A CHARACTERIZATION OF JACOBIANS BY THE EXISTENCE OF PICARD BUNDLES

ANA CRISTINA LÓPEZ MARTÍN, ERNESTO MISTRETTA, AND DARÍO SÁNCHEZ GÓMEZ

Abstract. Based on Matsusaka-Ran criterion we give a criterion to characterize when a principal polarized abelian variety is a Jacobian by the existence of Picard bundles.

1. Introduction

The problem of determining when an abelian variety is a Jacobian has been studied by many people along the years. Generalizing the classical criterion of Matsusaka, Ran gives in [12] a characterization of Jacobians by the existence of curves with minimal cohomology class in the abelian variety. This criterion is nowadays known as Matsusaka-Ran criterion.

More recently, G. Pareschi and M. Popa use the theory of Fourier-Mukai transforms as a useful tool in the study of the existence of subvarieties of a principal polarized abelian variety with minimal cohomology class. In this sense, they prove in [11] a cohomology criterion which claims that if (A, Θ) is an indecomposable principal polarized abelian variety and C is a geometrically non-degenerated reduced equidimensional curve in A such that the ideal sheaf $\mathcal{I}_C(\Theta)$ is a GV-sheaf, then (A, Θ) is the Jacobian of C and C has minimal cohomology class. In the same paper they conjecture that if the Index Theorem with index 0 holds for $\mathcal{I}_C(2\Theta)$, with respect to the Fourier-Mukai transform defined by the Poincaré bundle, then C has minimal cohomology class. Consequently, using Matsusaka-Ran criterion, this would give a different cohomogical criterion for detecting Jacobians.

In this paper, we show the existence of such curves of minimal class using Picard bundles. Our main result is the following:

Theorem. *Let (A, Θ) be an indescomposible p.p.a.v. of dimension g. If there exists a WIT_g sheaf $\mathcal{F}$ on A with Chern classes $c_i(\mathcal{F}) = (-1)^i \dfrac{\theta^i}{i!}$, then (A, Θ) is a Jacobian and $\mathcal{F}$ is a Picard bundle.*

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Picard bundles were introduced by Schwarzenberger in [13] and have been used by many authors in the study of the geometry of abelian varieties (c.f [5], [6]). Mukai studied Picard bundles by means of Fourier-Mukai transforms in (c.f. [7]). We generalize its definition of Picard bundles and study some properties of these sheaves in Proposition 3.3.

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2. Fourier-Mukai transforms for abelian varieties

In this section, we recall some of the terminology of Fourier-Mukai functors and the results that we will need in the rest of the paper.

Let A be an abelian variety of dimension g and $\widehat{A} = \operatorname{Pic}^0(A)$ the dual abelian variety. $\widehat{A}$ represents the Picard functor, so there exists a universal line bundle $\mathcal{P}$ on $A \times \widehat{A}$, called the Poincaré bundle. Thus, if $\alpha \in \widehat{A}$ corresponds to the line bundle $\mathcal{L}$ on A, one has

$$\mathcal{P}_\alpha := \mathcal{P}_{|A\times\{\alpha\}} \simeq \mathcal{L}$$

Analogously, if $x \in A$, we denote $\mathcal{P}_x := \mathcal{P}_{|\{x\}\times\widehat{A}}$. The Poincaré bundle can be normalised by the condition that $\mathcal{P}_{|\{0\}\times\widehat{A}}$ is the trivial line bundle on $\widehat{A}$.

Denote $\pi_A \colon A\times\widehat{A} \to A$ and $\pi_{\widehat{A}} \colon A\times\widehat{A} \to \widehat{A}$ the natural projections.

The following result was proved by Mukai.

Theorem 2.1. [7] *The integral functor $\Phi \colon D^b(A) \longrightarrow D^b(\widehat{A})$ defined by $\mathcal{P}$*

$$\Phi(\mathcal{E}^\bullet) := \boldsymbol{R}\pi_{\widehat{A}*}({\pi_A}^*(\mathcal{E}^\bullet)\otimes\mathcal{P})$$

is a Fourier-Mukai transform, that is, an equivalence of categories. Its quasi-inverse is the integral functor defined by $\mathcal{P}^*[g]$ where $\mathcal{P}^*$ denotes the dual of $\mathcal{P}$.

