Introduction to Anomalies in QFT

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Leaving diagrams behind: Anomalies through functional methods

Anomalies and Jacobians

Emmy Noether
(1882-1935)

Classically, continuous symmetries lead to conserved currents through Noether's theorem.

Take a theory with action $S[\phi_i]$ invariant under

$$\delta_\xi \phi_i(x) = \sum_j \xi_j F_{ij}(\phi_k)$$

The conserved current can be obtained using "Noether's trick". Taking $\xi_i(x)$ to depend on the position

$$\begin{aligned} S[\phi_i + \delta_\epsilon \phi_i] &= S[\phi_i] - \sum_i \int d^4x\, \partial_\mu \xi_i(x) j_i^\mu(x) \\ &= S[\phi_i] + \sum_i \int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) \end{aligned}$$

If the fields are on-shell, the action is invariant under any variation $\xi_i(x)$

$$\sum_i \int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) = 0 \quad \longrightarrow \quad \partial_\mu j_i^\mu(x) = 0$$

Let us move to the **quantum theory**. We look at a generic correlation function

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

We apply now a **change of variables** inside the integral

$$\phi_i'(x) = \phi_i(x) + \delta_\xi\phi_i(x)$$

that does not change its value

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{\frac{i}{\hbar}S[\phi_i']}$$

where $\mathscr{O}_i'(x)$ is the transformation of the operator $\mathscr{O}_i(x)$. At **first order**

$$\mathscr{O}_i'(x) = \mathscr{O}_i(x) + \delta_\xi\mathscr{O}_i(x)$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle=\frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)e^{\frac{i}{\hbar}S[\phi_i']}$$

$$\mathscr{O}_i'(x)=\mathscr{O}_i(x)+\delta_\xi\mathscr{O}_i(x)$$

$$S[\phi_i']=S[\phi_i]+\sum_i\int d^4x\,\xi_i(x)\partial_\mu j_i^\mu(x)$$

Combining these identities and expanding to **linear order**

$$\begin{aligned}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle&=\frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}\\&+\frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}\\&+\frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}\end{aligned}$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$
$$+\frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}$$
$$+\frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

Now we make a further assumption

$$\prod_i\mathscr{D}\phi_i' = \prod_i\mathscr{D}\phi_i$$

and arrive at the **Ward identity**:

$$\frac{i}{\hbar}\sum_k\int d^4k\,\xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\Big]|\Omega\rangle$$
$$= \sum_{a=1}^n\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

However, we can also have a **nontrivial Jacobian** in the functional integral

$$\prod_i\mathscr{D}\phi_i' = \left[1+\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\prod_i\mathscr{D}\phi_i$$

This introduces an extra term in the Ward identity

$$\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\int\left[\prod_i\mathscr{D}\phi_i\right]\mathscr{O}(x_1)\ldots\mathscr{O}(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{i}{\hbar}\frac{1}{Z}\sum_k\int d^4x\,\xi_k(x)\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\,e^{\frac{i}{\hbar}S[\phi_i]}$$

$$+\frac{1}{Z}\sum_{a=1}^n\int\left(\prod_i\mathscr{D}\phi_i'\right)\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

However, we can also have a **nontrivial Jacobian** in the functional integral

$$\prod_i\mathscr{D}\phi_i' = \left[1+\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\prod_i\mathscr{D}\phi_i$$

This introduces an extra term in the Ward identity

$$\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\int\left[\prod_i\mathscr{D}\phi_i\right]\mathscr{O}(x_1)\ldots\mathscr{O}(x_n)e^{\frac{i}{\hbar}S[\phi_i]}$$

This gives the **anomalous Ward identity**.

$$-\frac{i}{\hbar}\sum_k \int d^4k\, \xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j_k^\mu(x)\Big]|\Omega\rangle$$

$$= \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

$$+ \left[\sum_k \int d^4x\, \xi_k(x)\,\mathscr{J}_k(x)\right]\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

For the particular case in which $\mathscr{O}_i(x) \equiv \mathbb{I}$

$$\sum_k \int d^4x\, \xi_k(x)\langle\Omega|\partial_\mu j_k^\mu(x)|\Omega\rangle = i\hbar\sum_k \int d^4x\, \xi_k(x)\,\mathscr{J}_k(x)$$

$$\Downarrow \quad \forall\ \xi_k(x)$$

$$\langle\Omega|\partial_\mu j_k^\mu(x)|\Omega\rangle = i\hbar\,\mathscr{J}_k(x)$$

The anomaly is given by the functional Jacobian!

