Introduction to Anomalies in QFT

Miguel Á. Vázquez-Mozo
Universidad de Salamanca

Universidad Autónoma de Madrid, PhD Course.

Gauge anomalies

Prelude: quantum symmetries vs. gauge invariance

By Wigner's theorem, **global symmetries** are implemented on the Hilbert space by unitary or antiunitary operators:

$$\mathcal{U}(\alpha_i)|\psi\rangle = |\psi'\rangle \qquad \text{where, generically} \qquad |\psi\rangle \neq |\psi'\rangle$$

As an example, let us look at the hydrogen atom: a $\mathrm{SO}(3)$ rotation acts on a state as

$$\mathcal{U}(\theta,\varphi,\psi)|n,j,m\rangle = \sum_{m'=-j}^{j} \mathscr{D}^{(j)}_{mm'}(\theta,\varphi,\psi)|n,j,m'\rangle$$

Gauge invariance is very different from this. In a gauge theory, a physical state is represented by **infinitely many rays** in the Hilbert space.

The space of physical states is smaller than the "naive" Hilbert space of the theory

$$\mathscr{H}_{\text{phys}} = \mathscr{H}/\mathscr{G}$$

Thus, **gauge invariance is not a symmetry but a redundancy**. It is a technicality that allows to describe a spin-1 (or spin-2) theory in a way compatible with locality and Lorentz invariance.

Some of these redundant states, however, have negative norm, e.g.

$$|\Psi\rangle = A_0|\Omega\rangle \quad \longrightarrow \quad \langle\Psi|\Psi\rangle < 0$$

It is thanks to gauge invariance that these redundant states are eliminated from the physical spectrum

$$\delta_{\rm gauge}|\psi\rangle_{\rm phys} = 0$$

Since $\delta_{\rm gauge}A_0 = \dot{\epsilon}(x)$ we have

$$\delta_{\rm gauge}|\Psi\rangle \neq 0 \quad \longrightarrow \quad |\Psi\rangle \text{ is not a physical state}$$

The absence of ghost is preserved in time provided the theory is gauge invariant at the quantum level

$$[\delta_{\rm gauge}, H] = 0$$

which guarantees that

$$\delta_{\rm gauge}|\psi(0)\rangle = 0 \quad \longrightarrow \quad \delta_{\rm gauge}|\psi(t)\rangle = 0$$

i.e., the time evolution of a physical state is a physical state.

However, if **gauge invariant is anomalous** ghosts can be generated by time evolution

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the theory becomes **nonunitary**

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gauge anomalies should be **cancelled** in physical theories at all cost

Where can we expect gauge anomalies?

Since

$$\mathscr{P} : \psi_{R,L} \longrightarrow \psi_{L,R}$$

a parity-invariant theory contains as many right- and left-handed fermions in the same representation.

Thus, we can build **gauge-invariant mass terms** and the theory can be regularized using **Pauli-Villars** fields which preserve gauge invariance.

Gauge anomalies can arise only in **parity-violating** theories.

As a first example, we consider N Dirac fermions with charges Q_i **chirally coupled** to an external electromagnetic field

$$S=\sum_{i=j}^{N}\int d^4x\left[i\overline{\psi}_j\gamma^\mu\partial_\mu\psi_j+Q_i\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\mathscr{A}_\mu\right]$$

This theory has a gauge symmetry

$$\psi_j(x)\longrightarrow\frac{1+\gamma_5}{2}\psi_j(x)+e^{iQ_j\alpha(x)}\frac{1-\gamma_5}{2}\psi_j(x)$$

$$\mathscr{A}_\mu(x)\longrightarrow\mathscr{A}_\mu(x)+\partial_\mu\alpha(x)$$

where the associated conserved current is of the V-A type

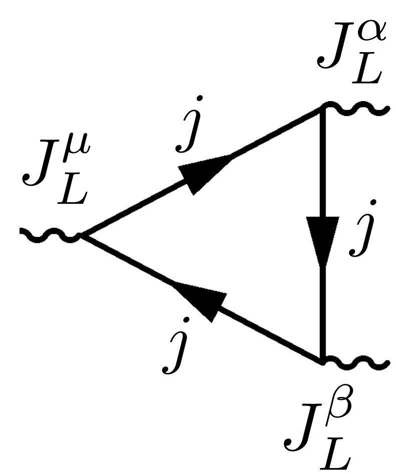
$$J^\mu_L=\sum_{j=1}^{N}Q_j\overline{\psi}_j\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi_j\qquad\text{with}\qquad\partial_\mu J^\mu_L=0$$

