aligned}$$

where the first isomorphism is the adjunction formula between the derived inverse and the derived direct images (see for instance [4, Prop A.80]), the second is [11, Proposition 5.16], the third follows from (C.2) in [4] taking into account that Y is Gorenstein of dimension n, and the fourth is obtained by applying Grothendieck's duality to the projective morphism $p_1\colon X\times Y\to X$. □

For any closed point $x\in X$ denote by $\mathbf{k}(x)$ the skyscraper sheaf supported at x. Notice that it is a perfect complex if X is a smooth scheme.

Lemma 2.2. *Let $j\colon Y\hookrightarrow X$ be a closed immersion of smooth, irreducible and quasi-projective k-schemes of dimensions n and m, respectively, and set $d:=m-n$. Let $0\neq\mathcal{K}^\bullet\in D^b_c(X)$ be a bounded complex with coherent cohomology sheaves. For any closed point $x\in X$ denote by $j_x\colon\{x\}\hookrightarrow X$ the natural immersion. Then, if*

$$\mathbf{L}^i j_x^*\mathcal{K}^\bullet\simeq\operatorname{Hom}^{m-i}_{D(X)}(\mathbf{k}(x),\mathcal{K}^\bullet)=0$$

unless $x\in Y$ and $i\in[0,d]$, then $\mathcal{K}^\bullet\simeq\mathcal{K}$ is a sheaf on X and $\operatorname{supp}\mathcal{K}=Y$ (topologically).

Proof. This follows from Proposition 1.5 of [5] (see also Proposition 1.6 in [18]). □

The following lemma was proved in [6] for projective schemes.

Lemma 2.3. *Let S and X be quasi-projective k-schemes of finite type, and let $\mathcal{F}$ be a coherent sheaf on $S\times X$ flat over S and schematically supported on a closed scheme $Z\hookrightarrow S\times X$. Suppose that for every closed point $s\in S$, the restriction $\mathcal{F}_s$ of $\mathcal{F}$ to the fiber X_s is topologically supported at a single point $x\in X$ and that*

$$\operatorname{Hom}_X(\mathcal{F}_s,\mathbf{k}(x))\simeq k\,.$$

Then $\mathcal{F}\simeq j_*\mathcal{L}$ for a line bundle $\mathcal{L}$ on Z.

Moreover, if $\operatorname{char}(k)=0$ and for all pairs of distinct points $s_1,s_2\in S$ the sheaves $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ are not isomorphic, then the Kodaira-Spencer map of the family $\mathcal{F}$ is injective at some point $s\in S$.

Proof. Let us see first that for any closed point $s\in S$ the sheaf $\mathcal{F}_s$ is the structure sheaf of a zero dimensional scheme with support at x. For this it is enough to prove that there is a surjective morphism $\mathcal{O}_X\to\mathcal{F}_s$. Indeed, consider non-zero morphisms

$g\colon \mathcal{F}_s \to \mathbf{k}(x)$ and $f\colon \mathcal{O}_X \to \mathcal{F}_s$. If the morphism f is not surjective, its cokernel is a non-zero sheaf topologically supported at x. Then, there is a non-zero morphism $\operatorname{Coker} f \to \mathbf{k}(x)$ that induces another non-zero morphism $h\colon \mathcal{F}_s \to \mathbf{k}(x)$ such that $h \circ f = 0$. Since $\operatorname{Hom}_X(\mathcal{F}_s, \mathbf{k}(x)) \simeq k$, one has that $g \circ f = 0$ so that any non surjective morphism f takes values on $\operatorname{Ker} g$. However, since we have the following exact sequence of sheaves supported at x

$$0 \to \operatorname{Ker} g \to \mathcal{F}_s \xrightarrow{g} \mathbf{k}(x) \to 0$$

there is some $f \in H^0(\mathcal{F}_s)$ such that $f \notin H^0(\operatorname{Ker} g)$. The corresponding morphism $f\colon \mathcal{O}_X \to \mathcal{F}_s$ is such that $g \circ f \neq 0$, and thus it is surjective.