This gives the **anomalous Ward identity**.

$$-\frac{i}{\hbar}\sum_k \int d^4k\, \xi_k(x)\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\partial_\mu j^\mu_k(x)\Big]|\Omega\rangle$$

$$= \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

$$+ \left[\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

For the particular case in which $\mathscr{O}_i(x)\equiv\mathbb{I}$

$$\sum_k\int d^4x\,\xi_k(x)\langle\Omega|\partial_\mu j^\mu_k(x)|\Omega\rangle = i\hbar\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)$$

$$\Downarrow\quad \forall\ \xi_k(x)$$

$$\langle\Omega|\partial_\mu j^\mu_k(x)|\Omega\rangle = i\hbar\,\mathscr{J}_k(x)$$

The anomaly is given by the functional Jacobian!

The fermion effective action

Foreword: Euclidean fermion fields

In Minkowski space, the Dirac matrices satisfy $[\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)]$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad \Longrightarrow \quad \begin{cases} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i \end{cases}$$

Dirac fermions are defined as objects transforming under the Lorentz group as

$$\psi' = e^{-\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu}}\psi \equiv U(\vartheta)\psi \qquad \text{where} \qquad \begin{cases} \sigma^{\mu\nu} = -\dfrac{i}{4}[\gamma^\mu,\gamma^\nu] \\ \sigma^{0i\dagger} = -\sigma^{0i}, \qquad \sigma^{ij\dagger} = \sigma^{ij}. \end{cases}$$

Since $\sigma^{\mu\nu}$ is not Hermitian, Hermitian conjugate spinors are not "contravariant"

$$\psi^{\dagger\prime} = \psi^\dagger e^{\frac{i}{2}\vartheta_{\mu\nu}\sigma^{\mu\nu\dagger}} \equiv \psi^\dagger U(\vartheta)^\dagger \neq \psi^\dagger U(\vartheta)^{-1}$$

$$\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0 \quad \Downarrow \quad \gamma^0 U(\vartheta)^\dagger\gamma^0 = U(\vartheta)^{-1}$$

$$\overline{\psi}' = \psi^{\dagger\prime}\gamma^0 = \psi^\dagger U(\vartheta)^\dagger\gamma^0 = \psi^\dagger\gamma^0 U(\vartheta)^{-1} = \overline{\psi}U(\vartheta)^{-1}$$

Euclidean space can be obtained by Wick rotation from Minkowski signature

$$x^0 = -ix^4 \quad \Longrightarrow \quad \eta_{\mu\nu} \longrightarrow -\delta_{\mu\nu}$$

while the new Dirac matrices are defined as

$$\left.\begin{array}{l}\widehat{\gamma}^4 = i\gamma^0 \\ \widehat{\gamma}^i = \gamma^i\end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{l}\{\widehat{\gamma}^\mu, \widehat{\gamma}^\nu\} = -2\delta^{\mu\nu}\mathbb{I} \\ \widehat{\gamma}^{\mu\dagger} = -\widehat{\gamma}^\mu\end{array}\right.$$

Euclidean Dirac fermions are objects transforming under SO(4) as

$$\psi' = e^{-\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu}}\psi \equiv O(\omega)\psi \quad \Longrightarrow \quad \left\{\begin{array}{l}\widehat{\sigma}^{\mu\nu} = \dfrac{i}{4}[\widehat{\gamma}^\mu, \widehat{\gamma}^\nu] \\ \widehat{\sigma}^{\mu\nu\dagger} = \widehat{\sigma}^{\mu\nu}\end{array}\right.$$

Now, Hermitian conjugate objects are **contravariant**

$$\psi'^\dagger = \psi^\dagger e^{\frac{i}{2}\omega_{\mu\nu}\widehat{\sigma}^{\mu\nu\dagger}} \equiv \psi^\dagger O(\omega)^\dagger = \psi^\dagger O(\omega)^{-1}$$