To study the quantum conservation of the gauge current, we have to compute

$$\partial_\mu \langle J_L^\mu(x)\rangle_{\mathscr{A}} = -\frac{1}{2}\int d^4y_1 d^4y_2 \langle 0|T[J_L^\mu(x)J_L^\alpha(y_1)J_L^\beta(y_2)]|0\rangle \mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2)$$

Diagrammatically, we have to evaluate a triangle diagram with three V-A currents at its vertices

$$\sum_{j=1}^{N}$$



(summing over all fermion species in the loop)

where **Bose symmetry** has to be imposed on **all three vertices**

Even before computing it, we see that the result should be proportional to the quantity

$$\partial_\mu \langle J_L^\mu\rangle_{\mathscr{A}} \sim \sum_{j=1}^{N} Q_j^3$$

To take advantage of our previous calculations, we write

$$J_L^\mu = \frac{1}{2}\Big(J_{\rm V}^\mu - J_{\rm A}^\mu\Big)$$

The anomaly is associated with the parity-violating part of the amplitude that contains the terms

$$\langle 0|T[J_A^\mu J_V^\alpha J_V^\beta]|0\rangle + \langle 0|T[J_V^\mu J_A^\alpha J_V^\beta]|0\rangle + \langle 0|T[J_V^\mu J_V^\alpha J_A^\beta]|0\rangle + \langle 0|T[J_A^\mu J_A^\alpha J_A^\beta]|0\rangle$$

Moving the γ_5's around, we find that the calculation reduces to the one of the axial anomaly. The final result is:

$$\partial_\mu \langle J_L^\mu(x)\rangle_{\mathscr{A}} = -\frac{1}{96\pi^2}\left(\sum_{j=1}^{N} Q_j^3\right)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

Gauge invariance is then anomalous unless

$$\sum_{j=1}^{N} Q_j^3 = 0$$

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A similar calculation for a left-handed theory

$$S = \sum_{i=j}^{N} \int d^4x \left[i\overline{\psi}_j \gamma^\mu \partial_\mu \psi_j + \widetilde{Q}_i \overline{\psi}_j \gamma^\mu \left(\frac{1+\gamma_5}{2} \right) \psi_j \mathscr{A}_\mu \right]$$

yields

$$\partial_\mu \langle J_R^\mu(x) \rangle_{\mathscr{A}} = \frac{1}{96\pi^2} \left(\sum_{j=1}^{N} \widetilde{Q}_j^3 \right) \epsilon^{\mu\nu\alpha\beta} \mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta}$$

Finally, for a theory with N_R right-handed and N_L left-handed fermions, the anomaly of the gauge current reads

$$\partial_\mu \langle J^\mu(x) \rangle_{\mathscr{A}} = \frac{1}{96\pi^2} \left(\sum_{j=1}^{N_R} \widetilde{Q}_j^3 - \sum_{j=1}^{N_L} Q_j^3 \right) \epsilon^{\mu\nu\alpha\beta} \mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta}$$

We analyze now the **non-Abelian** case

$$S=\int d^4x\left[i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathcal{L}_\mu\Big)\left(\frac{1-\gamma_5}{2}\right)\psi+i\overline{\psi}\gamma^\mu\Big(\partial_\mu-i\mathcal{R}_\mu\Big)\left(\frac{1+\gamma_5}{2}\right)\psi\right]$$

where we have introduced external gauge fields coupled respectively to the right- and left-handed component of the fermion

$$\mathcal{L}_\mu(x)=\mathcal{L}^a_\mu(x)T^a \qquad\qquad \mathcal{R}_\mu(x)=\mathcal{R}^a_\mu(x)T^a$$

