RELATIVE JACOBIANS OF ELLIPTIC FIBRATIONS WITH REDUCIBLE FIBERS

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Abstract. We prove that if X and S are smooth varieties and $f\colon X \to S$ is an elliptic fibration with singular fibers curves of types I_N with $N \geq 1$, II, III and IV, then the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ of f, defined as the relative moduli space of semistable pure dimension one sheaves of rank 1 and degree 0 on the fibers of f, is an elliptic fibration such that all its fibers are irreducible. This extends known results when fibers are integral or of type I_2.

1. Introduction

Let $f\colon X \to S$ be an elliptic fibration, that is, a proper flat morphism of schemes whose fibers are Gorenstein curves of arithmetic genus 1. If we have a relative ample sheaf $\mathcal{O}_X(1)$ on X, the relative Jacobian of f is defined as the Simpson moduli space $\hat{f}\colon \overline{M}_{X/S} \to S$ of semistable pure dimension one sheaves of rank 1 and degree 0 on the fibers of f with respect to the polarization induced by $\mathcal{O}_X(1)$. Since every torsion free rank 1 sheaf on an integral curve is stable, when $f\colon X \to S$ is an integral fibration, that is, all its fibers are geometrically integral curves, its relative Jacobian is simply the relative moduli space of torsion free rank 1 sheaves of relative degree 0. In this case, it is known that for any closed point $s \in S$ we have an isomorphism $(\overline{M}_{X/S})_s \simeq X_s$ between the fibers of both fibrations over s. Then this relative Jacobian is again an integral elliptic fibration. Furthermore if the fibration $f\colon X \to S$ has a section, $\overline{M}_{X/S}$ is not only locally but globally isomorphic to X over S (see [13]).

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On the other hand the geometry of such relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ is not known for a general elliptic fibration. In [7] we find some examples showing that when $f\colon X \to S$ has reducible fibers (exactly I_2 fibers), it is no longer true that the two fibrations f and $\hat{f}$ have isomorphic fibers. In fact, Căldăraru proves that if X_s is a fiber of type I_2, then the fiber $(\overline{M}_{X/S})_s$ is isomorphic to a rational curve with one node.

One should point out that elliptic fibrations have been used in string theory, notably in connection with mirror symmetry on Calabi-Yau manifolds and D-branes. Some of the classic examples of families of Calabi-Yau manifolds for which there is a Greene-Plesser [11] description of the mirror family produced by Candelas and others, are elliptic fibrations [5]. Moreover there is a relative Fourier-Mukai transform for most elliptic fibrations ([4, 13]) that can be understood in terms of duality in string theory ([8, 1] or D-brane theory. The latter application is due to the interpretation of B-type D-brane states as objects of the derived category $D(X)$ of coherent sheaves [16, 3, 9] and to Kontsevich's homological mirror symmetry proposal [16] that gives an equivalence between $D(X)$ and the Fukaya category [10]. According to that proposal, the monodromies around special points of the known-models of local moduli spaces of Lagrangian submanifolds should correspond to Fourier-Mukai transforms [1, 14]; this explains the importance of elliptic Calabi-Yau manifolds in string theory.

The aim of this paper is to study the structure of the relative Jacobian of an elliptic fibration $f\colon X \to S$ such that X and S are smooth projective varieties and the singular curves appearing as fibers of f are:

(I_1) : A rational curve with one node.

(II) : A rational curve with one cusp.

(III) : $C_1 \cup C_2$ where C_1 and C_2 are rational smooth curves with $C_1 \cdot C_2 = 2p$.

(IV) : $C_1 \cup C_2 \cup C_3$, where C_1, C_2, C_3 are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = p$

(I_N) : $C_1 \cup C_2 \cup \ldots \cup C_N$, where C_i, $i = 1, \ldots, N$, are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = \ldots = C_{N-1} \cdot C_N = C_N \cdot C_1 = 1$ if $N > 2$ and $C_1 \cdot C_2 = p_1 + p_2$ if $N = 2$.