Let $s \in S$ be a closed point. Since $\mathcal{F}_s$ is topologically supported at x, we may assume, up to shrinking S, that S is affine and that the support of $\mathcal{F}$ is included in $V \times S$ for a suitable affine neighborhood V of $x \in X$. Since $V \times S$ is affine and the natural map of sheaves $\mathcal{F} \to \mathcal{F}_s$ is surjective, then the restriction morphism $H^0(X \times S, \mathcal{F}) = H^0(V \times S, \mathcal{F}) \to H^0(V \times S, \mathcal{F}_s) = H^0(X_s, \mathcal{F}_s)$ is also surjective. Therefore, there exists a global section σ of $\mathcal{F}$ mapping to f. For a suitable open neighborhood U of s, $\sigma_U\colon \mathcal{O}_U \to \mathcal{F}_U$ is still surjective, that is $\mathcal{F}_U \simeq \mathcal{O}_{Z_U}$, where $Z_U = Z \cap (U \times X)$, which proves that $\mathcal{F}$ is a line bundle on its support.

By [10] the Hilbert scheme $\operatorname{Hilb}(X)$ of zero dimensional subschemes of X exists when X is a quasi-projective scheme. Then the sheaf $\mathcal{F}_U$ induces a map $\alpha\colon U \to \operatorname{Hilb}^P(X)$ to the Hilbert scheme of zero-dimensional subschemes of X whose Hilbert polynomial P is equal to that of the sheaf $\mathcal{F}_s$. The morphism α is characterized by the property $\mathcal{F} \simeq (\alpha \times \mathrm{id})^* \mathcal{Q}$ where $\mathcal{Q}$ is the universal sheaf on $\operatorname{Hilb}^P(X) \times X$. By the hypothesis, α is injective on closed points and, since the base field k has characteristic zero, the tangent map $T_s\alpha\colon T_sU \to T_{\alpha(s)}\operatorname{Hilb}^P(X)$ is injective at least at one point $s \in S$. Moreover, the Kodaira-Spencer map $KS_s(\mathcal{F})$ at s for $\mathcal{F}_U$ is the composition of the tangent map $T_s\alpha$ with the Kodaira-Specer map for the universal family $\mathcal{Q}$. Since the later is an isomorphism because of the universality of $\mathcal{Q}$, we conclude that $KS_s(\mathcal{F})$ is injective. □

Lemma 2.4. *Let X and Y be quasi-projective k-schemes and let $\mathcal{F}$ be a coherent sheaf on $X \times Y$ flat over X. Let x be a closed point of X and set $\mathcal{F}_x := \mathcal{F}_{|\{x\}\times Y}$. Then the associated integral functor $\Phi = \Phi^{\mathcal{F}}_{X\to Y}$ induces a morphism*

$$\Phi\colon \operatorname{Ext}^1_X(\mathbf{k}(x), \mathbf{k}(x)) \to \operatorname{Ext}^1(\Phi(\mathbf{k}(x)), \Phi(\mathbf{k}(x))) = \operatorname{Ext}^1_X(\mathcal{F}_x, \mathcal{F}_x)$$

which coincides with the Kodaira-Specer morphism for the family $\mathcal{F}$.

Proof. The proof is the same as in Lemma 4.4. of [6]. □

The main result of the paper is the following criterion (valid in characteristic zero) for an integral functor to be fully faithful. Its proof follows the line of Bondal-Orlov's criterion (see Theorem 1.1 of [5]) and its subsequent generalizations: Theorem 1.22 in [13], Theorems 3.6 and 3.8 [5] and Theorem 2.3 in [18].

Theorem 2.5. *Let X and Y be two quasi-projective schemes over an algebraically closed field k such that $\operatorname{char}(k) = 0$. Suppose that X is smooth and connected and Y is projective, connected and Gorenstein. Let $\mathcal{P}$ be a line bundle on $X \times Y$. Suppose that the following conditions are satisfied:*

1. $\operatorname{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)), \Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(y)) = 0$ *for any $x, y \in X$, unless $x = y$ and $0 \le i \le \dim X$.*
2. $\operatorname{Hom}^0_{D^b(Y)}(\Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)), \Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)) = k$ *for any $x \in X$.*

Then, the integral functor $\Phi^{\mathcal{P}}_{X\to Y}\colon D^b(X) \to D^b(Y)$ is fully faithful.