This theory has a $G_L\times G_R$ gauge invariance

$$\psi(x)\longrightarrow e^{i\epsilon^a_L(x)T^a}\left(\frac{1-\gamma_5}{2}\right)\psi(x)+e^{i\epsilon^a_R(x)T^a}\left(\frac{1+\gamma_5}{2}\right)\psi(x)$$

$$\mathcal{L}_\mu(x)\longrightarrow ie^{i\epsilon^a_L(x)T^a}\partial_\mu e^{-i\epsilon^a_L(x)T^a}+e^{i\epsilon^a_L(x)T^a}\mathcal{L}_\mu(x)e^{-i\epsilon^a_L(x)T^a}$$

$$\mathcal{R}_\mu(x)\longrightarrow ie^{i\epsilon^a_R(x)T^a}\partial_\mu e^{-i\epsilon^a_R(x)T^a}+e^{i\epsilon^a_R(x)T^a}\mathcal{R}_\mu(x)e^{-i\epsilon^a_R(x)T^a}$$

Alternatively, we can describe the theory in terms of **vector** and **axial gauge** fields

$$S = \int d^4x \left[i\overline{\psi}\gamma^\mu\Big(\partial_\mu - i\mathcal{V}_\mu - i\mathcal{A}_\mu\gamma_5\Big)\psi\right]$$

where $\mathcal{V}_\mu = \mathcal{V}_\mu^a T^a$ and $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$ are given by

$$\mathcal{V}_\mu = \frac{1}{2}\Big(\mathcal{L}_\mu + \mathcal{R}_\mu\Big) \qquad\qquad \mathcal{A}_\mu = \frac{1}{2}\Big(\mathcal{R}_\mu - \mathcal{L}_\mu\Big)$$

In terms of these fields, we have **vector and axial gauge transformations**

$$\psi(x) \longrightarrow e^{i\alpha^a(x)T^a}\psi(x)$$

$$\mathcal{V}_\mu(x) \longrightarrow ie^{i\alpha^a(x)T^a}\partial_\mu e^{-i\alpha^a(x)T^a} + e^{i\alpha^a(x)T^a}\mathcal{V}_\mu(x)e^{-i\alpha^a(x)T^a}$$

$$\mathcal{A}_\mu(x) \longrightarrow e^{i\alpha^a(x)T^a}\mathcal{A}_\mu(x)e^{-i\alpha^a(x)T^a}$$

$$\psi(x) \longrightarrow e^{i\beta^a(x)T^a\gamma_5}\psi(x)$$

$$\mathcal{V}_\mu(x) \longrightarrow e^{i\beta^a(x)T^a}\mathcal{V}_\mu(x)e^{-i\beta^a(x)T^a}$$

$$\mathcal{A}_\mu(x) \longrightarrow ie^{i\beta^a(x)T^a}\partial_\mu e^{-i\beta^a(x)T^a} + e^{i\beta^a(x)T^a}\mathcal{A}_\mu(x)e^{-i\beta^a(x)T^a}$$

A word of **warning**:

Formally, the theory in terms of axial and vector gauge fields seems to have a gauge symmetry

$$G_V \times G_A$$

However, **non-Abelian axial transformations do not close** so they do not define proper gauge invariance

$$e^{i\beta^a T^a \gamma_5} e^{i\beta'^b T^b \gamma_5} = e^{i(\beta^a + \beta'^a)T^a\gamma_5 + \frac{1}{2}\beta^a\beta'^b[T^a,T^b]+\ldots}$$

The transformations close only in the Abelian case.