If $f\colon X \to S$ is an elliptic fibration of this type, we prove that the moduli space of semistable pure dimension one sheaves of rank 1 and degree 0 on a fiber X_s is isomorphic to a smooth elliptic curve when X_s is smooth, to a rational curve with one node when X_s is I_N, $N \geq 1$ and to a rational curve with one cusp when X_s is II, III or IV. The result is then that the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ of f is an integral elliptic fibration (Theorem 4.1). In particular we deduce that, although $\overline{M}_{X/S}$ is irreducible, if the fibration f has reducible fibers, it cannot be isomorphic to X, even assuming that f has a section. For instance, if the variety X has dimension 2 or 3, then the results about contractibility of curves on smooth surfaces [2] and on smooth threefolds [21] allow us to to ensure the existence of singular points in the moduli space $\overline{M}_{X/S}$ that correspond to strictly semistable sheaves on the fibers of f. Then in these cases $\overline{M}_{X/S}$ is not isomorphic to the original variety X.

Since by Kodaira's work [15], we known that every elliptic surface $f\colon X \to S$ with reduced fibers is a fibration of this type, the theorem gives us in particular the structure of the relative Jacobian of any reduced elliptic surface.

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2. Preliminares

All the schemes considered in this paper are of finite type over an algebraically closed field κ of characteristic zero and all the sheaves are coherent.

2.1. **Some elliptic fibrations with reducible fibers.** Let $f\colon X \to S$ be an elliptic fibration. By this we mean a proper flat morphism of schemes whose fibers are geometrically connected Gorenstein curves of arithmetic genus 1. We denote by X_s the fibre of f over $s \in S$ and by $\Sigma(f)$ the *discriminant locus* of f, that is, the closed subset of points $s \in S$ such that X_s is not a smooth curve.

There are two important cases where the curves that can occur as singular fibers of an elliptic fibration $f\colon X \to S$ are known. When X is an elliptically fibred surface over a smooth curve S, the singular fibers of f are classified by Kodaira [15]. And if $f\colon X \to S$ is an elliptic threefold with X and S smooth projective varieties and the map f has a section, Miranda [18] studies the kinds of degenerated fibers of f that can appear. These two works allow us to ensure the existence of elliptic fibrations as in the following

Definition 2.1. *An elliptic fibration of type (∗) is an elliptic fibration $f\colon X \to S$, with X and S smooth projective varieties, without multiple fibers and such that if $s \in \Sigma(f)$, the fiber X_s is one of the following curves:*

(I_1) : A rational curve with one node.

(II) : A rational curve with one cusp.

(III) : $X_s = C_1 \cup C_2$ where C_1 and C_2 are rational smooth curves with $C_1 \cdot C_2 = 2p$.

(IV) : $X_s = C_1 \cup C_2 \cup C_3$, where C_1, C_2, C_3 are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = C_3 \cdot C_1 = p$

(I_N) : $X_s = C_1 \cup C_2 \cup \ldots \cup C_N$, where C_i, $i = 1, \ldots, N$, are rational smooth curves and $C_1 \cdot C_2 = C_2 \cdot C_3 = \ldots = C_{N-1} \cdot C_N = C_N \cdot C_1 = 1$ if $N > 2$ and $C_1 \cdot C_2 = p_1 + p_2$ if $N = 2$.

If S is a smooth curve and $f \colon X \to S$ is an elliptic surface such that all fibers are reduced, by Kodaira's classification, every singular fiber X_s of f is one of the above list. Then any smooth elliptic surface over S with reduced fibers is a elliptic fibration of type $(*)$.

If $f_0 \colon X_0 \to S_0$ is an elliptic fibration with X_0 and S_0 varieties of dimensions 3 and 2 respectively and f_0 has a section, Miranda constructs in [18] a flat model $f \colon X \to S$ with X and S smooth and such that the discriminant locus $\Sigma(f)$ is a curve with at worst ordinary double points as singularities. He proves that at a smooth point $s \in \Sigma(f)$ the singular fiber X_s is one on Kodaira's list and that the type of fiber is constant on the irreducible components of $\Sigma(f)^{\text{smooth}}$. Moreover he determines, case by case, the type of singular fiber over the collision points of $\Sigma(f)$. The results of Miranda imply that if the fibers of f over the smooth points of $\Sigma(f)$ are reduced and all collisions are of type $I_{N_1} + I_{N_2}$, then $f \colon X \to S$ is an elliptic fibration of type $(*)$.