Proof. By Lemma 2.1, $\Phi = \Phi^{\mathcal{P}}_{X\to Y}$ has as left adjoint $G = \Phi^{\mathcal{P}^{-1}\otimes p_2^*\omega_Y[n]}_{Y\to X}$, where n is the dimension of Y. By Proposition 1.18 in [13], to prove that Φ is fully faithful it is sufficient (and necessary) to prove that the composition $G \circ \Phi$ is fully faithful. This composition is still an integral functor; let us denote by $\mathcal{M}^\bullet \in D^b(X \times X)$ its kernel.

The strategy of the proof is as follows: first we are going to prove that condition (1) in the statement implies that the complex $\mathcal{M}^\bullet$ is isomorphic to a single sheaf $\mathcal{M}$ topologically supported on the image Δ of the diagonal immersion $\delta\colon X \hookrightarrow X \times X$. Next, using condition (2) in the statement, we show that $\mathcal{M}$ is the push-foward of a line bundle $\mathcal{N}$ on a closed scheme Z of $X \times X$. Then we prove that Z coincides with Δ. So, the functor $\Phi^{\mathcal{M}^\bullet} := \Phi^{\mathcal{M}^\bullet}_{X\to X}$ is given by tensoring by $\mathcal{N}$ and then it is fully faithful.

(a) *$\mathcal{M}^\bullet$ is a single sheaf $\mathcal{M}$ topologically supported on the image Δ of the diagonal morphism $\delta\colon X \hookrightarrow X \times X$.*

Indeed, by the convolution formula, one has that

$$\mathcal{M}^\bullet = \mathbf{R}\pi_{13,*}(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*(\mathcal{P}^{-1} \otimes p_2^*\omega_Y[n]))$$

where π_{12}, π_{23} and π_{13} are the natural projections of $X \times Y \times X$ onto the corresponding factors. Being π_{13} a base change of $q\colon Y \to \operatorname{Spec} k$, it is a projective morphism so that $\mathbf{R}\pi_{13,*}$ sends coherent sheaves to coherent sheaves, and then the complex $\mathcal{M}^\bullet$ belongs to $D^b_c(X \times X)$ and we can make use of Lemma 2.2.

For any closed point $x \in X$, denote by $j_x\colon \{x\} \hookrightarrow X$ and $i_x\colon \{x\} \times X \hookrightarrow X \times X$ the natural inclusions. Fix a closed point $(x_1, x_2) \in X \times X$. Then, we have

$$\begin{aligned}\mathcal{H}^i(\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1))^\vee) &\simeq \operatorname{Hom}^i_{D(\{x_2\})}(\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1)), \mathbf{k}(x_2)) \simeq \\ &\simeq \operatorname{Hom}^i_{D(X)}(\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1)), \mathbf{k}(x_2)) \simeq \\ &\simeq \operatorname{Hom}^i_{D(Y)}(\Phi(\mathbf{k}(x_1)), \Phi(\mathbf{k}(x_2)))\,.\end{aligned}$$

The first condition on the hypothesis implies that this is zero unless $x_1 = x_2$ and $0 \le i \le \dim X$. Since X is a smooth scheme, the skyscraper sheaf $\mathbf{k}(x_2)$ at x_2 is a l.c.i. zero cycle, so that the complex $\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1))$ is bounded with coherent cohomology because so is $\mathcal{M}^\bullet$. Being x_2 a zero-dimensional Gorenstein scheme, $\omega_{x_2} \simeq \mathbf{k}(x_2)$ is an injective object and then, using (1.7) in [13], we have

$$\mathcal{H}^i(\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1))^\vee) \simeq \mathcal{H}^i(\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1))^*) \simeq \\ \simeq \mathcal{H}^{-i}(\mathbf{L}j_{x_2}^*\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1)) \simeq \mathbf{L}^i j_{x_2}^*(\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1)))\,.$$

By applying Lemma 2.2 to the closed immersion $j_{x_1}\colon \{x_1\} \hookrightarrow X$, one has that the complex $\Phi^{\mathcal{M}^\bullet}(\mathbf{k}(x_1)) \simeq \mathbf{L}i_{x_1}^*\mathcal{M}^\bullet$ is a sheaf topologically supported at $\{x_1\}$. Then, using Lemma 4.3 in [11], we get that $\mathcal{M}^\bullet \simeq \mathcal{M}$ where $\mathcal{M}$ is a sheaf on $X \times X$ flat over X with respect to the first projection $\pi_1\colon X \times X \to X$. Moreover, since $\mathbf{L}i_x^*\mathcal{M} \cong i_x^*\mathcal{M}$ is topologically supported on $\{x\}$ for any closed point x of X, the sheaf $\mathcal{M}$ is topologically supported on the diagonal Δ (a closed subscheme of $X\times X$ because X is a separated scheme).