Let us denote by $\widehat{\Phi} \colon D^b(\widehat{A}) \longrightarrow D^b(A)$ the integral functor defined by $\mathcal{P}^*$. A straightforward consequence are the following isomorphisms:

$$\Phi\circ\widehat{\Phi} \simeq Id_{D^b(\widehat{A})}[-g] \quad \text{and} \quad \widehat{\Phi}\circ\Phi \simeq Id_{D^b(A)}[-g]$$

Remark 2.2. In his original paper [7], Mukai consider instead of $\widehat{\Phi}$ the integral functor $\mathscr{S} \colon D^b(\widehat{A}) \longrightarrow D^b(A)$ defined by $\mathcal{P}$. The relation between both functor is given by

$$\mathscr{S} \simeq \widehat{\Phi}\circ(-1_{\widehat{A}})^* .$$

For simplicity, we shall write $\Phi^j(\mathcal{F}^\bullet) = \mathcal{H}^j(\Phi(\mathcal{F}^\bullet))$ to denote the j-th cohomology sheaf of the complex $\Phi(\mathcal{F}^\bullet)$ and the same for the functor $\widehat{\Phi}$.

Definition 2.3. A coherent sheaf $\mathcal{F}$ on A is WIT_i with respect to Φ (WIT_i-Φ in short) if $\Phi^j(\mathcal{F}) = 0$ for all $j \neq i$, or equivalently if there exists a sheaf $\widehat{\mathcal{F}}$ on $\widehat{A}$ such that $\Phi(\mathcal{F}) \simeq \widehat{\mathcal{F}}[-i]$. The sheaf $\widehat{\mathcal{F}}$ is called the Fourier-Mukai transform of $\mathcal{F}$ with respect to Φ. When in addition $\widehat{\mathcal{F}}$ is locally free, we say that $\mathcal{F}$ is IT_i with respect to Φ.

We have analogous definitions of WIT and IT with respect the dual Fourier-Mukai functor $\widehat{\Phi}$.

The following proposition collect some easy properties about this special kind of sheaves.

Proposition 2.4. *Let $\mathcal{F}$ be a coherent sheaf on A. Then, the following holds:*

1. *$\mathcal{F}$ is IT_i-Φ if and only if $H^j(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$ for all $j \neq i$ and for all $\alpha \in \widehat{A}$.*
2. *$\mathcal{F}$ is IT_0-Φ if and only if $\mathcal{F}$ is WIT_0-Φ.*
3. *If $\mathcal{F}$ is WIT_i-Φ, then $\widehat{\mathcal{F}}$ is WIT_{g-i}-$\widehat{\Phi}$ and $\widehat{\widehat{\mathcal{F}}} \simeq \mathcal{F}$.*
4. *If $\mathcal{F}$ is WIT_g-Φ, then it is a locally free sheaf.*
5. *If $\mathcal{F}$ is an ample line bundle, then it is IT_0-Φ.*

Proof. Since $\mathcal{P}$ is a locally free sheaf, 1) and 2) follow straightforwardly from Grauert's cohomology base change theorem. Part 3) follows from the isomorphism $\widehat{\Phi} \circ \Phi \simeq [-g]$ and part 4) is a consequence of 3) and the definition of IT. Part 5) is a direct consequence of the vanishing results for ample line bundles (see for instance [9]) and 1). □

The relationship between the Chern characters of a WIT sheaf and those of its Fourier-Mukai transform is given by the following formula. **Mukai formula**([8, Corollary 1.18]): If $\mathcal{E}$ is a WIT_j-Φ sheaf, then

$$\mathrm{ch}_i(\widehat{\mathcal{E}}) = (-1)^{i+j} PD(\mathrm{ch}_{g-i}(\mathcal{E})) \tag{1}$$

where PD denotes the Poincaré duality isomorphism.

Definition 2.5. A *principally polarized abelian variety* (p.p.a.v. in short) is an abelian variety A endowed with an ample line bundle $\mathcal{L}$ such that $\chi(\mathcal{L}) = 1$.