In Euclidean space, the chirality matrix is defined as

$$\widehat{\gamma}_5 = -\widehat{\gamma}^1\widehat{\gamma}^2\widehat{\gamma}^3\widehat{\gamma}^4$$

satisfying

$$\widehat{\gamma}_5^\dagger = \widehat{\gamma}_5$$

A particularly important identity in the computation of anomalies is

$$\mathrm{Tr}\left(\widehat{\gamma}_5\widehat{\gamma}^\mu\widehat{\gamma}^\nu\widehat{\gamma}^\alpha\widehat{\gamma}^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta} \qquad \text{where} \qquad \epsilon^{1234} = 1$$

Comparing with its Minkowskian counterpart

$$\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4i\epsilon^{\mu\nu\alpha\beta} \qquad \text{with} \qquad \epsilon^{0123} = 1$$

we see how Euclidean chiral anomalies will have an **addition** factor of i.

Then, the Euclidean action for a Dirac fermion is

$$S_E = \int d^4x \, \psi^\dagger \Big(i\widehat{\gamma}^\mu \partial_\mu - m \Big) \psi$$

which leads to the propagator

$$\langle 0 | \psi_\alpha(x) \psi^\dagger_\beta(y) | 0 \rangle = \int \frac{d^4p}{(2\pi)^2} \frac{e^{-ip\cdot(x-y)}}{p_\mu \widehat{\gamma}^\mu - m}$$

This equation, however, is not homogeneous under Hermitian conjugation!

The way out to this problem is to take the Euclidean Dirac action

$$S_E = \int d^4x \, \overline{\psi} \Big(i\widehat{\gamma}^\mu \partial_\mu - m \Big) \psi$$

where $\psi(x)$ and $\overline{\psi}(x)$ are **independent** fields.

Thus, in Euclidean space $\overline{\psi}(x)$ transforms **contravariantly** and

$\overline{\psi}(x) \neq \psi(x)\widehat{\gamma}^0$ (despite the misleading notation)

Remember also that the representations of the Lorentz group $\mathrm{SO}(1,3)$ can be written as the **product of two copies** of $\mathrm{SU}(2)$

$$\mathcal{J}_k^{\pm} = \frac{1}{2}(J_k \pm iK_k) \qquad \begin{cases} J_k^\dagger = J_k & \text{rotations} \\ K_k^\dagger = K_k & \text{boosts} \end{cases}$$

These generators satisfy

$$[\mathcal{J}_k^{\pm}, \mathcal{J}_\ell^{\pm}] = i\epsilon_{k\ell j}\mathcal{J}_j^{\pm} \qquad\qquad [\mathcal{J}_k^{\pm}, \mathcal{J}_\ell^{\mp}] = 0$$

Thus, any representation of the Lorentz group can be written as a representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ labelled by

$$(\mathbf{s}_+, \mathbf{s}_-)$$

Since $\mathcal{J}_k^{\pm\dagger} = \mathcal{J}_k^{\mp}$ Hermitian conjugation interchanges the labels. In particular

$$(\frac{\mathbf{1}}{\mathbf{2}}, \mathbf{0}) \longrightarrow (\mathbf{0}, \frac{\mathbf{1}}{\mathbf{2}})$$

In the case of $\mathrm{SO}(4)$, its representations can also be written in terms of those of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ using the **'t Hooft symbols**:

$$\eta^a_{\mu\nu} = \varepsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}$$

$$\overline{\eta}^a_{\mu\nu} = \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4}$$

where $\varepsilon_{a\mu\nu}$ is the 3D antisymmetric symbol with $\varepsilon_{a\mu\nu} = 0$ whenever μ or ν take the value 4

The generators

$$N^a = \eta^a_{\mu\nu} J^{\mu\nu} \qquad \overline{N}^a = \overline{\eta}^a_{\mu\nu} J^{\mu\nu}$$

satisfy the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ algebra

$$[N^a, N^b] = i\varepsilon^{abc} N^c \qquad [\overline{N}^a, \overline{N}^b] = i\varepsilon^{abc}\overline{N}^c \qquad [N^a, \overline{N}^b] = 0$$

while N^a and $\overline{N}^a$ are **not related** by Hermitian conjugation.