(b) Denote by $\bar\delta\colon Z \hookrightarrow X \times X$ the schematic support of $\mathcal{M}$, so that $\mathcal{M} \simeq \bar\delta_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on Z. Let us see that $\mathcal{N}$ *is a line bundle.* For every closed point $x \in X$, we know that $i_x^*\mathcal{M} \simeq \Phi^{\mathcal{M}}(\mathbf{k}(x))$ is topologically supported at $\{x\}$ and by the hypothesis

$$\operatorname{Hom}_X(\Phi^{\mathcal{M}}(\mathbf{k}(x)), \mathbf{k}(x)) \simeq \operatorname{Hom}^0_{D^b(Y)}(\Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)), \Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x))) = k\,.$$

By Lemma 2.3, $\mathcal{N}$ is a line bundle.

(c) *The scheme Z coincides with* Δ.

The diagonal embedding $\delta\colon X \hookrightarrow X \times X$ factors through a closed immersion $\tau\colon X \hookrightarrow Z$ which topologically is a homeomorphism and we know that $\mathcal{M} \simeq \bar\delta_*\mathcal{N}$ is flat over X with respect to $\pi_1\colon X \times X \to X$. Then $\mathcal{N}$ is flat over X with respect to the composition $\bar\pi_1 = \pi_1 \circ \bar\delta$ and since $\bar\pi_1$ is a finite morphism, this implies that $\bar\pi_{1*}\mathcal{N} \simeq \pi_{1*}\mathcal{M}$ is a locally free sheaf. In order to show that $Z = \Delta$, it is enough to show that $\pi_{1*}\mathcal{M}$ is a line bundle. Since we know that $\pi_{1*}\mathcal{M}$ is a locally free sheaf and since X is connected, it is enough to show that $\Phi^{\mathcal{M}}(\mathbf{k}(x)) \simeq \mathbf{k}(x)$ for some closed point $x \in X$. Given a closed point $x \in X$, consider the morphism $\Phi^{\mathcal{M}}(\mathbf{k}(x)) \to \mathbf{k}(x) \to 0$ given by adjuntion and let us denote by $\mathcal{K}_x$ is kernel. It suffices to find a closed point $x \in X$ such that $\operatorname{Hom}_x(\mathcal{K}_x, \mathbf{k}(x)) = 0$. Take homomorphisms in $\mathbf{k}(x)$ from the exact sequence

$$0 \to \mathcal{K}_x \to \Phi^{\mathcal{M}}(\mathbf{k}(x)) \to \mathbf{k}(x) \to 0\,.$$

Since, by Lemma 2.4, the morphism

$$\Phi^{\mathcal{M}} : \operatorname{Hom}^1_{D(X)}(\mathbf{k}(x), \mathbf{k}(x)) \to \operatorname{Hom}^1_{D(X)}(\Phi^{\mathcal{M}}(\mathbf{k}(x)), \Phi^{\mathcal{M}}(\mathbf{k}(x)))$$

is the Kodaira-Spencer map for the family $\mathcal{M}$, it is injective at some point $x \in X$ by Lemma 2.3 (using $\operatorname{char}(k) = 0$) and the result follows. □

The hypothesis that $\operatorname{char}(k) = 0$ cannot be removed in the above Theorem 2.5, as it is observed in [13, Rmk. 1.25]. However, if we add one more assumption, then we get the following variant of Theorem 2.5 which is valid in arbitrary characteristic.

Theorem 2.6. *Let X and Y be two quasi-projective schemes over an algebraically closed field k (of arbitrary characteristic). Suppose that X is smooth and connected and that Y is projective, connected and Gorenstein of dimension n. Let $\mathcal{P}$ be a line bundle on $X \times Y$. Suppose that the following conditions are satisfied:*

1. $\operatorname{Hom}^i_{D^b(Y)}(\Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)), \Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(y))) = 0$ *for any $x, y \in X$, unless $x = y$ and $0 \le i \le \dim X$.*
2. $\operatorname{Hom}^0_{D^b(Y)}(\Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x)), \Phi^{\mathcal{P}}_{X\to Y}(\mathbf{k}(x))) = k$ *for any $x \in X$.*
3. *There exists a closed point $x \in X$ such that*

$$\left(\Phi^{\mathcal{P}^{-1}\otimes p_2^*\omega_Y[n]}_{Y\to X} \circ \Phi^{\mathcal{P}}_{X\to Y}\right)(\mathbf{k}(x)) \cong \mathbf{k}(x)\,.$$

Then, the integral functor $\Phi^{\mathcal{P}}_{X\to Y}\colon D^b(X) \to D^b(Y)$ is fully faithful.