2.2. **The Jacobian of a projective curve.** Let C be a projective curve. Let $\mathcal{L}$ be an ample invertible sheaf on C, let H be the associated polarization and let h denote the degree of H.

A sheaf F on C is *pure dimension one* if the support of any nonzero subsheaf of F is of dimension one. The (polarized) *rank* and *degree* with respect to H of F are the rational numbers $\mathrm{r}_H(F)$ and $\mathrm{d}_H(F)$ determined by the Hilbert polynomial

$$P(F, n, H) = \chi(F \otimes \mathcal{O}_C(nH)) = h\, \mathrm{r}_H(F) n + \mathrm{d}_H(F) + \mathrm{r}_H(F)\chi(\mathcal{O}_C).$$

The *slope* of F is defined by

$$\mu_H(F) = \frac{\mathrm{d}_H(F)}{\mathrm{r}_H(F)}$$

The sheaf F is *stable* (resp. *semistable*) with respect to H if F is pure of dimension one and for any proper subsheaf $F' \hookrightarrow F$ one has

$$\mu_H(F') < \mu_H(F) \text{ (resp. } \leq)$$

For every semistable sheaf F with respect to H there is a *Jordan-Hölder filtration*

$$0 = F_0 \subset F_1 \subset \ldots \subset F_n = F$$

with stable quotients F_i/F_{i-1} and $\mu_H(F_i/F_{i-1}) = \mu_H(F)$ for $i = 1, \ldots, n$. This filtration need not be unique, but *the graded object* $Gr(F) = \bigoplus_i F_i/F_{i-1}$ does not depend on the choice of the Jordan-Hölder filtration. Two semistable sheaves F and F' on C are said to be S*-equivalent* if $Gr(F) \simeq Gr(F')$. Observe that two stable sheaves are S-equivalent only if they are isomorphic. If F is a semistable sheaf on C, we will denote by $[F]$ its S-equivalence class.

By Simpson's work [20], there exists a projective moduli space of semistable pure dimension one sheaves on C of (polarized) rank 1 and degree 0.

If the curve is not integral, this moduli space can contain some components given by sheaves of (polarized) rank 1 whose restrictions to some irreducible components of C are concentrated sheaves. These components correspond to moduli spaces of higher rank sheaves on reducible curves (see [17] for some examples). Therefore, from here on by rank 1 sheaves we mean those sheaves having rank 1 on every irreducible component of C and we define *the Jacobian of* C as the moduli space $\overline{M}(C)$ of pure dimension one sheaves of rank 1 and degree 0 on C that are semistable with respect to the fixed polarization. Observe that if C is a reduced curve, $\overline{M}(C)$ is also a projective scheme because it coincides with Seshadri's compactification [19].

For certain projective curves, an explicit description of the structure of this Jacobian $\overline{M}(C)$ can be found in [17] where the author studies not only the case of degree 0 sheaves, but also these moduli spaces for arbitrary degree d sheaves.

2.3. **The relative Jacobian.** Let $f\colon X \to S$ be an elliptic fibration and let $\mathcal{O}_X(1)$ be a relative ample sheaf on X. Let $\overline{\mathcal{M}}_{X/S}$ be the functor which to any S-scheme T associates the space of S-equivalence classes of T-flat sheaves on $f_T\colon X \times_S T \to T$ whose restrictions to the fibers of f_T are semistable of rank 1 and degree 0 with respect to the induced polarization. Two such sheaves F and F' are said to be equivalent if $F' \simeq F \otimes f_T^* N$, where N is a line bundle on T.

Again by [20], there is a projective scheme $\overline{M}_{X/S} \to S$ which universally corepresents the functor $\overline{\mathcal{M}}_{X/S}$. Moreover there is an open subscheme $\overline{M}^s_{X/S} \subseteq \overline{M}_{X/S}$ that universally corepresents the subfunctor $\overline{\mathcal{M}}^s_{X/S} \subseteq \overline{\mathcal{M}}_{X/S}$ of families of stable sheaves.

