Introduction to Anomalies in QFT

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The connection with topology

From our previous discussion, we know that the axial anomaly is given by (taking $\beta(x)$ = constant)

$$\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\lim_{\epsilon\to 0}\mathrm{Tr}\,\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

Instead of a basis of plane waves, to compute the right-hand side we can use a basis of **eigenfunctions of the Dirac operator**

$$\not{D}(\mathscr{A})\psi_n(x) = \lambda_n\psi_n(x)$$

But the Dirac operator anticommutes with the chirality matrix γ_5

$$\not{D}(\mathscr{A})\gamma_5 = -\gamma_5\not{D}(\mathscr{A})$$

$$\not{D}(\mathscr{A})\gamma_5\psi_n(x) = -\gamma_5\not{D}(\mathscr{A})\psi_n(x) = -\lambda_n\gamma_5\psi_n(x)$$

$$\not{D}(\mathscr{A})\psi_n(x) = \lambda_n\psi_n(x) \qquad\qquad \not{D}(\mathscr{A})\gamma_5\psi_n(x) = -\lambda_n\gamma_5\psi_n(x)$$

For each eigenstate with $\lambda_n > 0$ there is another eigenstate of **opposite eigenvalue** $-\lambda_n < 0$

All **nonzero** eigenvectors of the Dirac operators are **paired**!

Moreover, since they have different (nonzero) eigenvalues, $\psi_n(x)$ and $\gamma_5\psi_n(x)$ are **orthogonal** (the Dirac operator is self-adjoint)

$$\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x) = 0 \qquad\qquad (\lambda_n \neq 0)$$

$$-2i\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\sum_{\lambda_n\neq 0}\int d^4x\, e^{-\epsilon\lambda_n^2}\overline{\psi}_n(x)\gamma_5\psi_n(x)$$

$$-2i\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$

In the limit $\epsilon \longrightarrow 0$ the sum over nonzero eigenvalues tends to zero (due to orthogonality)

Thus, the anomaly is only determined by the **zero modes** of the Dirac operator

$$\int d^4x\,\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -2i\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$

$$\int d^4x\, \partial_\mu \langle J^\mu_\mathrm{A}(x)\rangle_{\mathscr{A}} = -2i \sum_{\lambda_n=0} \int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

The zero modes of the Dirac operator can be classified into **positive** and **negative chirality**:

$$\gamma_5 \psi_n^{(\pm)}(x) = \pm\psi_n^{(\pm)}(x) \qquad\qquad (\lambda_n = 0)$$

Then, the sum over zero modes can be written as

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x) = \sum_{\lambda_n=0,+}\int d^4x\,\overline{\psi}{}_n^{(+)}(x)\psi_n^{(+)}(x) - \sum_{\lambda_n=0,-}\int d^4x\,\overline{\psi}{}_n^{(-)}(x)\psi_n^{(-)}(x)$$

and since the states are normalized

$$\sum_{\lambda_n=0}\int d^4x\,\overline{\psi}_n(x)\gamma_5\psi_n(x)$$
$$= (\#\text{ of }+\text{'ve chirality zero modes}) - (\#\text{ of }-\text{'ve chirality zero modes})$$

$$\int d^4x\, \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i \sum_{\lambda_n=0} \int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

The zero modes of the Dirac operator can be classified into **positive** and **negative chirality**:

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Then, the sum over zero modes can be written as

$$\sum_{\lambda_n=0} \int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x) = \sum_{\lambda_n=0,+} \overbrace{\int d^4x\, \overline{\psi}_n^{(+)}(x)\psi_n^{(+)}(x)}^{=1} - \sum_{\lambda_n=0,-} \overbrace{\int d^4x\, \overline{\psi}_n^{(-)}(x)\psi_n^{(-)}(x)}^{=1}$$

and since the states are normalized

$$\sum_{\lambda_n=0} \int d^4x\, \overline{\psi}_n(x)\gamma_5\psi_n(x)$$

$$= (\#\text{ of }+\text{'ve chirality zero modes}) - (\#\text{ of }-\text{'ve chirality zero modes})$$

Thus, the integrated axial anomaly is given by the difference between the number of **zero modes** of the Dirac operator with positive and negative chirality:

$$\int d^4x\,\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -2i\Big(n_+ - n_-\Big)$$

In fact, we can define the operators

$$D_\pm \equiv \not{D}(\mathscr{A})P_\pm \qquad \text{where} \qquad P_\pm = \frac{1}{2}\Big(\mathbb{I}\pm\gamma_5\Big)$$

we can write

$$n_+ = \dim\ker D_+ \qquad\qquad n_- = \dim\ker D_-$$

Using $\not{D}(\mathscr{A})P_\pm = P_\mp\not{D}(\mathscr{A})$ and self-adjointness of the Dirac operator,

$$D_- = D_+^\dagger \qquad \longrightarrow \qquad n_- = \dim\ker D_+^\dagger$$

With all this we have arrived at the result

$$\boxed{\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\Big(\dim\ker D_+ - \dim\ker D_+^\dagger\Big)}$$

The term inside the bracket on the right-hand side is known in Mathematics as the **index of the operator** D_+

$$\int d^4x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\,(\mathrm{ind}\,D_+)$$

In fact, the analysis is valid not on in $D=4$ but for **any dimension** $D=2n$

$$\boxed{\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\,(\mathrm{ind}\,D_+)}$$

This index only depends on **topological properties** of the manifold on which the operator is defined and the external gauge field $\mathscr{A}_\mu(x)$.

A short (non-sophisticated) excursion into Mathematics

Let us consider a nonabelian gauge theory defined on a **Euclidean closed even-dimensional manifold** M.

The gauge **connection** defines a **one-form** field taking values in the Lie algebra

$$\mathscr{A} = \mathscr{A}_\mu dx^\mu \qquad \text{where} \qquad \mathscr{A}_\mu = \mathscr{A}_\mu^a T^a$$

we should keep in mind that

$$\mathscr{A} \wedge \mathscr{A} = T^a T^b \mathscr{A}^a \wedge \mathscr{A}^b = \frac{1}{2}[T^a, T^b]\mathscr{A}^a \wedge \mathscr{A}^b = \frac{i}{2} f^{abc} \mathscr{A}^a \wedge \mathscr{A}^b T^c$$

The **field strength** is a **two-form** given by

$$\mathscr{F} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A} \qquad \text{with} \qquad \mathscr{F} = \frac{1}{2} \mathscr{F}_{\mu\nu} dx^\mu \wedge dx^\nu$$

Under gauge transformations, the connection transforms as

$$\mathscr{A} \longrightarrow g^{-1}dg + g^{-1}\mathscr{A}g$$

while the field strength two-form transforms in the adjoint representation of the gauge group

$$\mathscr{F} \longrightarrow g^{-1}\mathscr{F}g$$

Finally, computing

$$d\mathscr{F} = d\mathscr{A}\wedge\mathscr{A} - \mathscr{A}\wedge d\mathscr{A} \xrightarrow{d\mathscr{A} = \mathscr{F} - \mathscr{A}\wedge\mathscr{A}} \begin{aligned} d\mathscr{F} = {} & \mathscr{F}\wedge\mathscr{A} - \mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A} \\ & -\mathscr{A}\wedge\mathscr{F} + \mathscr{A}\wedge\mathscr{A}\wedge\mathscr{A} \end{aligned}$$

we get the **Bianchi identity**

$$d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0$$

We want to investigate the properties of **invariant polynomials** of the form ($\dim M = 2m$)

$$P(\mathscr{F}) = \sum_{n+j\leq m} c_{n,j}\Big(\operatorname{Tr}\mathscr{F}^n\Big)^j \qquad c_{n,j}\in\mathbb{C}$$

$$\mathscr{F}^n \equiv \mathscr{F}\wedge \overset{n}{\ldots} \wedge\mathscr{F}$$

- The polynomial is **gauge invariant**: $P(g\mathscr{F}g^{-1}) = P(\mathscr{F})$

$$\operatorname{Tr}\mathscr{F}^n \longrightarrow \operatorname{Tr}\left(g\mathscr{F}^n g^{-1}\right) = \operatorname{Tr}\mathscr{F}^n$$

- It is **closed**: $dP(\mathscr{F}) = 0$

$$d\operatorname{Tr}\mathscr{F}^n = \operatorname{Tr}\left(d\mathscr{F}\wedge\ldots\wedge\mathscr{F}\right) + \ldots + \operatorname{Tr}\left(\mathscr{F}\wedge\ldots\wedge d\mathscr{F}\right) = n\operatorname{Tr}\left(d\mathscr{F}\mathscr{F}^{n-1}\right)$$

using the Bianchi identity $d\mathscr{F} - \mathscr{F}\wedge\mathscr{A} + \mathscr{A}\wedge\mathscr{F} = 0$

$$d\operatorname{Tr}\mathscr{F}^n = n\operatorname{Tr}\left(\mathscr{F}\mathscr{A}\mathscr{F}^{n-1}\right) - n\operatorname{Tr}\left(\mathscr{A}\mathscr{F}^n\right) = 0$$

- $\displaystyle\int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n$ is invariant under deformations of the connection

Let us consider a continuous family of connections joining $\mathscr{A}_1$ and $\mathscr{A}_2$

$$\mathscr{A}_t = (1-t)\mathscr{A}_1 + t\mathscr{A}_2 \qquad\qquad (0 \leq t \leq 1)$$

$$\Downarrow$$

$$\mathscr{A}_t \longrightarrow g^{-1}dg + g^{-1}\mathscr{A}_t g$$

Defining $\mathscr{F}_t = d\mathscr{A}_t + \mathscr{A}_t \wedge \mathscr{A}_t$ we compute