Remark 2.6. If A is an abelian variety and we denote by τ_x the translation morphism by a point $x \in X$, recall that A is a p.p.a.v if and only if there exists an ample line bundle $\mathcal{L}$ on A such that the morphism $\phi_{\mathcal{L}}\colon A \to \widehat{A}$ defined as $\phi_{\mathcal{L}}(x) = \tau_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ is an isomorphism. Moreover, by Proposition 2.4, the polarization $\mathcal{L}$ is IT_0, and it satisfies

$$\phi_{\mathcal{L}}^*(\widehat{\mathcal{L}}) \simeq \mathcal{L}^{-1} \,. \tag{2}$$

3. Picard Bundles on Jacobians

Let C be a smooth curve of genus $g \geq 2$ and consider $J_d(C)$ the Picard scheme parameterizing line bundles of degree d on C. This is a fine moduli space. Denote by $\mathcal{P}_d$ the universal Poincaré line bundle on the direct product $C \times J_d(C)$ and $p\colon C \times J_d(C) \to C$ and $q\colon C \times J_d(C) \to J_d(C)$ the projections. Fixing a point $x_0 \in C$, it is normalized by imposing $\mathcal{P}_{d|\{x_0\}\times J_d} \simeq \mathcal{O}_{J_d}$. The higher direct images $\mathbb{R}^i q_*(\mathcal{P}_d)$ of $\mathcal{P}_d$ on $J_d(C)$ are known in the literature as degree d Picard sheaves.

Let us show how Picard sheaves can be seen in terms of the Fourier-Mukai transform. Let $J_0(C) = J(C)$ be the Jacobian of C, that is, the abelian variety that parametrizes the line bundles on C with degree zero. The Riemann theta divisor Θ is a natural polarization on $J(C)$ that defines a structure of principally polarized abelian variety of dimension g on $J(C)$. By Remark 2.2, this gives a natural identification between $J(C)$ and its dual abelian variety $\widehat{J(C)}$. With this identification, if we denote by

$$a\colon C \hookrightarrow J(C)$$

the Abel morphism, the normalized Poincaré bundle $\mathcal{P}_0$ is precisely the restriction $(a \times 1)^*\mathcal{P}$ of the universal line bundle $\mathcal{P}$ on $J(C) \times J(C)$. On the other hand, the line bundle $\mathcal{P}_d \otimes p^*\mathcal{O}_C(-dx_0)$ defines an isomorphism $\lambda_d\colon J_d(C) \xrightarrow{\sim} J(C)$ and by normalization of the Poincaré sheaves that we have considered, one has isomorphisms

$$\mathcal{P}_d \simeq (1 \times \lambda_d)^*\mathcal{P}_0 \otimes p^*\mathcal{O}_C(dx_0) \,.$$

Using the base-change and the projection formulas, the Picard sheaf $\mathbb{R}^i q_*(\mathcal{P}_d)$ is

$$\mathbb{R}^i q_*(\mathcal{P}_d) \simeq \lambda_d^*\Phi^i(a_*\mathcal{O}_C(dx_0)) \,.$$

Considering that all Jacobians are already identified and although the last isomorphism is no longer true for an arbitrary line bundle L of degree d, the above discussion justifies following notion of Picard sheaves.

Definition 3.1. Let L be a line bundle on C of degree d. The sheaves $\Phi^i(a_*L)$ are called the *degree d Picard sheaves.*

Remark 3.2. The use of Fourier-Mukai transforms in the study of Picard bundles is originally due to Mukai [7]. In this paper, he just considers the Picard sheaf $F_d = \Phi^1(a_*\mathcal{O}_C(dx_0))$ corresponding to the line bundle $\mathcal{O}_C(dx_0)$.