The fibration $\hat{f}\colon \overline{M}_{X/S} \to S$ is defined to be *the relative Jacobian* of $f\colon X \to S$. Points of $\overline{M}_{X/S}$ represent semistable sheaves of rank 1 and degree 0 on the fibers of f and the natural map $\hat{f}\colon \overline{M}_{X/S} \to S$ sends a sheaf supported on the fiber X_s to the corresponding point $s \in S$. In particular, for any closed point $s \in S$ one has that the fiber $\hat{f}^{-1}(s)$ is isomorphic to the (absolute) moduli space $\overline{M}(X_s)$ of semistable rank 1 degree 0 sheaves on X_s, that is, the Jacobian of the curve X_s.

In order to study the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ we have then to know the structure of the corresponding Jacobians of the curves that can appear as fibers of the elliptic fibration $f\colon X \to S$. We do this in the following section when $f\colon X \to S$ is an elliptic fibration of type $(*)$ (Definition 2.1).

3. The Jacobians of the fibers

Let $f\colon X \to S$ be an elliptic fibration of type $(*)$ and let $\mathcal{O}_X(1)$ be a line bundle on X ample relative to S.

Let C denote a fiber of f and let H be the induced polarization on C. If C is an integral curve (a smooth elliptic curve, a rational curve with one node or a rational curve with one cusp), it is well known that the moduli space $\overline{M}(C)$ is isomorphic to C. However if C is a fiber of type I_2, that is, two projective lines meeting transversely at two points, Căldăraru [7] has proved that the moduli space $\overline{M}(C)$ is isomorphic to a rational curve with one node. Following his argument and as a consequence of the descriptions given in [17], in this section we prove that for every reducible fiber C of $f\colon X \to S$, the moduli space $\overline{M}(C)$ is isomorphic either to a rational curve with one node or to a rational curve with one cusp.

Let C be any reducible fiber of $f\colon X \to S$, that is, a curve of type III, IV or I_N with $N \geq 2$. Let us denote by C_i the irreducible components of C. In the following lemma we collect some properties of rank 1 degree 0 sheaves on C that we will use later (see [17] for the proof).

Lemma 3.1. *If C is a curve of type III, IV or I_N with $N \geq 2$, it holds that:*

1. *The (semi)stability of a pure dimension one sheaf of rank 1 and degree 0 on C does not depend on the polarization.*
2. *A degree 0 line bundle L on C is stable if and only if $L|_{C_i} \simeq \mathcal{O}_{\mathbb{P}^1}$ for all i.*
3. *If F is a stable pure dimension one sheaf of rank 1 and degree 0 on C, then F is a line bundle.*

(4) *If L is a line bundle on C of degree 0, then L is strictly semistable if and only if $L|_{C_i} \simeq \mathcal{O}_{\mathbb{P}^1}(r)$ where $r = -1, 0$ or 1 in such a way that when we remove the components C_i for which $r = 0$ there are neither two consecutive $r = 1$ nor two consecutive $r = -1$.*

(5) *If F is a strictly semistable pure dimension one sheaf of rank 1 and degree 0 on C, then its graded object is $Gr(F) = \oplus_i \mathcal{O}_{\mathbb{P}^1}(-1)$*

Let q be a fixed smooth point of C and let us denote by C_0 the irreducible component of C on which q lies.

If $\Delta \subseteq C \times C$ denotes the diagonal and $\mathcal{J}_\Delta$ is its ideal sheaf, define $\mathcal{O}_{C\times C}(\Delta) = \mathcal{H}\mathrm{om}(\mathcal{J}_\Delta, \mathcal{O}_{C\times C})$ as the dual of $\mathcal{J}_\Delta$.

Consider the sheaf

$$\mathcal{E} = \mathcal{O}_{C\times C}(\Delta) \otimes \pi_1^* \mathcal{O}_C(-q)$$

where $\pi_1 \colon C \times C \to C$ is the projection on the first component. This sheaf is flat over C via the projection $\pi_2 \colon C \times C \to C$ (see [6], for details) and we have the following

Proposition 3.2. *For any point $p \in C$, the restriction $\mathcal{E}_p$ of $\mathcal{E}$ to $C \times \{p\}$ is a semistable pure dimension one sheaf of rank 1 and degree 0. Moreover, if p is not a smooth point of C_0, then all sheaves $\mathcal{E}_p$ define the same point of the moduli space $\overline{M}(C)$.*

Proof. Since C is Gorenstein, we have that $\mathrm{Ext}^1(\mathcal{O}_P, \mathcal{O}_C(-q)) = \kappa$, so that the restriction $\mathcal{E}_p$ is the unique non trivial extension

$$0 \to \mathcal{O}_C(-q) \to \mathcal{E}_p \to \mathcal{O}_p \to 0\,.$$

Using this exact sequence one easily proves that $\mathcal{E}_p$, which is precisely $\mathcal{J}_p^* \otimes \mathcal{O}_C(-q)$, is a pure dimension one sheaf of rank 1 and degree 0.