Notation **WARNING**

From now on, Euclidean gamma matrices will be "**hatless**"

The fermion effective action

From now on we work in **Euclidean space**.

In the computation of anomalies, it is convenient to work with the **one-loop fermion effective action**. In the case of QED, this is

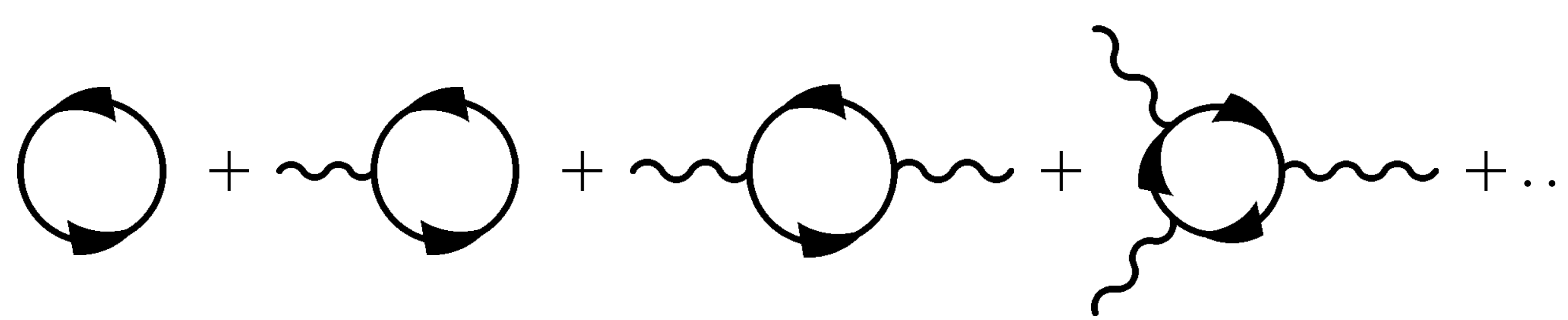
$$e^{-\Gamma[\mathscr{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + e\mathscr{A}\!\!\!/\Big)\psi\right]$$

This effective action is a **nonlocal** functional.

we are integrating out a massless fermion

Expanding the integrand in powers of the electric charge e, the effective action can be written as the sum of **one-loop** diagrams:

$\Gamma[\mathscr{A}] =$



Consider a massless Dirac fermion coupled to **external** axial and vector Abelian gauge fields

$$S = \int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi$$

This theory has two types of **local** invariances:

Vector	**Axial-vector**
$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x)$	$\psi(x) \longrightarrow e^{i\beta(x)\gamma_5}\psi(x)$
$\overline{\psi}(x) \longrightarrow \overline{\psi}(x)e^{-i\alpha(x)}$	$\overline{\psi}(x) \longrightarrow \overline{\psi}(x)e^{i\beta(x)\gamma_5}$
$\mathcal{V}_\mu(x) \longrightarrow \mathcal{V}_\mu(x) + \partial_\mu\alpha(x)$	$\mathcal{V}_\mu(x) \longrightarrow \mathcal{V}_\mu(x)$
$\mathcal{A}_\mu(x) \longrightarrow \mathcal{A}_\mu(x)$	$\mathcal{A}_\mu(x) \longrightarrow \mathcal{A}_\mu(x) + \partial_\mu\beta(x)$
$J^\mu_{\rm V} = \overline{\psi}\gamma^\mu\psi$	$J^\mu_{\rm A} = \overline{\psi}\gamma^\mu\gamma_5\psi$

For this theory, the fermion effective action is defined as

$$e^{-\Gamma[\mathcal{V},\mathcal{A}]} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

To find why this definition is useful, let us take the functional derivative

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)\exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

$$= -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,J^\mu_{\mathrm{A}}(x)\exp\left[-\int d^4y\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

while the left-hand side can be written as

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -e^{-\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}]$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -e^{\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]}$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, J^\mu_{\rm A}(x)\exp\left[-\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -e^{\Gamma[\mathcal{V},\mathcal{A}]}\frac{\delta}{\delta\mathcal{A}_\mu(x)}e^{-\Gamma[\mathcal{V},\mathcal{A}]}$$