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = n\mathrm{Tr}\left(\dot{\mathscr{F}}_t\mathscr{F}_t^{n-1}\right)$$

Now we can use $\dot{\mathscr{F}}_t = d\dot{\mathscr{A}}_t + \dot{\mathscr{A}}_t \wedge \mathscr{A}_t + \mathscr{A}_t \wedge \dot{\mathscr{A}}_t$ to write

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = n\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)$$

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = n\mathrm{Tr}\left(d\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{A}_t\mathscr{F}_t^{n-1}\right) + n\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\mathscr{A}_t\right)$$

Applying the Bianchi identity recursively, one can easily prove

$$\frac{\partial}{\partial t}\mathrm{Tr}\,\mathscr{F}_t^n = nd\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right)$$

Integrating over the parameter t shows that

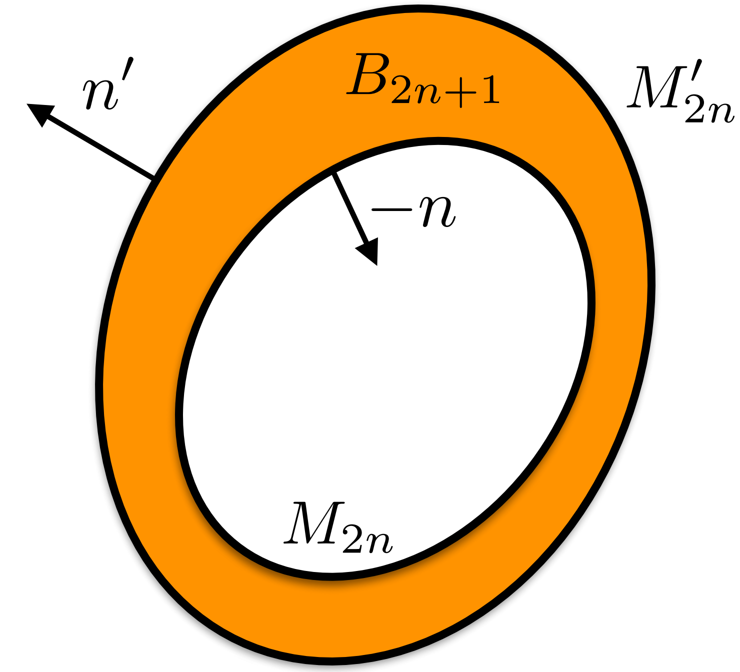
$$\mathrm{Tr}\,\mathscr{F}_2^n - \mathrm{Tr}\,\mathscr{F}_1^n = nd\int_0^1 dt\,\mathrm{Tr}\left(\dot{\mathscr{A}}_t\mathscr{F}_t^{n-1}\right) \equiv d\int_0^1 dt\,Q_{2n-1}^0(\mathscr{A}_t,\mathscr{F}_t)$$

Thus, given any closed $2n$-dimensional surface submanifold $M_{2n}\subset M$

$$\int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_1^n = \int_{M_{2n}}\mathrm{Tr}\,\mathscr{F}_2^n$$

and the result of the integral is **independent of the connection**.

- $\displaystyle\int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n$ is invariant under deformations of the submanifold M_{2n}



Let M'_{2n} be a deformation of M_{2n} and let

$$\partial B_{2n+1} = M'_{2n} - M_{2n}$$

Then, using $d\mathrm{Tr}\,\mathscr{F}^n = 0$

$$\int_{M'_{2n}} \mathrm{Tr}\,\mathscr{F}^n - \int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n = \int_{\partial B_{2n+1}} \mathrm{Tr}\,\mathscr{F}^n = \int_{B_{2n+1}} d\mathrm{Tr}\,\mathscr{F}^n = 0$$

and we conclude that the integral does not change under deformations of the submanifold

$$\int_{M'_{2n}} \mathrm{Tr}\,\mathscr{F}^n = \int_{M_{2n}} \mathrm{Tr}\,\mathscr{F}^n$$

We have seen how using invariant polynomials we can construct **topological invariants** both with respect to deformation of the **gauge field** and of the **manifold**.

At this point we introduce two examples:

- **Chern classes**: given a $\mathrm{U}(n)$ gauge bundle, the **total Chern class** is defined as

$$c(\mathscr{F}) = \det\left(1 + \frac{i}{2\pi}\mathscr{F}\right)$$

to write it in terms of invariant polynomials, we notice that since $\mathscr{F}$ is Hermitian [it lives in the algebra of $\mathrm{U}(n)$], we can diagonalize it

$$\frac{i}{2\pi}\mathscr{F} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

$$c(\mathscr{F}) = \det\left(1+\frac{i}{2\pi}\mathscr{F}\right) \qquad \frac{i}{2\pi}\mathscr{F} = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$$

The total Chern class is written then as

$$c(\mathscr{F}) = \prod_{i=1}^{n}(1+x_i) = 1 + \sum_{i=1}^{n} x_i + \sum_{i<j}^{n} x_i x_j + \ldots + \prod_{i=1}^{n} x_i$$

so we can identify the ***i*-th Chern class**

$$c_1(\mathscr{F}) = \sum_{i=1}^{n} x_i = \frac{i}{2\pi}\operatorname{Tr}\mathscr{F}$$

$$c_2(\mathscr{F}) = \sum_{i<j}^{n} x_i x_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right] = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\left[(\operatorname{Tr}\mathscr{F})^2 - \operatorname{Tr}\mathscr{F}^2\right]$$

$$\vdots$$

$$c_n(\mathscr{F}) = \det\left(\frac{i}{2\pi}\mathscr{F}\right)$$

- **Chern character**: given again a $\mathrm{U}(n)$ gauge bundle, we define

$$\mathrm{ch}(\mathscr{F}) = \mathrm{Tr}\,\exp\left(\frac{i}{2\pi}\mathscr{F}\right)$$

Formally expanding the exponential, we find the ***i*-th Chern character**

$$\mathrm{Tr}\,\exp\left(\frac{i}{2\pi}\mathscr{F}\right) = \sum_{k=0}^{m}\frac{1}{k!}\left(\frac{i}{2\pi}\mathscr{F}\right)^{k}$$

$$\Downarrow$$

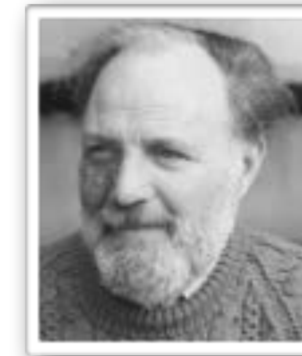
$$\mathrm{ch}_0(\mathscr{F}) = r$$

$$\mathrm{ch}_j(\mathscr{F}) = \frac{1}{j!}\left(\frac{i}{2\pi}\right)^{j}\mathrm{Tr}\,\mathscr{F}^{j} \qquad 2 \leq 2j \leq \dim M$$

Atiyah-Singer index theorem

(first version)

Sir Michael Atiyah
(b. 1929)



Isadore Singer
(b. 1924)

Let be a vector bundle defined on an **even-dimensional flat manifold without boundary** M.

The index of the Weyl operator $D_{\pm} \equiv \not{D}(\mathscr{A})P_{\pm}$ is given by

$$\boxed{\operatorname{ind} D_{+} = \int_M [\operatorname{ch}(\mathscr{F})]_{\rm vol}}$$

where $\operatorname{ch}(\mathscr{F}) = \operatorname{Tr}\, \exp\left(\frac{i}{2\pi}\mathscr{F}\right)$

In particular, if $\dim M = 2m$

$$\operatorname{ind} D_{+} = \int_M \operatorname{ch}_m(\mathscr{F}) = \frac{1}{m!}\left(\frac{i}{2\pi}\right)^m \int_M \operatorname{Tr}\,\mathscr{F}^m$$

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\,(\mathrm{ind}\,D_+) \qquad\qquad \mathrm{ind}\,D_+ = \int_M \mathrm{ch}_m(\mathscr{F}) = \frac{1}{m!}\left(\frac{i}{2\pi}\right)^m\int_M \mathrm{Tr}\,\mathscr{F}^m$$

Using the Atiyah-Singer index theorem, the axial anomaly in $D = 2n$ is given by

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{2\pi}\right)^n\int_M \mathrm{Tr}\,\mathscr{F}^n$$

To rewrite the right-hand side, we use $\mathscr{F} = \frac{1}{2}\mathscr{F}_{\mu\nu}dx^\mu\wedge dx^\nu$