To prove that it is semistable, let us consider two cases: when p is a smooth point and when it is a singular point of C. In the first case, $\mathcal{E}_p$ is the line bundle $\mathcal{O}_C(p - q)$ and we have the following:

(1) If $p \in C_0$, the restrictions of $\mathcal{E}_p$ to all irreducible components of C are of degree 0. Then, by (2) of the previous Lemma, the sheaf $\mathcal{E}_p$ is stable.

(2) If $p \notin C_0$, let C_1 be the irreducible component of C on which p lies. Since the restriction of $\mathcal{E}_p$ to C_0 has degree -1, to C_1 degree 1 and to the others components degree 0, (4) in Lemma 3.1 implies that the sheaf $\mathcal{E}_p$ is strictly semistable.

In the second case, since p is a singular point of C, $\mathcal{E}_p$ is not an invertible sheaf and then, by (3) in Lemma 3.1, it is not stable. Let us see that it is semistable. Let $\mathcal{G} \hookrightarrow \mathcal{E}_p$ be a proper subsheaf. We have the exact sequence

$$0 \to \mathcal{O}_C(-q) \to \mathcal{E}_p \to \mathcal{O}_p \to 0\,.$$

Bearing in mind that every proper connected subcurve D of C has arithmetic genus 0 and $D \cdot \overline{D} = 2$, it is easy to deduce from Lemma 3.4 in [17] that the line bundle $\mathcal{O}_C(-q)$ is stable. Consider the composition map $g\colon \mathcal{G} \to \mathcal{O}_p$ that can be either zero or surjective. If g is zero, $\mathcal{G} \hookrightarrow \mathcal{O}_C(-q)$ and then $\mu_H(\mathcal{G}) < -1$. If g is surjective and we denote by $\mathcal{H}$ its kernel, we have that $\mathcal{H}$ is a subsheaf of $\mathcal{O}_C(-q)$ and then $\mu_H(\mathcal{H}) < -1$. Since the degree of $\mathcal{H}$ is integer, from the exact sequence

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{O}_p \to 0\,,$$

we conclude that $\mu_H(\mathcal{G}) \leq 0$. Then $\mathcal{E}_p$ is a strictly semistable sheaf.

For the second part of the statement, it is enough to note that if p is not a smooth point of C_0, $\mathcal{E}_p$ is a strictly semistable sheaf and then, by (5) in 3.1, its graded object is isomorphic to $\oplus_i \mathcal{O}_{\mathbb{P}^1}(-1)$. Hence all these sheaves are in the same S-equivalence class and the proof is complete. □

The restriction of the family $\mathcal{E}$ to $C \times C_0$ gives, by the universal property of $\overline{M}(C)$, a map

$$\phi\colon C_0 \to \overline{M}(C)$$

defined as $\phi(p) = [\mathcal{E}_p]$.

The same proof that Căldăraru gives in [7] when C is a curve of type I_2 proves the following

Proposition 3.3. *If C is any reducible fiber of a elliptic fibration of type $(*)$, then the sheaves $\mathcal{E}_p$ satisfy*

$$\operatorname{Ext}^i(\mathcal{E}_p, \mathcal{E}_{p'}) = \begin{cases} \kappa & \text{if } p = p' \text{ and } i = 0 \\ 0 & \text{if } p \neq p' \text{ and all } i \end{cases}$$

In particular the moduli space $\overline{M}(C)$ has dimension 1 and is smooth at $[\mathcal{E}_p]$ for any smooth point $p \in C_0$.