Combining these two identities, we arrive at

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Moreover, the variation of the effective axion under axial-vector transformations are

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = \int d^4x\,\delta_\beta\mathcal{A}_\mu(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = \int d^4x\,\partial_\mu\beta(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}]$$

and integrating by parts

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\frac{\delta}{\delta\mathcal{A}_\mu(x)}\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Thus, the (integrated) **anomaly of the axial current** is given by the **variation** of the effective action under **axial-vector transformations**

$$\delta_\beta \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\, \beta(x) \partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Similarly, we can compute the variation of the effective action under vector gauge transformations

$$\frac{\delta}{\delta \mathcal{V}_\mu(x)} e^{-\Gamma[\mathcal{V},\mathcal{A}]} = -\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, J^\mu_{\rm V}(x) \exp\left[-\int d^4y\, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right]$$

Proceeding as with the axial-vector current, we arrive at

$$\delta_\alpha \Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\, \alpha(x) \partial_\mu \langle J^\mu_{\rm V}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

Thus, the **anomaly of the vector current** is given by the **variation** of the fermion effective action under **vector gauge transformations**.

This expression of the anomaly can be connected with the existence of a **nontrivial Jacobian**. (**Fujikawa's method**)

Let us consider, for example, an axial-vector gauge transformation

$$\mathcal{V}'_\mu(x) = \mathcal{V}_\mu(x) \qquad\qquad \mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + \partial_\mu \beta(x)$$

The transformed effective action is

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\,'\gamma_5\Big)\psi\right]$$

However, this change in the action can be "undone" by a change of variables in the functional integral

$$\psi'(x) = e^{-i\beta(x)\gamma_5}\psi(x) \qquad\qquad \overline{\psi}'(x) = \overline{\psi}(x) e^{-i\beta(x)\gamma_5}$$

such that

$$\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\,'\gamma_5\Big)\psi = \int d^4x \, \overline{\psi}'\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi'$$

The problem arises because of the existence of a Jacobian

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \int \mathscr{D}\overline{\psi}\mathscr{D}\psi \, \exp\left[\int d^4x \, \overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/'\gamma_5\Big)\psi\right]$$

$$= \int \mathscr{D}\overline{\psi}'\mathscr{D}\psi' \, \mathcal{J}[\beta] \exp\left[\int d^4x \, \overline{\psi}'\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi'\right]$$

Now, the Jacobian is a field-independent c-number that can be taken outside the integral

$$e^{-\Gamma[\mathcal{V},\mathcal{A}']} = \mathcal{J}[\beta] e^{-\Gamma[\mathcal{V},\mathcal{A}]} \quad \longrightarrow \quad \Gamma[\mathcal{V},\mathcal{A}'] - \Gamma[\mathcal{V},\mathcal{A}] = -\log\mathcal{J}[\beta]$$

Considering now infinitesimal axial-vector gauge transformations

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x \, \beta(x) \left(\frac{1}{\mathcal{J}[\beta]}\frac{\delta\mathcal{J}[\beta]}{\delta\beta(x)}\right)\Bigg|_{\beta=0}$$

$$\mathcal{J}[0] = 1$$

$$\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}} = \frac{\delta\mathcal{J}[\beta]}{\delta\beta(x)}\Bigg|_{\beta=0}$$

Kazuo Fujikawa
(b. 1942)

We will use Fujikawa's method in a different way…

Using the usual identity for Gaussian functional integrals with Grassmann fields

$$\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, e^{-\int d^4y\overline{\psi}\mathscr{O}\psi} = \det\mathscr{O}$$

the fermion effective action can be written as a **functional determinant**:

$$\begin{aligned} e^{-\Gamma[\mathcal{V},\mathcal{A}]} &= \int \mathscr{D}\overline{\psi}\mathscr{D}\psi\, \exp\left[-\int d^4x\,\overline{\psi}\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big)\psi\right] \\ &= \det\Big(i\partial\!\!\!/ + \mathcal{V}\!\!\!/ + \mathcal{A}\!\!\!/\gamma_5\Big) \end{aligned}$$

and therefore

$$\Gamma[\mathcal{V},\mathcal{A}] = -\log\det\Big[iD\!\!\!\!/(\mathcal{V}) + \mathcal{A}\!\!\!/\gamma_5\Big]$$

where we have written

$$iD\!\!\!\!/(\mathcal{V}) = i\partial\!\!\!/ + \mathcal{V}\!\!\!/$$

How to compute a functional determinant (in three slides)