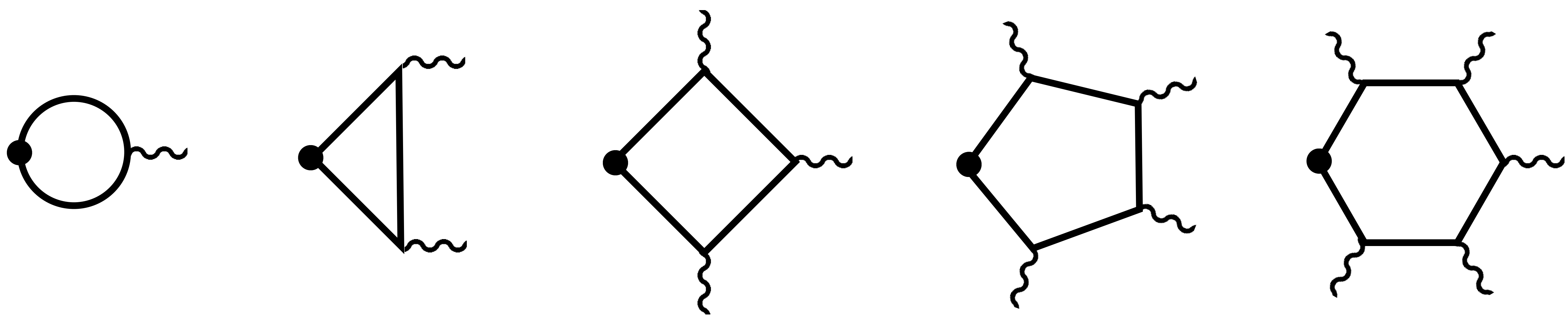
$$\begin{aligned}\mathscr{F}^n &= \frac{1}{2^n}\mathscr{F}_{\mu_1\nu_n}\ldots\mathscr{F}_{\mu_n\nu_n}dx^{\mu_1}\wedge dx^{\nu_1}\wedge\ldots\wedge dx^{\mu_n}\wedge dx^{\nu_n} \\ &= \frac{1}{2^n}\epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}d^{2n}x\end{aligned}$$

$$\int d^{2n}x\,\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{4\pi}\right)^n\int d^{2n}x\,\epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\mathrm{Tr}\left(\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}\right)$$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{2i}{n!}\left(\frac{i}{4\pi}\right)^n \epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\,\mathrm{Tr}\left(\mathscr{F}_{\mu_1\nu_1}\ldots\mathscr{F}_{\mu_n\nu_n}\right)$$

The **axial anomaly** in $D=2n$ has the following **properties**:

- It is **determined** by the one-loop, $(n+1)$-gon diagram with one **axial-vector** current and n **vector currents**

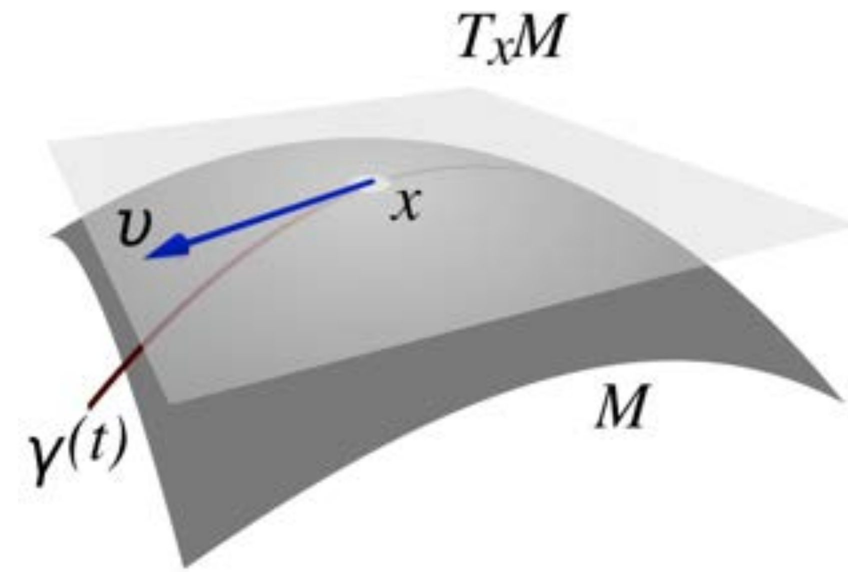


$D=2$ $\quad$ $D=4$ $\quad$ $D=6$ $\quad$ $D=8$ $\quad$ $D=10$

- The anomaly is **exact** at one loop.

But remember that **gravity** also contributes to the axial anomaly…

On the $2n$-dimensional Euclidean manifold, we have the **freedom** to choose an **orthonormal basis** of the tangent (and cotangent) space at each point **independently**



$$TM_x: \quad \mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$$

$$TM_x^*: \quad \theta^a(\mathbf{e}_b) = \delta^a_b$$

These relations are left invariant by $\mathrm{SO}(2n)$ rotations of the frame. It is with respect to these transformations that **Dirac spinors** are defined:

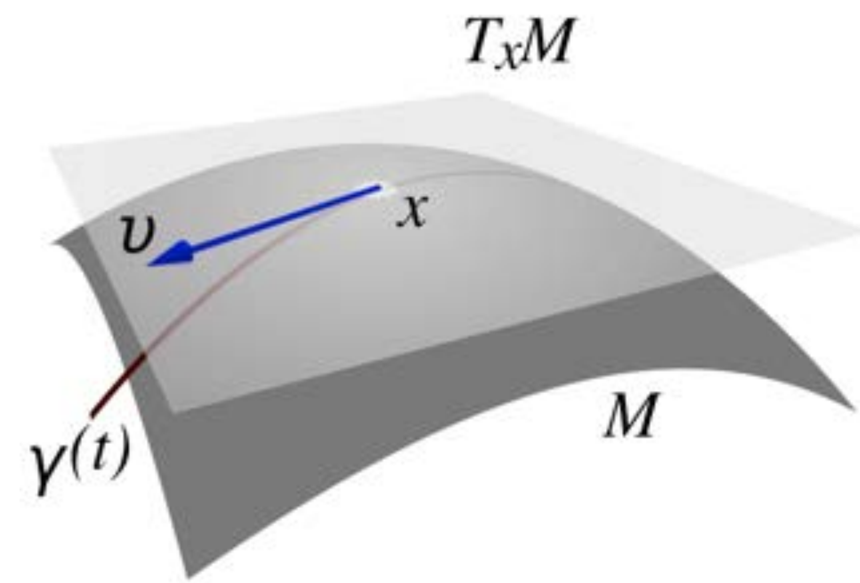
$$\{\gamma^a, \gamma^b\} = -2\delta^{ab}\mathbb{I} \qquad\qquad \gamma^{a\dagger} = -\gamma^a$$

$$\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$$

$$\psi'(x) = e^{-\frac{i}{2}\xi_{ab}(x)\sigma^{ab}}\psi(x)$$

General relativity can be seen as a $\mathrm{SO}(2n)$ **gauge theory** for the choice of **local frames** [$\Rightarrow$ $\mathrm{SO}(1,2n-1)$ in Lorentzian signature]

To define the notion of **parallel transport** along a curve γ(t) we introduce the **one-form spin connection** ω_{ab}



For a general field $\Phi(x)$ transforming in some representation Σ^{ab} of the local $\mathrm{SO}(2n)$ group,

$$\nabla_{\mathbf{v}}\Phi = d\Phi(\mathbf{v}) + \frac{1}{2}\omega_{ab}(\mathbf{v})\Sigma^{ab}\Phi$$

$$\nabla_{\mathbf{v}}\Phi = 0$$

In the case of a **spinor**, the representation is $\Sigma^{ab} \equiv \sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$ and

$$\nabla_{\mathbf{v}}\psi = d\psi(\mathbf{v}) + \frac{1}{2}\omega_{ab}(\mathbf{v})\sigma^{ab}\psi$$

In terms of the spin connection, the **curvature two-form** is defined by

$$\mathscr{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

Taking the exterior derivative

$$d\mathscr{R}^a{}_b = d\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge d\omega^c{}_b$$

and using

$$d\omega^a{}_b = \mathscr{R}^a{}_b - \omega^a{}_c \wedge \omega^c{}_b$$

we arrive at the **Bianchi identity**

$$d\mathscr{R}^a{}_b - \mathscr{R}^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \mathscr{R}^c{}_b = 0$$

Gauge theories

$$\mathscr{F} = d\mathscr{A} + \mathscr{A} \wedge \mathscr{A}$$

$$d\mathscr{F} - \mathscr{F} \wedge \mathscr{A} + \mathscr{A} \wedge \mathscr{F} = 0$$

Gravity

$$\mathscr{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

$$d\mathscr{R}^a{}_b - \mathscr{R}^a{}_c \wedge \omega^c{}_b + \omega^a{}_c \wedge \mathscr{R}^c{}_b = 0$$

The curvature two-form can be expressed in the basis of differentials

$$\mathscr{R}^a{}_b = \frac{1}{2}\mathscr{R}^a{}_{b,\mu\nu} dx^\mu \wedge dx^\nu$$

Lorentz indices can be turned into Einstein ones by using the **vielbein**

$$\mathbf{e}_a = e_a{}^\mu(x)\partial_\mu \qquad\qquad \partial_\mu = e_\mu{}^a(x)\mathbf{e}_a$$

which satisfy

$$\delta_{ab} = e_a{}^\mu(x) e_b{}^\nu(x) g_{\mu\nu}(x) \qquad\qquad g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x)\eta_{ab}$$

In terms of the vielbein, the Einstein components of the curvature tensor are

$$\mathscr{R}^\mu{}_{\nu,\alpha\beta} = e_a{}^\mu e_\nu{}^b \mathscr{R}^a{}_{b,\alpha\beta}$$

Given the transformation properties of $\omega^a{}_b$ and $\mathscr{R}^a{}_b$

$$\omega \longrightarrow U^{-1}dU + U^{-1}\omega U \qquad\qquad \mathscr{R} \longrightarrow U^{-1}\mathscr{R}\,U$$

We can define invariant polynomials as we did with gauge theories

$$P(\mathscr{R}) = \sum_{n+j\leq m} a_{n,j}\Big(\mathrm{Tr}\,\mathscr{R}^n\Big)^j \qquad\qquad a_{n,j}\in\mathbb{R}$$

$$(\mathscr{R}^n)^a{}_b = \mathscr{R}^a{}_{c_1}\wedge\mathscr{R}^{c_1}{}_{c_2}\wedge\ldots\wedge\mathscr{R}^{c_{n-1}}{}_b$$

where the trace is over $\mathrm{SO}(2n)$ indices.