Let $\Delta\colon D^b(J(C)) \to D^b(J(C))$ be the dualizing functor defined by $\Delta(\mathcal{F}^\bullet) = \mathbf{R}\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_{J(C)})[g]$. From Grothendieck duality, there is an isomorphism of functors

$$\Delta \circ \Phi \simeq ((-1)^* \circ \Phi \circ \Delta)[g]\,.$$

Taking into account that if L is a line bundle on C, its derived dual is $\Delta(a_*L) \simeq a_*(L^* \otimes \omega_C)[1]$ where ω_C is dualizing sheaf of C, one has an isomorphism

$$\mathbf{R}\mathcal{H}om((\Phi(a_*L), \mathcal{O}_{J(C))})) \simeq (-1)^*\Phi(a_*(L^* \otimes \omega_C))[1]\,. \tag{3}$$

which, in some cases, gives a duality relation between degree d and degree $2g-2-d$ Picard bundles.

Applying the theory of Fourier-Mukai transforms, we get some properties of Picard sheaves that we summarize in the following proposition. Compare it with Theorem 4.2 and Proposition 4.3 in [7] where Mukai studies its $F_d = \Phi^1(a_*\mathcal{O}_C(dx_0))$.

Proposition 3.3. *The following holds:*

1. $\Phi^i(a_*L)$ *are zero for* $i \neq 0, 1$.
2. *For* $d < 0$, $\Phi^0(a_*L) = 0$ *and* $\Phi^1(a_*L)$ *is simple locally free of rank* $g-d-1$. *There is an isomorphism*
$$\Phi^1(a_*L) \simeq (-1)^*\mathcal{H}om((\Phi^0(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}))\,.$$
3. *For* $0 \leq d < g-1$, $\Phi^1(a_*L)$ *is supported on* $J(C)$.
4. *For* $g-1 \leq d < 2g-1$, $\Phi^0(a_*L)$ *and* $\Phi^1(a_*L)$ *are both non-zero.*
5. *For* $d \geq 2g-1$, $\Phi^0(a_*L)$ *is a simple locally free sheaf of rank* $d+1-g$ *and* $\Phi^1(a_*L) = 0$. *There is an isomorphism*
$$\Phi^0(a_*L) \simeq (-1)^*\mathcal{H}om(\Phi^1(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)})\,.$$

Proof. The first part is because the support of L has dimension 1. If $d < 0$, from Grauert's base-change theorem $\Phi^0(a_*L) = 0$ and $\Phi^1(a_*L)$ is a locally free sheaf of rank $g-d-1$. Since Φ is an equivalence of categories and L is simple, $\Phi(a_*L)[1] = \Phi^1(a_*L)$ is simple as well. Analogously, one gets the corresponding statements in 5). In both cases, the duality relation between degree d and degree $2g-2-d$ Picard bundles is a consequence of the equation (3). Let us show 3). By cohomology base-change, one has that $\Phi^1(a_*L)_\alpha \simeq H^1(C, L \otimes \mathcal{P}_\alpha)$ which, being $L \otimes \mathcal{P}_\alpha$ of degree d, is non-zero for every $\alpha \in J(C)$ because

$\chi(L \otimes \mathcal{P}_\alpha) < 0$ by Riemman-Roch theorem. Now we prove 4). By the equation (3), there is an isomorphism

$$\Phi^0(a_*L) \simeq (-1)^* \mathcal{H}om^{-1}((\Phi(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}))\,.$$

From 3), the sheaf $\Phi^1(a_*(L^* \otimes \omega_C))$ is supported on $J(C)$, and then $\mathcal{H}om^0(\Phi^1(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}))$ is non-zero. From the spectral sequence for local homomorphims

$$E_2^{p,q} = \mathcal{H}om^p(\Phi^{-q}(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)})) \Rightarrow \\ E_\infty^{p+q} = \mathcal{H}om^{p+q}(\Phi(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}))\,.$$

and the above isomorphism we easily conclude that $\Phi^0(a_*L)$ is non-zero. Finally, to show that $\Phi^1(a_*L)$ is also non-zero, consider the line bundle $L \otimes \mathcal{O}_C(-dx_0)$ where x_0 is the point of C that we have fixed to normalize the Poincaré bundle. This is a line bundle of degree zero and then $L \otimes \mathcal{O}_C(-dx_0) \simeq \mathcal{P}_\alpha$ for some $\alpha \in J(C)$. By Theorem 4.2 in [7], there is a point $\kappa \in J(C)$ in the support of the sheaf $\Phi^1(\mathcal{O}_C(dx_0))$. Using again cohomology base-change, one obtains that

$$H^1(C, \mathcal{O}_C(dx_0) \otimes \mathcal{P}_\kappa) \simeq H^1(C, L \otimes \mathcal{P}_{\kappa-\alpha})$$

is non-zero. Hence, the point $\kappa - \alpha$ belongs to the support of $\Phi^1(a_*L)$ and we have the result. □