Let us focus on a **positive definite** differential operator $\mathscr{O}$ satisfying the eigenvalue equation ($n = 1,2,\ldots$)

$$\mathscr{O} w_n(x) = \lambda_n w_n(x) \qquad\qquad \lambda_n > 0$$

Its determinant is **formally** defined as

$$\det \mathscr{O} = \prod_{n=1}^{\infty} \lambda_n$$

In our case, we are in fact interested in computing

$$\log \det \mathscr{O} = \sum_{n=1}^{\infty} \log \lambda_n$$

Thus, we need to find a useful representation of the logarithm…

Let us look at the definition of the exponential integral

$$E_1(z)=\int_z^\infty \frac{dt}{t}e^{-t}$$

which around $z=0$ this function admits the expansion

$$E_1(z)=-\gamma-\log z-\sum_{\ell=1}^{\infty}\frac{(-z)^\ell}{\ell\ell!}$$

Now, computing

$$\begin{aligned}\int_\epsilon^\infty \frac{dt}{t}e^{-xt} &= E_1(\epsilon x)\\ &= -\log x-\gamma-\log\epsilon-\sum_{\ell=1}^{\infty}\frac{(-\epsilon x)^\ell}{\ell\ell!}\end{aligned}$$

we arrive at

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{dt}{t}e^{-xt}=-\log x+x\text{-independent divergent constant}$$

Eventually, we will be interested in gauge variations of the determinant (this eliminates the divergent constant). Thus, we can use the following "definition" of the **logarithm**

$$\log x = -\int_{\epsilon}^{\infty} \frac{dt}{t} e^{-xt} \qquad\qquad \epsilon \longrightarrow 0^{+}$$

With this we can write

$$\begin{aligned} \log\det \mathscr{O} &= \sum_{n=1}^{\infty} \log \lambda_n \\ &= -\int_{\epsilon}^{\infty} \frac{dt}{t} \sum_{n=1}^{\infty} e^{-t\lambda_n} \end{aligned}$$

that is,

$$\log\det \mathscr{O} = -\int_{\epsilon}^{\infty} \frac{dt}{t} \mathrm{Tr}\, e^{-t\mathscr{O}}$$

Back to the fermion effective action

Remember that we wanted to compute

$$\Gamma[\mathcal{V},\mathcal{A}] = -\log\det\left[i\not{D}(\mathcal{V}) + \not{\mathcal{A}}\gamma_5\right]$$

To make the operator **positive definite**, we compute instead

$$\Gamma[\mathcal{V},\mathcal{A}] = -\frac{1}{2}\log\det\left[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\right]^2$$

$$= \frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\,\mathrm{Tr}\, e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}$$

Since the anomaly of the axial current is given by

$$\delta_\beta\Gamma[\mathcal{V},\mathcal{A}] = -\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}}$$

we are left with

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}} = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta\,\mathrm{Tr}\, e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}$$

$$\int d^4x\, \beta(x)\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}} = -\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta {\rm Tr}\, e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}$$

We compute then the variation of the trace

$$-\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta {\rm Tr}\, e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}$$

$$= \int_\epsilon^\infty dt\, {\rm Tr}\left\{\delta_\beta\Big[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\Big]\Big[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\Big]e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}\right\}$$