- The polynomials are invariant under $\mathrm{SO}(2n)$ transformations:

$$P(\mathscr{R}) = P(U^{-1}\mathscr{R}U)$$

- They are closed:

$$dP(\mathscr{R}) = 0$$

- The integrals $\displaystyle\int_{M_{2m}}\mathrm{Tr}\,\mathscr{R}^m$ are topological invariants.

- The first invariant polynomial we define is the **Pontrjagin index**

$$p(\mathscr{R}) = \det\left(1 + \frac{1}{2\pi}\mathscr{R}\right)$$

The curvature two-form takes values in the Lie algebra of $\mathrm{SO}(2n)$, i.e. it is an **antisymmetric** matrix. To diagonalize it requires a complex transformation.

However, there exist **real** similarity transformations bringing the curvature to the form

$$\frac{1}{2\pi}\mathscr{R} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix} \qquad\qquad x_i \in \mathbb{R}$$

then

$$p(\mathscr{R}) = \prod_{i=1}^{n}(1 + x_i^2) = 1 + \sum_{i=1}^{n} x_i^2 + \sum_{i<j}^{n} x_i^2 x_j^2 + \ldots + \prod_{i=1}^{n} x_i^2$$

To write the Pontrjagin index in a more useful form, we notice

$$\frac{1}{2\pi}\mathscr{R} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$\mathrm{Tr}\left(\frac{1}{2\pi}\mathscr{R}\right)^{2k} = 2(-1)^k\sum_{i=1}^{n} x_i^{2k}$$

writing

$$p(\mathscr{R}) = 1 + p_1(\mathscr{R}) + p_2(\mathscr{R}) + \ldots + p_n(\mathscr{R})$$

with

$$p_1(\mathscr{R}) = \sum_{i=1}^{n} x_i^2 = -\frac{1}{8\pi^2}\mathrm{Tr}\,\mathscr{R}^2$$

$$p_2(\mathscr{R}) = \sum_{i<j}^{n} x_i^2 x_j^2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i^2\right)^2 - \sum_{i=1}^{n} x_i^4\right] = \frac{1}{128\pi^4}\left[(\mathrm{Tr}\,\mathscr{R}^2)^2 - 2\mathrm{Tr}\,\mathscr{R}^4\right]$$

$$\vdots$$

$$p_n(\mathscr{R}) = \prod_{i=1}^{n} x_i^2 = \left(\frac{1}{2\pi}\right)^n \det\mathscr{R}$$

- The **Â-genus (A-roof)** is defined as

$$\widehat{A}(M) = \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}\prod_{i=1}^{n} x_i^2 + \frac{7}{5760}\prod_{i=1}^{n} x_i^4 + \ldots$$

or using again $\mathrm{Tr}\left(\frac{1}{2\pi}\mathscr{R}\right)^{2k} = 2(-1)^k \sum_{i=1}^{n} x_i^{2k}$

$$\widehat{A}(M) = 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\mathrm{Tr}\,\mathscr{R}^2 + \frac{1}{(4\pi)^4}\left[\frac{1}{288}(\mathrm{Tr}\mathscr{R}^2)^2 + \frac{1}{360}\mathrm{Tr}\,\mathscr{R}^4\right] + \ldots$$

- **Euler class**

$$e(M) = \prod_{i=1}^{n} x_i$$

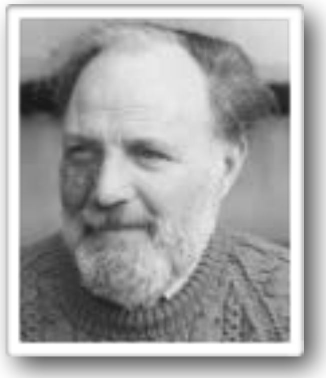
E.g., in a four-dimensional manifold, this is the "square root" of $p_2(\mathscr{R})$

$$p_2(\mathscr{R}) = e(M) \wedge e(M)$$

Atiyah-Singer index theorem

(second version)

Sir Michael Atiyah (b. 1929)



Isadore Singer (b. 1924)

Let be a vector bundle defined on an **even-dimensional curved manifold without boundary** M.

The index of the Weyl operator $D_{\pm} \equiv \not{D}(\mathscr{A})P_{\pm}$ is given now in terms of the Chern class and the Â-genus as

$$\boxed{\operatorname{ind} D_{+} = \int_M [\widehat{A}(M)\mathrm{ch}(\mathscr{F})]_{\mathrm{vol}}}$$

In four dimensions, the index has two contributions

$$\operatorname{ind} D_{+} = -\frac{1}{8\pi^2}\int_M \left(\operatorname{Tr}\mathscr{F}^2 + \frac{r}{12}\operatorname{Tr}\mathscr{R}^2\right)$$

Global anomalies

Convention **warning!**

We change convention and take the Euclidean Dirac matrices hermitian

So far, we have considered **anomalies** with respect to **infinitesimal** gauge transformations…

In **compactified four-dimensional Euclidean space**, gauge transformations are maps

$$g(x) : S^4 \longrightarrow \mathscr{G}$$

Then, the topology of gauge transformations is classified by the **fourth homotopy group** of the gauge group, $\pi_4(\mathscr{G})$

For some "popular groups", we have

$$\pi_4[\mathrm{SU}(3)] = 0$$

$$\pi_4[\mathrm{SU}(2)] = \mathbb{Z}_2$$

$$\pi_4[\mathrm{U}(1)] = 0$$

Thus, in the **standard model**, we can have transformations of SU(2) which are not contractible to the identity (they **wrap once** around the gauge group).

"Large" gauge transformations are important. They are **not taken care** of by the **Fadeev-Popov** trick in the functional integral. E.g., for SU(2)

Since the space of connections is contractible:

$$\int \mathscr{D}\mathscr{A}_\mu e^{-\frac{1}{4}\int d^4x \mathrm{Tr}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}} \longrightarrow$$ overcount by a factor of 2

In the **absence** of chiral fermions this is **harmless**, since the factor cancel out in expectation values.

In the case of a **Dirac fermion**

$$\begin{aligned} Z &= \int \mathscr{D}\mathscr{A}_\mu \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{-\int d^4x\left(\frac{1}{4}\mathrm{Tr}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}+\overline{\psi}i\not{D}\psi\right)} \\ &= \int \mathscr{D}\mathscr{A}_\mu \det(i\not{D}) e^{-\frac{1}{4}\int d^4x \mathrm{Tr}\,\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}} \end{aligned}$$

No problem: the determinant of the Dirac operator can be defined unambiguously and the result is gauge invariant.

What about a $\mathrm{SU}(2)$ gauge theory with fundamental **chiral** fermions?

(Witten, 1982)

Let us decompose the Dirac fermion into two Weyl spinors

$$\psi = \psi_+ + \psi_-$$

and write ψ_- in terms of a charge-conjugated spinor

$$\psi = \psi_+ + (\chi_+)^c$$

The Dirac action is now

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x \left[\overline{\psi}_+ i\not{D}\psi_+ + \overline{(\chi_+)^c} i\not{D}(\chi_+)^c\right]$$

But since the **fundamental** representation of $\mathrm{SU}(2)$ is **real** we can drop the charge conjugation symbol

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x \left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)$$

$$\int d^4x\,\overline{\psi} i\not{D}\psi = \int d^4x\left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)$$

Then, a Dirac fermion in the fundamental of $\mathrm{SU}(2)$ is equivalent to two positive chirality Weyl fermions.

As a consequence,

$$\det(i\not{D}) = \int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ \int \mathscr{D}\overline{\chi}_+\mathscr{D}\chi_+\, e^{-\int d^4x\left(\overline{\psi}_+ i\not{D}\psi_+ + \overline{\chi}_+ i\not{D}\chi_+\right)}$$

$$= \int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ e^{-\int d^4x\,\overline{\psi}_+ i\not{D}\psi_+} \int \mathscr{D}\overline{\chi}_+\mathscr{D}\chi_+\, e^{-\int d^4x\,\overline{\chi}_+ i\not{D}\chi_+}$$

and we arrive at

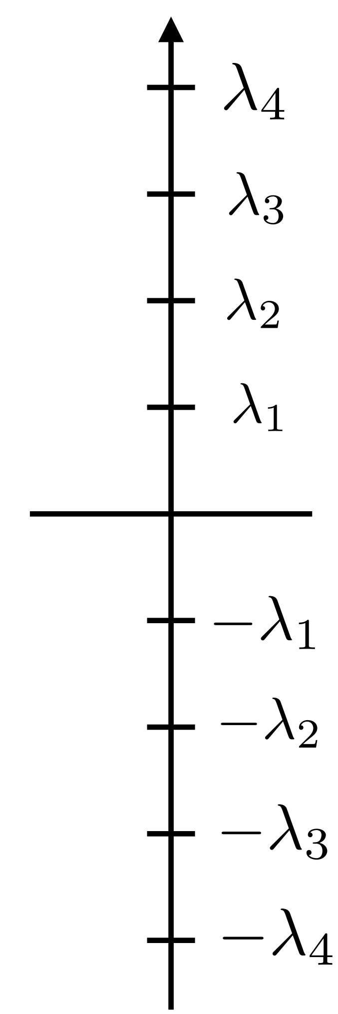
$$\int \mathscr{D}\overline{\psi}_+\mathscr{D}\psi_+ e^{-\int d^4x\,\overline{\psi}_+ i\not{D}\psi_+} = \pm[\det i\not{D}]^{\frac{1}{2}}$$

$$\int \mathscr{D}\overline{\psi}_+ \mathscr{D}\psi_+ e^{-\int d^4x\, \overline{\psi}_+ i\not{D}\psi_+} = \pm[\det(i\not{D})]^{\frac{1}{2}}$$

There is an **ambiguity** in the sign of the square root but, can we **fix** it once and for all?