Consider now the line bundle $L = \mathcal{O}_C(2\Theta) \in J_{2g}(C)$. By the last proposition, the Picard sheaf $\Phi^0(a_*\mathcal{O}_C(2\Theta)) = a_*\widehat{\mathcal{O}_C(2\Theta)}$ is a vector bundle on $J(C)$.

The aim of this section is to show some of the properties that this Picard bundle has. Namely, if $\mathcal{F} = a_*\widehat{\mathcal{O}_C(2\Theta)}$, then

1. $\mathcal{F}$ is a quotient of $\widehat{\mathcal{O}_{J(C)}(2\Theta)}$.
2. $\mathcal{F}$ is WIT_g-$\widehat{\Phi}$.
3. $c_i(\mathcal{F}) = (-1)^i \dfrac{\theta^i}{i!}$.
4. $\mathcal{F}$ is simple.

Let us consider the exact sequence

$$0 \longrightarrow \mathcal{I}_C(2\Theta) \longrightarrow \mathcal{O}_{J(C)}(2\Theta) \longrightarrow a_*\mathcal{O}_C(2\Theta) \longrightarrow 0 \qquad (4)$$

Since Θ is an ample divisor, the line bundle $\mathcal{O}_{J(C)}(2\Theta) \otimes \mathcal{P}_\alpha$ is also ample for any $\alpha \in J(C)$. Thus, by applying the vanishing results for ample line bundles (see for instance [9]), we get that

$$H^i(J(C), \mathcal{O}_{J(C)}(2\Theta) \otimes \mathcal{P}_\alpha) = 0 \text{ for all } i > 0 \text{ and all } \alpha \in J(C)$$

and, by Proposition 2.4, one concludes that $\mathcal{O}_{J(C)}(2\Theta)$ is IT_0-Φ.

On the other hand, Theorem 4.1 in [10] proves that the sheaf $\mathcal{I}_C(2\Theta)$ is also IT$_0$-Φ. In this particular situation, this can be proved directly as follows. If $g = 2$, C is an effective divisor on $J(C)$ and $[C] = \Theta$. Then $\mathcal{I}_C(2\Theta) \simeq \mathcal{O}_{J(C)}(\Theta)$ and we conclude as above. Suppose now that $g > 2$. It suffices to show that $H^1(J(C), \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha) = 0$ for every $\alpha \in J(C)$. Since

$$H^1(J(C), \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha) \simeq \operatorname{Ext}^1(\mathcal{O}_{J(C)}, \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha)$$

it is enough to prove that any extension

$$0 \to \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha \to \mathcal{E} \to \mathcal{O}_{J(C)} \to 0 \tag{5}$$

is trivial. Since C and $J(C)$ are smooth, the Abel morphism is a regular embedding and then a standard local computation using the Koszul complex yields $\mathcal{E}xt^i(\mathcal{O}_C, \mathcal{O}_{J(C)}) = 0 \quad$ for all $i \neq g-1$. Thus $\mathcal{I}_C^* \simeq \mathcal{O}_{J(C)}$. Dualizing twice the extension (5), one has the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_{J(C)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_{J(C)}(2\Theta) \otimes \mathcal{P}_\alpha & \longrightarrow & \mathcal{E}^{**} & \longrightarrow & \mathcal{O}_{J(C)} & \longrightarrow & 0 \end{array}$$

and, using the snake lemma, an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}^{**} \to a_*\mathcal{O}_C(2\Theta) \otimes \mathcal{P}_\alpha \to 0\,.$$

Since $\mathcal{O}_{J(C)}(2\Theta)$ is IT$_0$, $\mathcal{E}^{**}$ is the trivial extension and so is $\mathcal{E}$ which proves the statement in this case.