Proof. In the proof of Theorem 2.5, the only place where the assumption $\operatorname{char}(k) = 0$ is used, is in finding a closed point $x \in X$ such that $\Phi^{\mathcal{M}}(\mathbf{k}(x)) \cong \mathbf{k}(x)$. Since $\Phi^{\mathcal{M}} = \Phi^{\mathcal{P}^{-1}\otimes p_2^*\omega_Y[n]}_{Y\to X} \circ \Phi^{\mathcal{P}}_{X\to Y}$ by definition, the existence of such a closed point x is now guaranteed by the condition (3). □

3. Application to the autoduality of compactified jacobians

Let X be a smooth, irreducible and projective curve over a field k. It is a classical result that the Jacobian J_X of X is "autodual", that is, canonically isomorphic to its dual abelian variety. This is equivalent to the existence of a Poincaré line bundle $\mathcal{P}$ on $J_X \times J_X$ which is universal as a family of topologically trivial line bundles on J_X. In [17] S. Mukai proved that the Fourier-Mukai functor associated to $\mathcal{P}$ is an equivalence of categories. When working with singular curves instead of dealing with the Jacobian of X, which is no longer an abelian variety, one should deal with the compactied Jacobian of X. For integral curves with double point singularities the autoduality of the compactified Jacobian was proved by Esteves, Gagné and Kleiman in [9]. More generally, for integral curves with planar singularities the corresponding autoduality result is due to Arinkin ([2], [3]). Very recently, the autoduality of the compactified Jacobian for reduced, projective curves with locally planar singularities has been proved in [16]. In that article, the autoduality is deduced from the following fact: if X is a reduced curve with locally planar singularities, the integral functor defined by a Poincaré sheaf from the derived category of the generalized Jacobian of X to the derived category of any fine compactified Jacobian of X is fully faithful (Theorem A in [16]). In this section we use our Theorem 2.6 to provide a new proof of that Theorem A. Notice that X being singular, the generalized Jacobian of X is rarely complete, so that our criteria for non-projective schemes is useful.

Let X be a reduced and projective curve. Denote by J_X its generalized Jacobian, that is, the moduli space of line bundles on X that have degree zero on each irreducible component of X. This is a smooth algebraic group of dimension equal to the arithmetic genus g of X which, in general, is not complete. The problem of finding a compactification is a classical problem studied by many authors (see for instance [1, 7, 8, 19, 14]). Here we consider fine compactified Jacobians constructed by Esteves [8]. Let $\underline{q}$ be a polarization on X, that is, a tuple of rational numbers $\underline{q} = \{\underline{q}_{C_i}\}$, one for each irreducible component C_i of X and let $\bar{J}_X(\underline{q})$ be Esteves's fine compactified Jacobian of X parametrizing torsion free sheaves of rank 1 that are $\underline{q}$-semistable (see [8] for the definitions). It is known ([15], Theorem A) that if X has only locally planar singularities and the polarization $\underline{q}$ is *general*, then $\bar{J}_X(\underline{q})$ is a reduced and connected scheme with locally complete intersection singularities and trivial dualizing sheaf. Its smooth locus coincides with the open subset $J_X(\underline{q})$ of line bundles. Since $\bar{J}_X(\underline{q})$ is a fine moduli space, there exist a universal sheaf $\mathcal{I}$ on $X \times \bar{J}_X(\underline{q})$. Using this universal sheaf and the theory of determinant of cohomology, a Poincaré sheaf $\mathcal{P}$ on $J_X \times \bar{J}_X(\underline{q})$ is constructed in Section 5 of [16].