Let us take a gauge field for which the Dirac operator has **no zero modes** [in other words, $\det(i\not{D}) \neq 0$]. Then, the **square root** can be defined as

$$[\det(i\not{D})]^{\frac{1}{2}} = \prod_{\lambda_n > 0} \lambda_n$$



Remember: the eigenvalues of the Dirac operator are **paired** (λ_n, $-\lambda_n$)

Now we consider a family of connections

$$\mathscr{A}_\mu^t = (1-t)\mathscr{A}_\mu + t\mathscr{A}_\mu^U \qquad 0 \leq t \leq 1$$

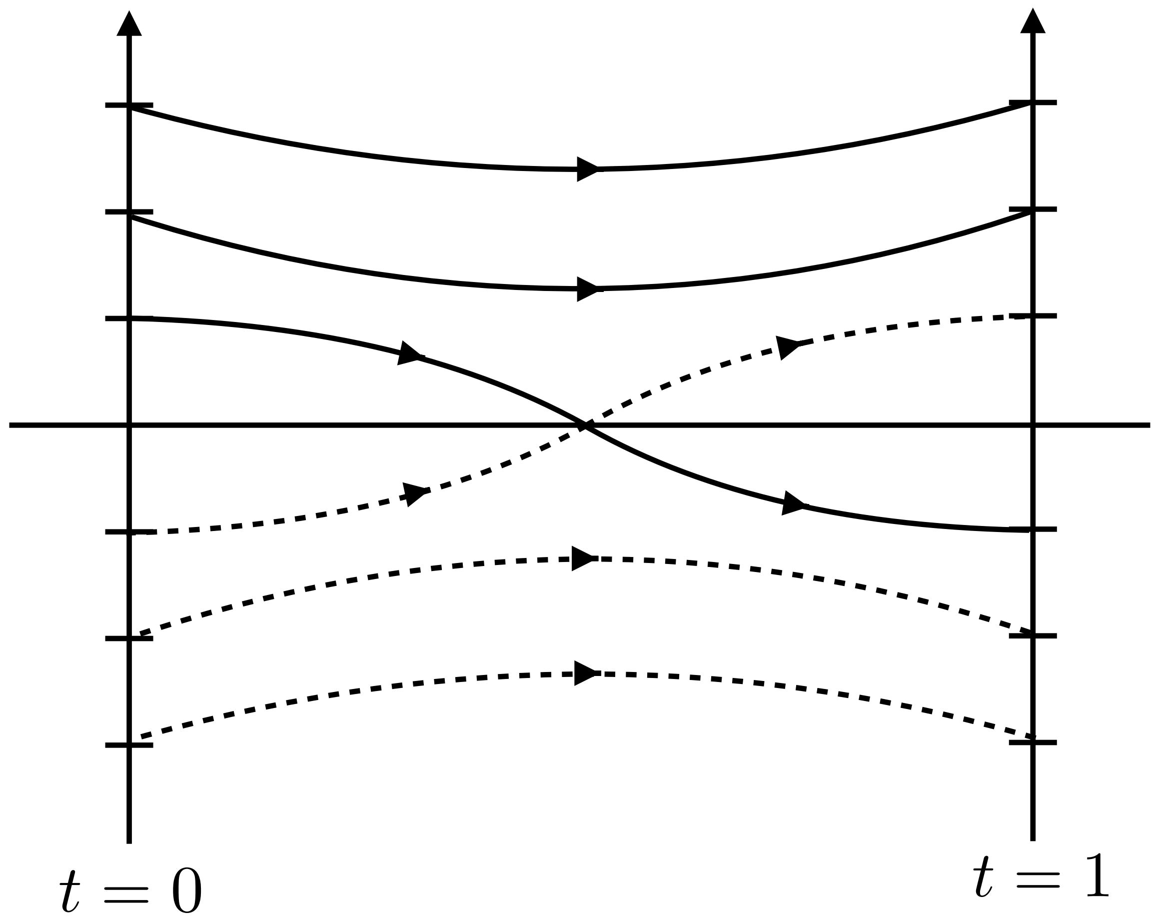
where U is a topologically nontrivial gauge transformation.

How do the positive eigenvalues "move" with t?

$$\mathscr{A}^t_\mu = (1-t)\mathscr{A}_\mu + t\mathscr{A}^U_\mu \qquad\qquad 0 \leq t \leq 1$$

Varying t induces a **spectral flow** of eigenvalues in which some may change sign:

$$[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}} = \prod_{\lambda_n>0} \lambda_n$$



$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = \prod_{\lambda_n>0} \lambda_n^U$$

If an **odd number** of positive eigenvalues change sign, then

$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}}$$

$$[\det i\not{D}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{D}(\mathscr{A})]^{\frac{1}{2}}$$

This would be a disaster, since after integrating over **all gauge fields**, the correlation functions of any gauge invariant operators vanish!

$$\int \mathscr{D}\mathscr{A}_\mu \det[i\not{D}(\mathscr{A})]^{\frac{1}{2}} \mathscr{O}_1 \ldots \mathscr{O}_n e^{-\frac{1}{4}\int d^4x \mathrm{Tr}\, \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}} = 0$$

and the theory become "empty".

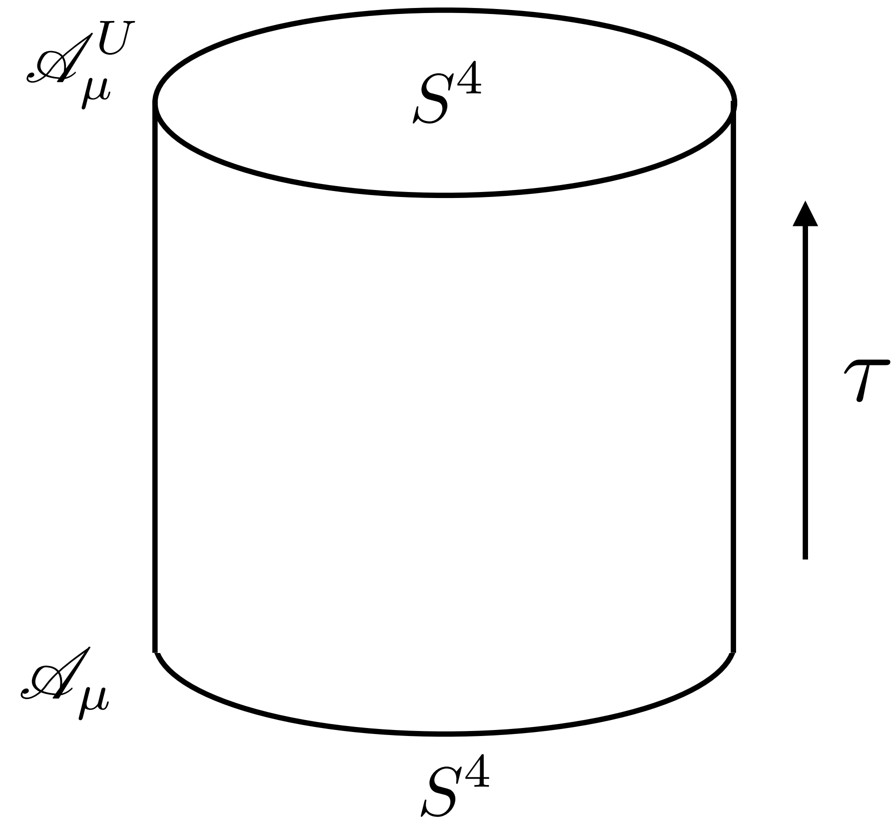
If the theory contains n $\mathrm{SU}(2)$ doublets, then the result of the fermionic integration is

$$\det[i\not{D}(\mathscr{A})]^{\frac{n}{2}}$$

and the conclusion is avoided if n is **even**.

But, is there really a **global** $\mathrm{SU}(2)$ **anomaly**?