Thus if p and p' are two different smooth points of C_0, since the sheaves $\mathcal{E}_p$ and $\mathcal{E}_{p'}$ are stable and $\operatorname{Hom}(\mathcal{E}_p, \mathcal{E}_{p'}) = 0$, we have that $\phi(p) \neq \phi(p')$. Then the map ϕ is injective in $C_0 \setminus \overline{C_0}$ where $\overline{C_0}$ denotes the complementary subcurve of C_0 in C. Since C_0 is irreducible and $[\mathcal{O}_C] \in \operatorname{Im}\phi$, the map ϕ factors as

$$\begin{array}{ccc} C_0 & \longrightarrow & \overline{M}(C) \\ \downarrow & \nearrow & \\ M'(C) & & \end{array}$$

where $M'(C)$ is an irreducible component of $\overline{M}(C)$ that contains the point $[\mathcal{O}_C]$. But since $M'(C)$ is of dimension 1 and smooth at $[\mathcal{O}_C]$, it is the unique irreducible component of $\overline{M}(C)$ containing $[\mathcal{O}_C]$ and the map $\phi\colon C_0 \to M'(C)$ is also surjective.

In this point, we distinguish to cases:

1- If C is a curve of type I_N with $N \geq 2$, let $\{r_1, r_2\}$ be the two intersection points between C_0 and $\overline{C_0}$. By Proposition 3.2, we have that $\phi(r_1) = \phi(r_2)$ and then $M'(C)$ is a rational curve with one node (Figure 1).

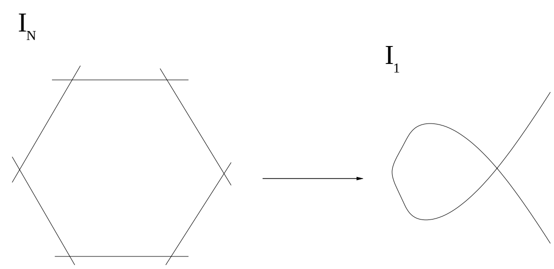

Figure 1.

In fact, since by Proposition 5.13 in [17], stable line bundles on C of degree 0 are given by the group exact sequence

$$0 \to \mathbb{G}_m \to \overline{M}^s(C) \to \prod_{i=1}^{N} \mathrm{Pic}^0(C_i) \to 0$$

and, by Corollary 6.7 in [17], there is only one extra point in $\overline{M}(C)$ corresponding to any strictly semistable sheaf, we can conclude that $\overline{M}(C) = M'(C)$. Thus the moduli space $\overline{M}(C)$ is in this case isomorphic to a rational curve with one node.

2- If C is a curve of type III or IV and r is the intersection point of C_0 and $\overline{C_0}$, we know that $\phi(r)$ is the unique singular point of $M'(C)$ and that it is a cusp (Figure 2).

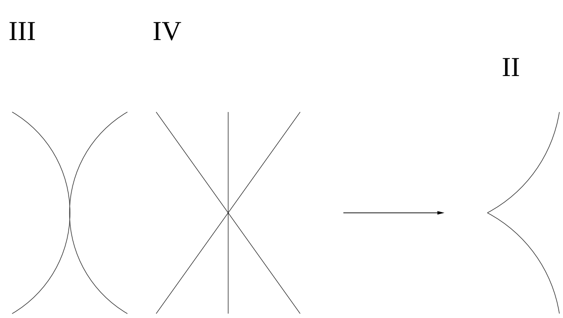

Figure 2.

Moreover, Propositions 5.6 and 5.8 of [17] imply that stable line bundles on C of degree 0 are determined by the group exact sequence:

$$0 \to \mathbb{G}_a \to \overline{M}^s(C) \to \prod_{i=1}^{N} \operatorname{Pic}^0(C_i) \to 0$$

and as above there is only one extra point in $\overline{M}(C)$. Then we conclude that $\overline{M}(C) = M'(C)$ is isomorphic to a cuspidal irreducible curve.

In this case we don't know a priori the arithmetic genus of $\overline{M}(C)$. However in the next section we will prove that the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ of f is a flat morphism, and then all its fibers have arithmetic genus 1. This allows us to conclude that the moduli space $\overline{M}(C)$ is isomorphic to a rational curve with one cusp.