Theorem 3.1 (Theorem A in [16])**.** *Let X be a reduced curve with locally planar singularities over an algebraically closed field k. Let J_X be the generalized Jacobian of X and let $\bar{J}_X(\underline{q})$ be a fine compactified Jacobian of X. Let $\mathcal{P}$ be a Poincaré line bundle on $J_X \times \bar{J}_X(\underline{q})$. Then the integral functor of kernel $\mathcal{P}$*

$$\Phi^{\mathcal{P}} = \Phi^{\mathcal{P}}_{J_X \to \bar{J}_X(\underline{q})} \colon D^b(J_X) \to D^b(\bar{J}_X(\underline{q}))$$

is fully faithful.

Proof. Let us use the criterion given in Theorem 2.6. Notice that J_X is a smooth, connected and quasi-projective scheme and $\bar{J}_X(\underline{q})$ is a connected and Gorenstein scheme. Let us check that the three conditions in Theorem 2.6 are satisfied. Denote by p_1 and p_2 the projections of $J_X \times \bar{J}_X(\underline{q})$ onto its factors. Notice first that for any point $x \in J_X$ one has that $\Phi^{\mathcal{P}}(\mathbf{k}(x)) = \mathbf{R}p_{2_*}(\mathcal{P}|_{\{x\}\times \bar{J}_X(\underline{q})})$ is the line bundle on $\bar{J}_X(\underline{q})$ denoted $\mathcal{P}_x$ in [16]. Then, if $x, y \in J_X$ are two different points, one has

$$\begin{aligned} \operatorname{Hom}^i_{D^b(\bar{J}_X(\underline{q}))}(\Phi^{\mathcal{P}}(\mathbf{k}(x)), \Phi^{\mathcal{P}}(\mathbf{k}(y))) &\simeq \operatorname{Ext}^i_{\bar{J}_X(\underline{q})}(\mathcal{P}_x, \mathcal{P}_y) \simeq \\ &\simeq H^i(\bar{J}_X(\underline{q}), \mathcal{P}_x^{-1} \otimes \mathcal{P}_y) \end{aligned}$$

which is equal to zero for any i by Corollary B in [16]. This shows the first condition in Theorem 2.6. Similarly, for any $x \in J_X$ we have

$$\begin{aligned} \operatorname{Hom}^0_{D^b(\bar{J}_X(\underline{q}))}(\Phi^{\mathcal{P}}(\mathbf{k}(x)), \Phi^{\mathcal{P}}(\mathbf{k}(x))) &\simeq \operatorname{Hom}_{\bar{J}_X(\underline{q})}(\mathcal{P}_x, \mathcal{P}_x) \simeq \\ &\simeq H^0(\bar{J}_X(\underline{q}), \mathcal{O}_{\bar{J}_X(\underline{q})}) \end{aligned}$$

which is equal to k because $\bar{J}_X(\underline{q})$ is a reduced scheme. This shows the second condition in Theorem 2.6.

To finish we show that the last condition in Theorem 2.6 is satisfied for the origin $0 = [\mathcal{O}_X] \in J_X$. Indeed, if $i\colon J_X \to J_X$ is the involution $i(x) = -x$, one has that

$\mathcal{P}^{-1} \simeq (i \times Id)^*\mathcal{P}$. Thus, taking into account that the dualizing sheaf of $\bar{J}_X(\underline{q})$ is trivial and $\Phi^{\mathcal{P}}(\mathbf{k}(0)) = \mathcal{P}_0 = \mathcal{O}_{\bar{J}_X(\underline{q})}$, one gets

$$\begin{aligned}\left(\Phi_{\bar{J}_X(\underline{q})\to J_X}^{\mathcal{P}^{-1}\otimes p_2^*\omega_{\bar{J}_X(\underline{q})}[g]} \circ \Phi^{\mathcal{P}}\right)(\mathbf{k}(0)) &\simeq \Phi_{\bar{J}_X(\underline{q})\to J_X}^{\mathcal{P}^{-1}[g]}(\mathcal{O}_{\bar{J}_X(\underline{q})}) \simeq \\ &\simeq \mathbf{R}p_{1_*}((id \times i)^*(\mathcal{P})[g]) \simeq \\ &\simeq i^*(\mathbf{R}p_{1_*}(\mathcal{P})[g]) \simeq \\ &\simeq i^*(\mathbf{k}(0)[-g])[g] \simeq \mathbf{k}(0)\,.\end{aligned}$$

where the third isomorphism is flat base-change and the fourth is Theorem 8.1 in [16].

□

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