We study a **five-dimensional** problem on the cylinder $S^4 \times \mathbb{R}$ with an **instanton-like** configuration…



… and define the Dirac operator

$$\not{D}^{(5)} = \gamma^\tau \frac{\partial}{\partial \tau} + \not{D}$$

The zero-mode equation is

$$\not{D}^{(5)} \psi = 0 \quad \longrightarrow \quad \frac{\partial \psi}{\partial \tau} = -\gamma^\tau \not{D} \psi$$

The operators $\not{D}$ and $\gamma^\tau \not{D}$ have the **same spectrum**:

$$\not{D} \psi_n = \lambda_n \psi_n \quad \longrightarrow \quad \gamma^\tau \not{D} (\mathbb{I} - \gamma^\tau) \psi_n = \lambda_n (\mathbb{I} - \gamma^\tau) \psi_n$$

where $\{\gamma^\tau, \gamma^\mu\} = 0$ and $(\gamma^\tau)^2 = \mathbb{I}$

$$\frac{\partial\psi}{\partial\tau} = -\gamma^\tau \not{D}\psi$$

Now, we assume that gauge field $\mathscr{A}_\mu(x,\tau)$ varies **adiabatically** with respect to τ

$$\psi(x,\tau) = F(\tau)\psi_\tau(x) \qquad \text{where} \qquad \gamma^\tau \not{D}\psi_\tau(x) = \lambda(\tau)\psi_\tau(x)$$

In the adiabatic approximation, the zero-mode equation $\not{D}^{(5)}\psi = 0$ reads

$$\frac{\partial\psi}{\partial\tau} = -\gamma^\tau \not{D}\psi \quad \longrightarrow \quad F'(\tau) = -\lambda(\tau)F(\tau)$$

$$\Downarrow$$

$$F(\tau) = F(0)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

and the zero-modes of the five-dimensional Dirac operator are

$$\psi(x,\tau) = F(0)\psi_\tau(x)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

$$\psi(x,\tau) = F(0)\psi_\tau(x)\exp\left[-\int_0^\tau dt'\lambda(t')\right]$$

This mode is normalizable only if:

$$\lambda(\tau) > 0 \qquad \text{for} \qquad \tau \longrightarrow \infty$$

$$\lambda(\tau) < 0 \qquad \text{for} \qquad \tau \longrightarrow -\infty$$

With this adiabatic argument, we have shown that:

- The zero-modes of $\not{D}^{(5)}$ are in **one-to-one correspondence** with the eigenvectors of $\not{D}(\mathscr{A})$ **changing sign** with the spectral flow.

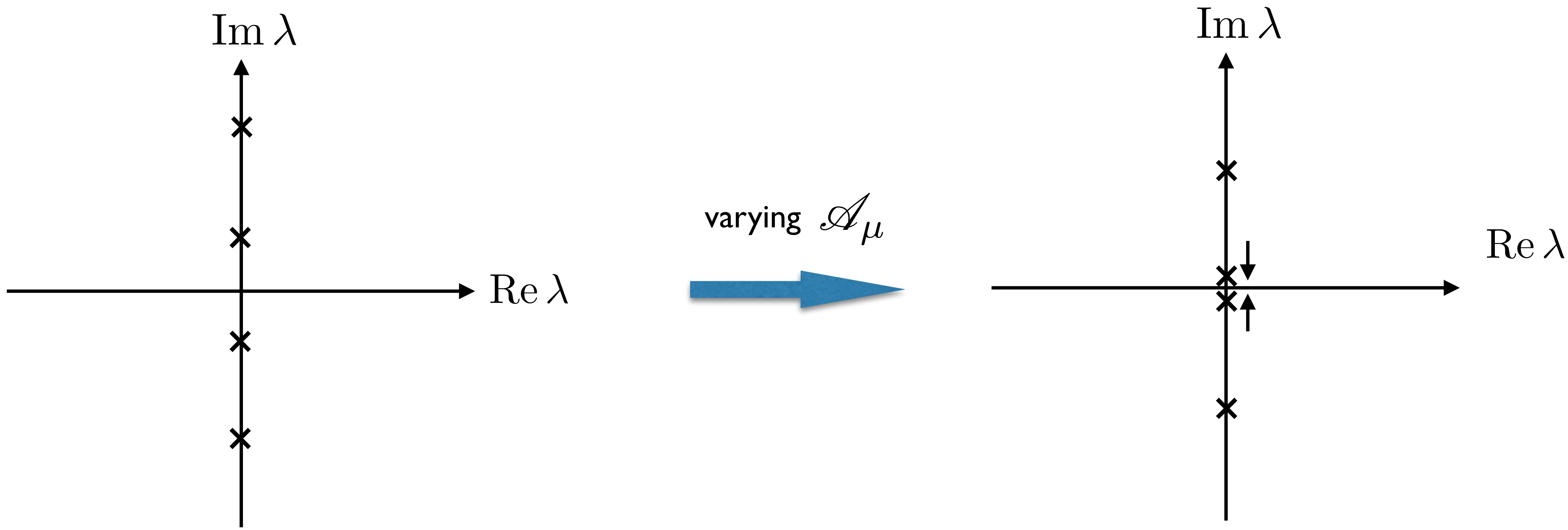
The question now is:

How many zero-modes does $\not{D}^{(5)}$ have?

⬇

mod 2 Atiyah-Singer index theorem

The operator $\not{D}^{(5)}$ is real and antisymmetric. Its eigenvalues are either **zero** or **purely imaginary** and come in complex conjugate **pairs**.



The number or zero-modes changes with a pair of **complex conjugate** eigenvalues moves towards or away the real axis.

The number of zero-modes of $\not{D}^{(5)}$ **mod 2** is a topological invariant.

The number (mod 2) of zero-modes of $\not{\mathbb{D}}^{(5)}$ can be computed using the **mod 2 Atiyah-Singer index theorem** for the gauge instanton-like configuration:

$$\#\text{ of zero-modes of } \not{\mathbb{D}}^{(5)} = 1 \ (\mathrm{mod}\ 2)$$

Thus, there is an **odd number** of eigenvalues of $\not{\mathbb{D}}(\mathscr{A})$ changing sign as we deform the connection from $\mathscr{A}_\mu$ to $\mathscr{A}_\mu^U$

$$[\det i\not{\mathbb{D}}(\mathscr{A}^U)]^{\frac{1}{2}} = -[\det i\not{\mathbb{D}}(\mathscr{A})]^{\frac{1}{2}}$$

A theory with an **odd number** of **chiral** fermions transforming as **doublets** of $\mathrm{SU}(2)$ is anomalous!

Fortunately, the **standard model** is **safe**!

$$\left.\begin{matrix} \begin{pmatrix} e \\ \nu_e \end{pmatrix}_L & \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}_L & \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}_L \\ & & \\ \begin{pmatrix} u \\ d \end{pmatrix}_L & \begin{pmatrix} c \\ s \end{pmatrix}_L & \begin{pmatrix} t \\ b \end{pmatrix}_L \end{matrix}\right\} \quad 6\ \mathrm{SU}(2)_L \text{ doublets}$$

Both **leptons** and **quarks** are required to cancel the anomaly!

The **MSSM** is also **safe** due to the second Higgsino doublet

$$\left.\begin{matrix} \begin{pmatrix} \widetilde{h}_1^0 \\ \widetilde{h}_1^- \end{pmatrix}_L \\ \\ \begin{pmatrix} \widetilde{h}_2^+ \\ \widetilde{h}_2^0 \end{pmatrix}_L \end{matrix}\right\} \quad +\ 2\ \mathrm{SU}(2)_L \text{ doublets}$$

An index theorem computation of the gauge anomaly

Convention **warning!**

We change convention and take the Euclidean Dirac matrices hermitian

We have managed to reformulate the problem of computing the axial anomaly into the calculation of the index of the Dirac-Weyl operator $D_+ \equiv \not{D}(\mathscr{A})P_+$

Let us try to do the same for the **gauge anomaly** in the simplest case of a chiral theory with a single **right-handed Weyl spinor** with gauge group G

We begin by computing the one-loop fermionic effective action in Euclidean space

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^4x \, \overline{\psi} i\not{D}(\mathscr{A})P_+\psi\right]$$

The gauge anomaly is given by the gauge variation of the effective action

$$\delta_\alpha \Gamma[\mathscr{A}] = -\int d^4x \, \alpha(x) D_\mu \langle J_R^\mu(x)\rangle_{\mathscr{A}}$$

To find the origin of the anomaly, let us consider the fermion in a **complex** representation R of the gauge group.

This theory is anomalous

$$\delta_\alpha \Gamma_R[\mathscr{A}] \neq 0$$

The same happens if the fermion is in the complex conjugate representation $\overline{R}$

$$\Gamma_{\overline{R}}[\mathscr{A}] = \Gamma_R[\mathscr{A}]^* \longrightarrow \delta_\alpha \Gamma_{\overline{R}}[\mathscr{A}] \neq 0$$

The theory with two fermions in the representations R and $\overline{R}$ is, however, anomaly free

$$\Gamma_{R\oplus\overline{R}}[\mathscr{A}] = \Gamma_R[\mathscr{A}] + \Gamma_{\overline{R}}[\mathscr{A}] \longrightarrow \delta_\alpha \Gamma_{R\oplus\overline{R}}[\mathscr{A}] = 0$$

Thus, **only the imaginary part** of the effective action is **anomalous**

$$\delta_\alpha\Big(\mathrm{Re}\,\Gamma_R[\mathscr{A}]\Big) = 0 \qquad \delta_\alpha\Big(\mathrm{Im}\,\Gamma_R[\mathscr{A}]\Big) \neq 0$$

$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^4x \, \overline{\psi} i \not{D}(\mathscr{A}) P_+ \psi\right]$$

Being naive, we would just write

$$\Gamma[\mathscr{A}] = \log\det D_+ \qquad\qquad (D_+ = \not{D} P_+)$$

The problem is that **this determinant does not exist**…

The identity

$$\gamma_5 D_+ \equiv \gamma_5 \not{D}(\mathscr{A}) P_+ = -\not{D}(\mathscr{A})\gamma_5 P_+ = -\not{D}(\mathscr{A}) P_+ = -D_+$$

shows that D_+ maps positive chirality into negative chirality spinors

$$D_+ : S_+ \otimes E \longrightarrow S_- \otimes E \qquad\qquad E = \text{gauge bundle}$$

Since it is not an endomorphism, there is no eigenvalue problem and the **determinant cannot be defined**.