The interesting thing is that the integrand can be written now as a total derivative

$$-\frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\delta_\beta {\rm Tr}\, e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}$$

$$= -\int_\epsilon^\infty dt\,\frac{d}{dt}{\rm Tr}\,\left\{\frac{\delta_\beta\Big[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\Big]}{\Big[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\Big]}e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}\right\}$$

$$-\frac{1}{2}\int_{\epsilon}^{\infty}\frac{dt}{t}\delta_{\beta}\mathrm{Tr}\,e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}=-\int_{\epsilon}^{\infty}dt\,\frac{d}{dt}\mathrm{Tr}\left\{\frac{\delta_{\beta}\Big[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\Big]}{\Big[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\Big]}e^{-t[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}\right\}$$

We are left with the evaluation of the variation of the operator. Recalling

$$\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\longrightarrow e^{-i\beta(x)\gamma_5}\Big[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\Big]e^{-i\beta(x)\gamma_5}$$

and infinitesimally,

$$\delta_{\beta}\Big[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\Big]=\{-i\beta\gamma_5,\not{D}(\mathcal{V})\}$$

$$\int d^4x\,\beta(x)\partial_{\mu}\langle J^{\mu}_{\mathrm{A}}(x)\rangle_{\mathcal{V},\mathcal{A}}=\mathrm{Tr}\left\{\frac{\{-i\beta\gamma_5,\not{D}(\mathcal{V})\}}{\Big[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5\Big]}e^{-\epsilon[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}\right\}$$

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathcal{V},\mathcal{A}} = {\rm Tr}\left\{\frac{\{-i\beta\gamma_5, \not{D}(\mathcal{V})\}}{\left[\not{D}(\mathcal{V}) - i\not{\mathcal{A}}\gamma_5\right]}e^{-\epsilon[\not{D}(\mathcal{V})-i\not{\mathcal{A}}\gamma_5]^2}\right\}$$

The introduction of the axial-vector gauge field was a **computational trick**. To recover our result for the axial anomaly we set

$$\mathcal{V}_\mu = e\mathscr{A}_\mu \qquad\qquad \mathcal{A}_\mu = 0$$

The integrated Euclidean axial anomaly is then given by

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = {\rm Tr}\left\{\frac{\{-i\beta\gamma_5, \not{D}(\mathscr{A})\}}{\not{D}(\mathscr{A})}e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

$$= -2i{\rm Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

$$\int d^4x\, \beta(x)\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

To compute the trace, we introduce a basis $|\phi_k\rangle$ in the space of functions

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4k\, \langle\phi_k|\beta\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}|\phi_k\rangle$$

$$= -2i\int d^4x\int d^4x'\int d^4k\,\langle\phi_k|x\rangle\,\langle x|\beta\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}|x'\rangle\,\langle x'|\phi_k\rangle$$

Using locality, we write

$$\langle x|\beta\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}|x'\rangle = \delta^{(4)}(x-x')\beta(x)\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

$$\Downarrow$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int d^4k\,\phi_k(x)^*\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}\phi_k(x)$$

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int d^4k\,\phi_k(x)^*\text{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}\phi_k(x)$$

Since we can use any complete set of functions, we choose a set of plane waves

$$\phi_k(x) = \frac{1}{(2\pi)^2}e^{ik\cdot x}$$

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int \frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\text{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

Now we have to compute the trace over the Dirac indices:

$$\text{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

$$[\not{D}(\mathscr{A})]^2 = \gamma^\mu\gamma^\nu D_\mu(\mathscr{A})D_\nu(\mathscr{A})$$

$$= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}D_\mu(\mathscr{A})D_\nu(\mathscr{A}) + \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})]$$

Using the Euclidean Dirac algebra $\{\gamma^\mu,\gamma^\nu\} = -2\delta^{\mu\nu}$

$$[\not{D}(\mathscr{A})]^2 = -[D(\mathscr{A})]^2 + \frac{1}{2}\gamma^\mu\gamma^\nu[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})]$$

while the second term gives the **background field strength**

$$[D_\mu(\mathscr{A}),D_\nu(\mathscr{A})] = -ie\mathscr{F}_{\mu\nu}$$

$$[\not{D}(\mathscr{A})]^2 = -[D(\mathscr{A})]^2 - \frac{ie}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

$$\mathrm{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = \mathrm{Tr}\left\{\gamma_5 e^{\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\mathrm{Tr}\left\{\gamma_5 e^{\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}e^{ik\cdot x}$$