By applying the Fourier-Mukai transform Φ to exact sequence (4) we then obtain

$$0 \longrightarrow \widehat{\mathcal{I}_C(2\Theta)} \longrightarrow \widehat{\mathcal{O}_{J(C)}(2\Theta)} \longrightarrow a_*\widehat{\mathcal{O}_C(2\Theta)} \longrightarrow 0 \tag{6}$$

The next step is to compute the Chern classes of $a_*\widehat{\mathcal{O}_C(2\Theta)}$. In fact, this is an old computation originally due to Schwarzenberger [13]. Here we are going to deduce it in an easy way using the Fourier-Mukai transform.
The Chern characters of $a_*\mathcal{O}_C(2\Theta)$ can be obtained using Grothendieck Riemann Roch theorem for the Abel morphism $C \overset{a}{\hookrightarrow} J(C)$

$$\operatorname{ch}(a_*\mathcal{O}_C(2\Theta)) \cdot \operatorname{td}(J(C)) = a_*(\operatorname{ch}(\mathcal{O}_C(2\Theta_{|C})) \cdot \operatorname{td}(C))$$

Remember that C has minimal cohomology class, that is,

$$[C] = \frac{\theta^{g-1}}{(g-1)!}$$

and the Todd class of $J(C)$ is trivial because it is an abelian variety. Then, one gets

$$\mathrm{ch}_j(a_*\mathcal{O}_C(2\Theta)) = \begin{cases} 0 & j < g-1 \\ \dfrac{\theta^{g-1}}{(g-1)!} & j = g-1 \\ g+1 & j = g \end{cases} \tag{7}$$

By applying Mukai formula (1) we may compute the Chern characters of Fourier-Mukai transform of $a_*\mathcal{O}_C(2\Theta)$. Thus we get

$$\mathrm{ch}_j(a_*\widehat{\mathcal{O}_C(2\Theta)}) = \begin{cases} g+1 & j = 0 \\ -\Theta & j = 1 \\ 0 & j > 1 \end{cases} \tag{8}$$

Lemma 3.4. *Let $\mathcal{E}$ be a vector bundle on a smooth variety X. The following are equivalent:*

a) $\mathrm{ch}_j(\mathcal{E}) = 0$ *for all $j \geq 2$.*

b) $\mathrm{c}_i(\mathcal{E}) = \dfrac{c_1(\mathcal{E})^i}{i!}$ *for all i.*

Proof. By definition the total Chern class of $\mathcal{E}$ is

$$\mathrm{c}_t(\mathcal{E}) = \mathrm{c}_0(\mathcal{E}) + \mathrm{c}_1(\mathcal{E})t + \mathrm{c}_2(\mathcal{E})t^2 + \cdots + \mathrm{c}_r(\mathcal{E})t^r = \prod_{i=1}^{r}(1 + a_i t)$$

and the Chern character is

$$\mathrm{ch}(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i t} = \sum \Big(1 + a_i t + \frac{(a_i t)^2}{2!} + \cdots\Big) = r + \mathrm{ch}_1(\mathcal{E})t + \cdots$$

So, if $\mathrm{c}_i(\mathcal{E}) = \dfrac{c_1(\mathcal{E})^i}{i!}$ we get that $\mathrm{c}_t(\mathcal{E}) = e^{\mathrm{c}_1(\mathcal{E})t}$. Then

$$\begin{aligned} \mathrm{c}_1(\mathcal{E})t =& \log(\mathrm{c}_t(\mathcal{E})) = \sum \log(1 + a_i t) = \\ =& \sum (a_i t - \frac{(a_i t)^2}{2} + \frac{(a_i t)^3}{3} + \cdots) = \\ =& \mathrm{ch}_1(\mathcal{E})t - \mathrm{ch}_2(\mathcal{E})t^2 + 2\,\mathrm{ch}_3(\mathcal{E})t^3 - 3\,\mathrm{ch}_4(\mathcal{E})t^4 + \cdots \end{aligned}$$

Hence $\mathrm{ch}_j(\mathcal{E}) = 0$ for any $j \geq 2$.