Instead, we work with a different **operator**

$$\widehat{D}:\Big(S_+\oplus S_-\Big)\otimes E\longrightarrow\Big(S_+\oplus S_-\Big)\otimes E$$

where

$$\widehat{D}=\begin{pmatrix}0 & \partial\!\!\!/_- \\ D_+ & 0\end{pmatrix}\qquad(\,\partial\!\!\!/_-\equiv\partial\!\!\!/\,P_-\,)$$

This operator has a **well-defined eigenvalue problem** and the determinant can be computed.

This modification of the Weyl operator does not affect the anomaly, since **does not couple** to the **gauge field**

$$\boxed{D\!\!\!\!/=\begin{pmatrix}0 & D_- \\ D_+ & 0\end{pmatrix}}$$

Its **modulus** is gauge invariant

$$|\det\widehat{D}|^2=\det\widehat{D}\,\det\widehat{D}^\dagger=\det\Big(\partial\!\!\!/_+\partial\!\!\!/_-\Big)\det\Big(D_+D_-\Big)=\det\Big(\partial\!\!\!/_+\partial\!\!\!/_-\Big)\det D\!\!\!\!/$$

$$\Gamma[\mathscr{A}]=-\log\det\widehat{D}(\mathscr{A})\qquad\qquad \mathrm{Re}\,\Gamma[\mathscr{A}]=-\log|\det\widehat{D}(\mathscr{A})|$$

Towards a topological interpretation of the gauge anomaly

(Álvarez-Gaumé & Ginsparg 1984)

Let us **compactify** our $2n$-dimensional Euclidean space

$$\mathbb{R}^{2n} \cup \{\infty\} \longrightarrow S^{2n}$$

and consider a **one-parameter family** of gauge transformations

$$g(x,\theta) \in G \qquad\qquad g(x,0) = g(x,2\pi) = \mathbb{I}$$

This defines a family of gauge transformations

$$\mathscr{A}^{\theta} = g(x,\theta)^{-1}(d+\mathscr{A})g(x,\theta)$$

where $\mathscr{A}$ is a **reference** connection such that $\not{D}(\mathscr{A})$ has **no zero modes**.

The transformation of $\det \widehat{D}(\mathscr{A})$ is

$$|\det \widehat{D}(\mathscr{A}^{\theta})| = |\det \widehat{D}(\mathscr{A})|$$

$$\det \widehat{D}(\mathscr{A}^{\theta}) = |\det \widehat{D}(\mathscr{A})| e^{iw(\theta,\mathscr{A})} = \sqrt{\det \not{D}(\mathscr{A})}\; e^{iw(\theta,\mathscr{A})}$$

$$\det \widehat{D}(\mathscr{A}^\theta) = \sqrt{\det \not{\!\!D}(\mathscr{A})}\; e^{iw(\theta,\mathscr{A})}$$

The **anomaly** is then given by the variation of the phase

$$\Gamma[\mathscr{A}] = -\log\det \widehat{D}(\mathscr{A}) \quad\longrightarrow\quad \delta_\alpha\Gamma[\mathscr{A}] = -i\delta\theta\frac{\partial}{\partial\theta}w(\theta,\mathscr{A})$$

The phase of the determinant defines a map

$$e^{iw(\theta,\mathscr{A})} : S^1 \longrightarrow S^1$$

classified by its **winding number**

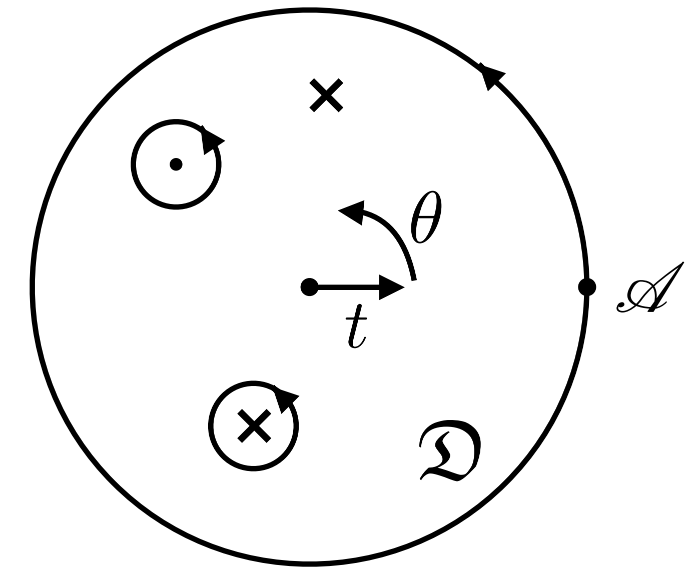
$$m = \frac{1}{2\pi}\int_0^{2\pi} d\theta\frac{\partial}{\partial\theta}w(\theta,\mathscr{A}) \in \mathbb{Z}$$

Thus, the anomaly is given by the winding number density!

The **gauge anomaly** admits a **topological interpretation**.

Is the winding number related with some kind of **index theorem**?

Let us consider the following connection defined on the manifold $S^{2n} \times \mathfrak{D}$

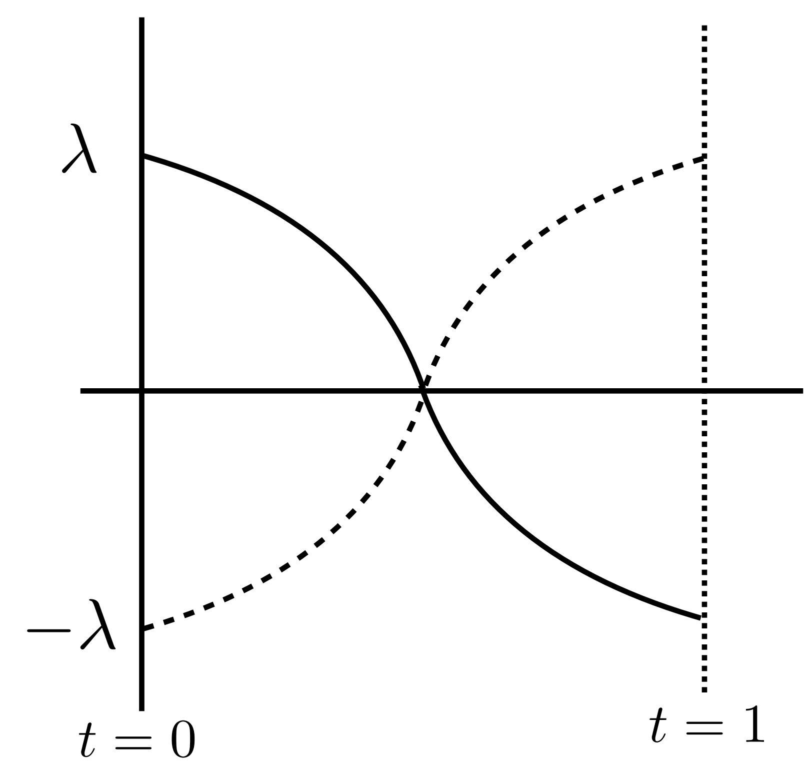


$$\mathscr{A}^{t,\theta}(x) = t\mathscr{A}^{\theta}(x) = g(x,\theta)^{-1}[d+\mathscr{A}(x)]g(x,\theta)$$

with $t \in [0,1]$.

By hypothesis, $\det \widehat{D}(\mathscr{A})$ does not vanish at $t=1$. However, it may vanish at various points in the interior of $\mathfrak{D}$

$$\mathscr{A}^{t,\theta}(x) = t\mathscr{A}^{\theta}(x)$$ is not a gauge transformation!



The vanishing of $\det \widehat{D}(\mathscr{A})$ occurs when a pair of eigenvalues of the Dirac operator crosses zero.