4. The Flatness of the relative Jacobian

Let $f\colon X \to S$ be an elliptic fibration of type $(*)$ and $\hat{f}\colon \overline{M}_{X/S} \to S$ its relative Jacobian. From the previous section we know that this relative Jacobian is a projective morphism whose fibers are integral curves. We are now going to prove that it is a flat morphism.

Using Corollary 15.2.3 in [12] about the flatness of universally open morphisms with reduced fibers, in order to prove that $\hat{f}$ is flat, we only need to see that it is universally open. Taking into account that S is a smooth variety and then geometrically unibranch, if $\hat{f}$ is equidimensional we conclude thanks to Chevalley's criterion (Corollary 14.4.4 in [12]). Since $\hat{f}$ is surjective and all its fibers have the same dimension, to show that it is equidimensional it is enough to prove that $\overline{M}_{X/S}$ is irreducible.

Notice first that, since for all $s \in S$ the sheaf $\mathcal{O}_{X_s}$ is stable, the flat family $\mathcal{O}_X$ defines a natural section $\sigma\colon S \hookrightarrow \overline{M}_{X/S}$ of $\hat{f}\colon \overline{M}_{X/S} \to S$ whose image is contained in only one irreducible component M' of $\overline{M}_{X/S}$. Indeed, since S is irreducible, $\operatorname{Im}\sigma$ is contained in some irreducible component of $\overline{M}_{X/S}$. But for all $s \in S$, $\sigma(s) = [\mathcal{O}_{X_s}]$ is a smooth point of $\overline{M}_{X/S}$ so that this irreducible component M' is unique.

Let us see that $\overline{M}_{X/S} = M'$. Actually, since $(\overline{M}_{X/S})_s$ is irreducible for every $s \in S$, it is contained in some irreducible component of $\overline{M}_{X/S}$, namely $\tilde{M}$. If $\tilde{M} \neq M'$, the point $[\mathcal{O}_{X_s}]$, which lies on $(\overline{M}_{X/S})_s$ and on the image of the section σ, belongs to $\tilde{M} \cap M'$. But this is not possible because this is a smooth point of $\overline{M}_{X/S}$. Hence all fibers $(\overline{M}_{X/S})_s$ are contained in the irreducible component M' and so it is the whole moduli space $\overline{M}_{X/S}$.

The final result is then the following

Theorem 4.1. *If $f\colon X \to S$ is an elliptic fibration of type $(*)$ (Definition 2.1), the moduli space of semistable pure dimension one sheaves of rank 1 and degree 0 on a fiber X_s is isomorphic to:*

(1) *A smooth elliptic curve if X_s is smooth.*
(2) *A rational curve with one node if X_s is of type I_N with $N \geq 1$.*
(3) *A rational curve with one cusp if X_s is of type II, III, or IV.*

Thus the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ of an elliptic fibration of type $(*)$ is an integral elliptic fibration which always has a global section even if the original fibration has no sections. Here the difference between the integral case and the case with reducible fibers is the following. If the integral elliptic fibration has a global section, we know that it is globally isomorphic to its relative Jacobian, in contrast if the original fibration has reducible fibers even if it has a section, we are not able to ensure that it is globally isomorphic to its relative Jacobian because, as we will see now, $\overline{M}_{X/S}$ can be a singular space.

In fact, this theorem shows that as long as the original fibration $f\colon X \to S$ has reducible fibres, to get its relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ we have to contract to a point all but one irreducible components of every reducible fibre X_s of f, that is, for every reducible fibre of f we have to contract to a point $q \in \overline{M}_{X/S}$ a linear chain $\cup_i C_i$ of smooth rational curves.

When $f\colon X \to S$ is a smooth elliptic surface, being $C_i^2 = -2$ for every i, we know by [2] that the contraction of a such chain is a singular point. Since the discriminant locus of f is a finite number of points, we conclude that the relative Jacobian $\hat{f}\colon \overline{M}_{X/S} \to S$ is an integral elliptic surface with at worst a finite number of singular points. When $f\colon X \to S$ is a smooth elliptic threefold, by [18] every irreducible component of these chains has length 1, and then using the results in [21] we have that the contraction point q is also a singular point. However in this case, since the discriminant locus of f has dimension one, the singular locus of the relative Jacobian $\overline{M}_{X/S}$ has dimension less or equal to 1.

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