Conversely, if we assume that $\mathrm{ch}_j(\mathcal{E}) = 0$ for all $j \geq 2$, then one obtains

$$\log(\prod_{i=1}^{r}(1 + a_i t)) = \mathrm{c}_1(\mathcal{E})t$$

which implies the condition b). □

Using Equation (8) and Lemma 3.4, one obtains that

$$c_i(a_*\widehat{\mathcal{O}_C(2\Theta)}) = (-1)^i\frac{\theta^i}{i!} \tag{9}$$

Finally, notice that the sheaf $\mathcal{F}$ is simple because

$$\mathrm{Hom}_{D(J(C))}(a_*\widehat{\mathcal{O}_C(2\Theta)}, a_*\widehat{\mathcal{O}_C(2\Theta)}) \simeq \mathrm{Hom}_C(\mathcal{O}_C(2\Theta), \mathcal{O}_C(2\Theta))\,,$$

in particular, it is an indecomposable sheaf.

We summarize everything up in the following:

Proposition 3.5. *Let C be a smooth curve of genus $g \geq 2$, and $(J(C), \Theta)$ its Jacobian. Then, there exits a Picard bundle $\mathcal{F}$ on the abelian variety $J(C)$ such that the following holds:*

1. *$\mathcal{F}$ is WIT_g-$\widehat{\Phi}$.*
2. *$c_i(\mathcal{F}) = (-1)^i\frac{\theta^i}{i!}$.*
3. *$\mathcal{F}$ is a quotient of $\widehat{\mathcal{O}_{J(C)}(-2\Theta)}$.*
4. *$\mathcal{F}$ is simple.*

4. A Characterization of Jacobians via Picard Bundles

In this section we shall use Matsusaka-Ran criterion to prove that the existence of a sheaf satisfying the two first properties of the Picard bundle in Proposition 3.5 is enough to ensure that an indecomposable p.p.v.a. is the Jacobian of a curve.

Let us introduce some necessary notions and recall Matsusaka-Ran criterion.

Definition 4.1. A curve C on an abelian variety A is said to generate A, if A is the smallest abelian variety containing C. More generally, an effective algebraic 1-cycle $\sum n_iC_i$ on A, with $n_i > 0$ for all n_i, generates A, if the union of the curves C_i generates A.

The following result is the criterion of Matsusaka-Ran criterion [12]. This precise statement can be found in [1].

Theorem 4.2 (Matsusaka-Ran criterion)**.** *Suppose (A, Θ) is a polarized abelian variety of dimension g and $C = \sum_{i=1}^{r} n_iC_i$ is an effective 1-cycle generating A with $[C]\cdot\Theta = g$. Then $n_i = 1$ for all $1 \leq i \leq r$, the curves C_i are smooth, and (A, Θ) is isomorphic to the product of the canonically polarized Jacobians of the C_i's:*

$$(A, \Theta) \simeq (J(C_1), \Theta_1) \times \cdots \times (J(C_r), \Theta_r)$$

In particular, if C is an irreducible curve generating A with $[C]\cdot\Theta = g$, then C is smooth and (A,Θ) is the Jacobian of C.

Thus, the criterion that characterizes Jacobians by the existence of Picard bundles is the following

Theorem 4.3. *Let (A,Θ) be an indescomposible p.p.a.v. of dimension g. Suppose that there exists a sheaf $\mathcal{F}$ on A such that satisfies the following conditions:*

1. *$\mathcal{F}$ is WIT_g-$\widehat{\Phi}$.*
2. $c_i(\mathcal{F}) = (-1)^i \dfrac{\theta^i}{i!}$.

Then there exists a smooth curve C in A such that $(A,\Theta) \simeq (J(C),\Theta)$. Moreover, if the sheaf $\mathcal{F}$ is indecomposable, then it is a simple degree $\mathrm{rk}(\mathcal{F}) + g - 1$ Picard bundle with $\mathrm{rk}(\mathcal{F}) \geq g$.