We can define an extension of the Dirac operator to the interior of the disk. Introducing the $(D+2)$-dimensional gauge field

$$\mathfrak{a}_C(x,\theta,t) = (\mathscr{A}_\mu^{\theta,t},0,0) \qquad\qquad C=1,\ldots,D+2$$

the new Dirac operator takes the form

$$\not{D}(\mathfrak{a}) = \sum_{C=1}^{2n+2} \Gamma^C(\partial_C + \mathfrak{a}_C)$$

where

$$\Gamma^\mu = \sigma_1 \otimes \gamma^\mu$$

$$\Gamma^{2n+1} = \sigma_2 \otimes \gamma^\mu$$

$$\Gamma^{2n+2} = \sigma_1 \otimes \gamma_5$$

and

$$\Gamma_5 = \sigma_3 \otimes \mathbb{I}$$

It can be shown (long calculation) that the zero modes of $\not{D}_{2n+2}(\mathfrak{a})$ are in one-to-one correspondence with the zeroes of $\det \not{D}_{2n}(A^{\theta,t})$

The total winding number of the phase of $\det \not{D}_{2n}(A^{\theta,t})$ is the **sum of the winding numbers** of the vanishing eigenvalues.

$$m = \sum_i m_i$$

where

$$m_i = \pm 1$$

and moreover, m_i equals the chirality of the corresponding zero mode of the (2n+2)-dimensional Dirac operator $\not{D}_{2n+2}(\mathfrak{a})$

$$\Downarrow$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta, \mathscr{A}) = \operatorname{ind} \not{D}_{2n+2}(\mathfrak{a})$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta,\mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}(\mathfrak{a})$$

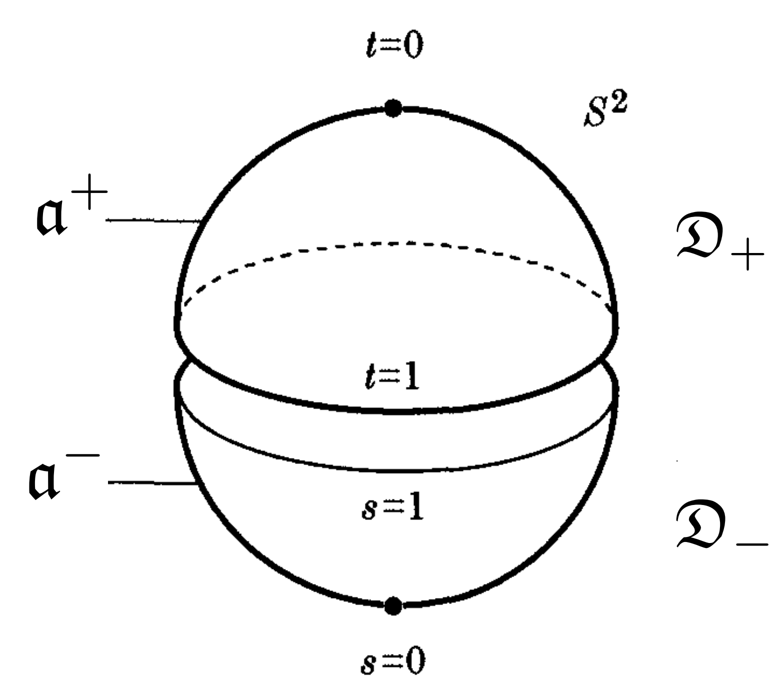
However, the Atiyah-Singer index theorem is **not applicable** because our manifold has a **boundary**!

We have two options:

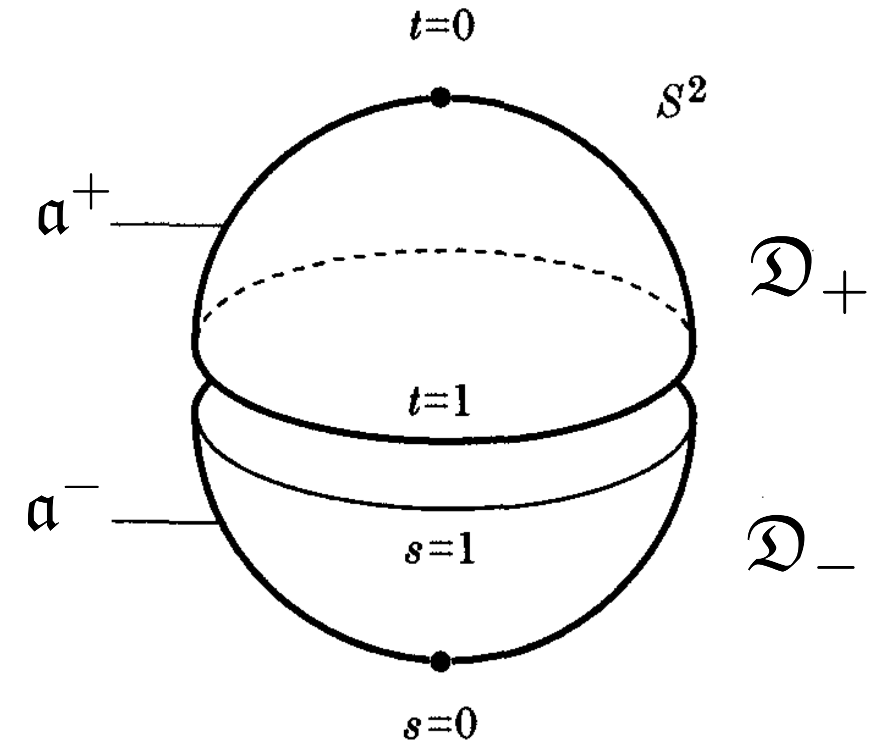
- Use the **Atiyah-Patodi-Singer index theorem** (valid for manifold with boundary).
- Set the boundary conditions by **gluing** two disks together to define the Dirac operator on the **closed manifold**

$$[S^{2n}\times\mathfrak{D}_+]\underset{\downarrow}{\cup}[S^{2n}\times\mathfrak{D}_-] \longrightarrow S^{2n}\times S^2$$

nontrivial transition functions on $S^{2n}\times S^1$



Take the connection on the **upper hemisphere** to be ($d_\theta \equiv d\theta\partial_\theta$)



$$\mathfrak{a}^+(x,\theta,t) = tg(x,\theta)^{-1}d_\theta g(x,\theta) + \mathscr{A}^{t,\theta}(x)$$

while in the **lower hemisphere** we have

$$\mathfrak{a}^-(x,\theta,s) = \mathscr{A}(x)$$

At the hemisphere $t=s=1$, both connections are related by a **gauge transformation**

$$\mathfrak{a}^+(x,\theta,1) = g(x,\theta)^{-1}d_\theta g(x,\theta) + g(x,\theta)^{-1}dg(x,\theta) + g(x,\theta)^{-1}\mathfrak{a}^-(x)g(x,\theta)$$

The field strength is given by

$$\mathfrak{f}^+ = (d + d_\theta + d_t)\mathfrak{a}^+ + \mathfrak{a}^+ \wedge \mathfrak{a}^+$$

$$\mathfrak{f}^- = (d + d_\theta + d_t)\mathfrak{a}^- + \mathfrak{a}^- \wedge \mathfrak{a}^-$$

We can apply now the Atiyah-Singer index theorem to the Dirac operator in $S^{2n}\times S^2$

$$\operatorname{ind} \not{D}_{2n+2}(\mathfrak{a}) = \int_{S^{2n}\times S^2}[\operatorname{ch}(\mathfrak{f})]_{\mathrm{vol}} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{S^{2n}\times S^2}\operatorname{Tr}\mathfrak{f}^{n+1}$$

The integral has to be computed as

$$\int_{S^{2n}\times S^2}\operatorname{Tr}\mathfrak{f}^{m+1} = \int_{S^{2n}\times\mathfrak{D}_+}\operatorname{Tr}(\mathfrak{f}^+)^{m+1} + \int_{S^{2n}\times\mathfrak{D}_-}\operatorname{Tr}(\mathfrak{f}^-)^{m+1}$$

Locally, $\operatorname{Tr}\mathfrak{f}^{n+1}$ is **exact** and on each hemisphere we have

$$\operatorname{Tr}\mathfrak{f}^{n+1} = dQ_{2n+1}$$

and using Gauß' theorem, the integral gives in terms of the Chern-Simons form

$$\int_{S^{2n}\times S^2}\operatorname{Tr}\mathfrak{f}^{m+1} = \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^-,\mathfrak{f}^-)\Big|_{s=1}$$

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\left[\int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^-,\mathfrak{f}^-)\Big|_{s=1}\right]$$

$$\Downarrow \qquad \mathfrak{a}^- = \mathscr{A}$$

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\left[\int_{S^{2n}\times S^1} Q_{2n+1}(\mathfrak{a}^+,\mathfrak{f}^+)\Big|_{t=1} - \int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A},\mathscr{F})\right]$$

We should recall that

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta,\mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}$$

so we are only interested in those terms proportional to $d\theta$. Taking into account that

$$\mathfrak{a}^+ = \mathscr{A}^\theta + g^{-1}d_\theta g \equiv \mathscr{A}^\theta + \widehat{v} \qquad\qquad \mathfrak{f}^+(\mathscr{A}^\theta + \widehat{v}) = \mathfrak{f}^+(\mathscr{A}^\theta) \equiv \mathscr{F}^\theta$$

"Russian" formula

$$\Downarrow$$

$$\mathrm{ind}\, \not{D}_{2n+2} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta} w(\theta,\mathscr{A}) = \mathrm{ind}\, \not{D}_{2n+2}$$

$$\mathrm{ind}\, \not{D} = \frac{1}{(n+1)!}\left(\frac{i}{2\pi}\right)^{n+1}\int_{S^{2n}\times S^1} Q_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$$

From here we conclude

$$i d_\theta w(\theta,\mathscr{A}) = \frac{i^{n+2}}{(2\pi)^n (n+1)!}\int_{S^{2n}} Q^1_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$$

where $Q^1_{2n+1}(\mathscr{A}^\theta + \widehat{v}, \mathscr{F}^\theta)$ is the part of the Chern-Simons form linear in $\widehat{v}$

At the end of the calculation we can set $\theta = 0$ and $g(0,x) = \mathbb{I}$.

Thus, the **gauge anomaly** in D=$2n$ can be recast as the **axial anomaly** for a Dirac operator in D=$2n$+2

We have reached the **end of the course**…

There are a number of things **we did not have time** to discuss. For example:

- **Covariant vs. consistent** anomalies.
- **Wess-Zumino** terms.
- **Gravitational anomalies** in $D = 2, 6,$ and $10.$

Green-Schwarz cancellation mechanism

- Other **advanced topics** (parity anomaly, anomalies on the lattice, anomaly inflow, etc.)

Thank you