To further simplify, we use

$$[D_\mu(\mathscr{A}), e^{ik\cdot x}] = ik_\mu e^{ik\cdot x} \quad\longrightarrow\quad D_\mu(\mathscr{A})e^{ik\cdot x} = e^{ik\cdot x}[D_\mu(\mathscr{A}) + ik_\mu]$$

and this leads to:

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{\epsilon[\{D(\mathscr{A})+ik\}^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right)$$

$$k_\mu \to \frac{1}{\sqrt{\epsilon}}k_\mu$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{i}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\,\beta(x)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\text{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}e^{ik\cdot x}$$

$$\text{Tr}\left\{\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = \text{Tr}\left\{\gamma_5 e^{\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}$$

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\text{Tr}\left\{\gamma_5 e^{\epsilon[D(\mathscr{A})^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right\}e^{ik\cdot x}$$

To further simplify, we use

$$[D_\mu(\mathscr{A}), e^{ik\cdot x}] = ik_\mu e^{ik\cdot x} \quad\longrightarrow\quad D_\mu(\mathscr{A})e^{ik\cdot x} = e^{ik\cdot x}[D_\mu(\mathscr{A})+ik_\mu]$$

and this leads to:

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -2i\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left(\gamma_5 e^{\epsilon[\{D(\mathscr{A})+ik\}^2+\frac{i}{2}\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}]}\right)$$

$$k_\mu \to \frac{1}{\sqrt{\epsilon}}k_\mu$$

$$-2i\text{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left(\gamma_5 e^{\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{i}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

Now we take the limit $\epsilon \longrightarrow 0$ remembering that

$$\mathrm{Tr}\,\gamma_5 = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\right) = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta}.$$

Thus, the first term contributing is in the expansion in ϵ is

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\left[\frac{1}{2}\left(\frac{ie\epsilon}{2}\right)^2\int d^4x\beta(x)\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{-k^2}\right.$$
$$\left. + \mathcal{O}(\epsilon^4)\right]$$

$\epsilon \longrightarrow 0$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\epsilon^2}\int d^4x\beta(x)\int\frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma_5 e^{\{[\sqrt{\epsilon}D(\mathscr{A})+ik]^2+\frac{ie}{2}\epsilon\gamma^\mu\gamma^\nu\mathscr{F}_{\mu\nu}\}}\right)$$

Now we take the limit $\epsilon \longrightarrow 0$ remembering that

$$\mathrm{Tr}\,\gamma_5 = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\right) = 0, \qquad \mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = -4\epsilon^{\mu\nu\alpha\beta}.$$

Thus, the first term contributing is in the expansion in ϵ is

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = -\frac{2i}{\cancel{\epsilon^2}}\left[\frac{1}{2}\left(\frac{ie\cancel{\epsilon}}{2}\right)^2\int d^4x\beta(x)\mathrm{Tr}\left(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\int\frac{d^4k}{(2\pi)^4}e^{-k^2} + \mathcal{O}(\epsilon^4)\right]$$

$\Downarrow$ $\epsilon \longrightarrow 0$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$-2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

Since the anomaly is given by

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = -2i\mathrm{Tr}\left\{\beta\gamma_5 e^{-\epsilon[\not{D}(\mathscr{A})]^2}\right\}$$

we arrive at the known Adler-Bell-Jackiw anomaly in **Euclidean space**

$$\int d^4x\,\beta(x)\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\int d^4x\,\beta(x)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

$$\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

mind the i !

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{ie^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)$$

With just a few changes, this calculation is easily generalized to the case of the **singlet anomaly**

- Take the vector gauge field $\mathcal{V}_\mu(x)$ to be non-Abelian, while keeping $\mathcal{A}_\mu(x)$Abelian
- Add group theory traces in all expressions
- Set at the end $\mathcal{V}_\mu(x) = g\mathscr{A}_\mu(x)$ and $\mathcal{A}_\mu(x) = 0$

$$\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = \frac{ig^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}{\rm Tr}\left[\mathscr{F}_{\mu\nu}(x)\mathscr{F}_{\alpha\beta}(x)\right]$$

mind the i ! (again)