Proof. Consider $\widehat{\mathcal{F}}$ the Fourier-Mukai transform of $\mathcal{F}$. Denote by $Z = \mathrm{supp}(\widehat{F})$ the support of $\widehat{\mathcal{F}}$ and $i\colon Z \hookrightarrow A$ the natural inclusion. Then, $\widehat{\mathcal{F}} \simeq i_*\mathcal{G}$ for some sheaf $\mathcal{G}$ on Z.

Using Lemma 3.4 and Mukai formula (1), we compute Chern characters of $\widehat{\mathcal{F}}$ getting that

$$\mathrm{ch}_j(\widehat{\mathcal{F}}) = \begin{cases} 0 & j < g-1 \\ \dfrac{\theta^{g-1}}{(g-1)!} & j = g-1 \\ \mathrm{rk}(\mathcal{F}) & j = g \end{cases} \tag{10}$$

This proves that Z is a subscheme of codimension $g-1$. Define now the 1-cycle

$$Z_1(\widehat{\mathcal{F}}) = \sum_{\dim V = 1} l_V(\widehat{\mathcal{F}})[V]$$

where the sum is over all 1-dimensional subvarieties in Z and $l_V(\widehat{F})$ is the length of the stalk of $\widehat{\mathcal{F}}$. As a consequence of Grothendieck-Riemman-Roch theorem (c.f. [4, Theorem 18.3, Example 18.3.11]), it is known that

$$\mathrm{ch}(\widehat{\mathcal{F}}) = Z_1(\widehat{\mathcal{F}}) + \text{higher degree terms}\,.$$

Hence the effective 1-cycle $Z_1(\widehat{\mathcal{F}})$ on A satisfies that $Z_1(\widehat{\mathcal{F}}) = \dfrac{\theta^{g-1}}{(g-1)!}$. This implies that this cycle generates A, by Corollary II.2 and Corollary II.3 in [12]. Finally, since (A,Θ) is indecomposable, then $[Z] = \dfrac{\theta^{g-1}}{(g-1)!}$ is irreducible via the Poincaré duality. The Matsusaka-Ran criterion allows us to conclude that the abelian variety (A, Θ) is the Jacobian of a smooth curve which proves the first part of the theorem.

Assume now that the sheaf $\mathcal{F}$ is indecomposable. According to the previous discussion the support $Z = C \sqcup W$ where C is the smooth curve and W is a 0-dimensional closed subscheme. Since $\widehat{\mathcal{F}} \simeq i_*\mathcal{G}$ is also indecomposable, $Z = C$ the inclusion i is $\pm a\colon C \hookrightarrow J(C)$ where a is the Abel morphism, and $\mathcal{G}$ is a torsion free sheaf on C. By applying Grothendieck-Riemman-Roch theorem to $\widehat{\mathcal{F}} \simeq i_*\mathcal{G}$, we get

$$i_*(\mathrm{rk}(\mathcal{G})) = \frac{\theta^{g-1}}{(g-1)!} \quad \text{and} \quad i_*(c_1(\mathcal{G}) - \frac{1}{2}\mathrm{rk}(\mathcal{G})K_C) = \mathrm{rk}(\mathcal{F})$$

K_C being a canonical divisor of C. Thus, $i = a$ and $\mathcal{G}$ is a line bundle on C of degree $\mathrm{rk}(\mathcal{F}) + g - 1$. From Proposition 2.4 $a_*\mathcal{G}$ is IT_0-Φ and then $\mathcal{F}$ is simple and $\mathrm{rk}(\mathcal{F}) \geq g$ by Proposition 3.3. □

Remark 4.4. The same proof shows that when (A, Θ) is decomposable, then it is isomorphic to the direct product of the Jacobians of the irreducible components of $Z_1(\widehat{\mathcal{F}})$.

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Departamento de Matemáticas and Instituto Universitario de Física Fundamental y Matemáticas (IUFFYM), Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca, Spain

E-mail address: anacris@usal.es

E-mail address: dario@usal.es

Institut de Mathématiques de Jussieu, 175 rue du Chevaleret - 75013 Paris

E-mail address: ernesto@math.jussieu.fr