Thus, the only *bona fide* gauge fields of the theory are the ones associated with

$$G_L \qquad G_R \qquad G_V$$

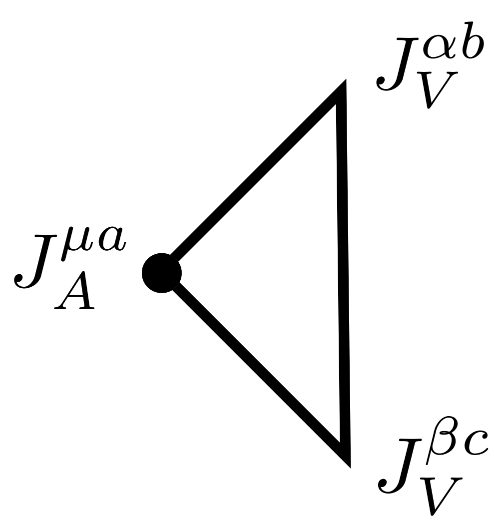
The classical conservation equations for the vector and axial-vector currents are

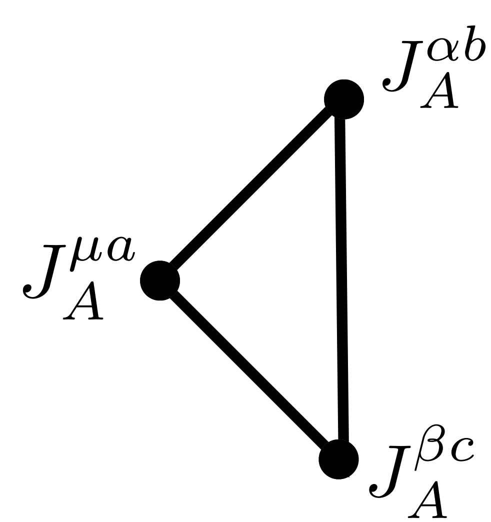
$$(\mathcal{D}_\mu J^\mu_{\rm A})^a = 0 \quad \longrightarrow \quad \begin{aligned} \partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A} = 0 \\ (D_\mu J^\mu_{\rm A})^a + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A} = 0 \end{aligned}$$

To find the anomaly we have to calculate

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{A},\mathcal{V}} = \frac{1}{Z}\int \mathscr{D}\psi \mathscr{D}\overline{\psi}\Big(\partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A}\Big) e^{i\int d^4x [i\overline{\psi}\gamma^\mu(\partial_\mu - i\mathcal{V}_\mu - i\mathcal{A}_\mu\gamma_5)\psi]}$$

Expanding in perturbation theory, the terms with two gauge fields come as usual from the triangle diagram. The parity-violating ones are

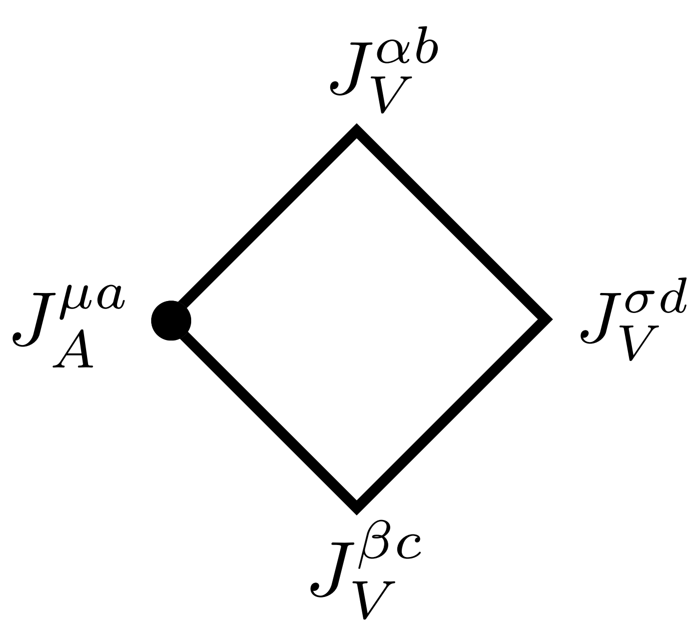


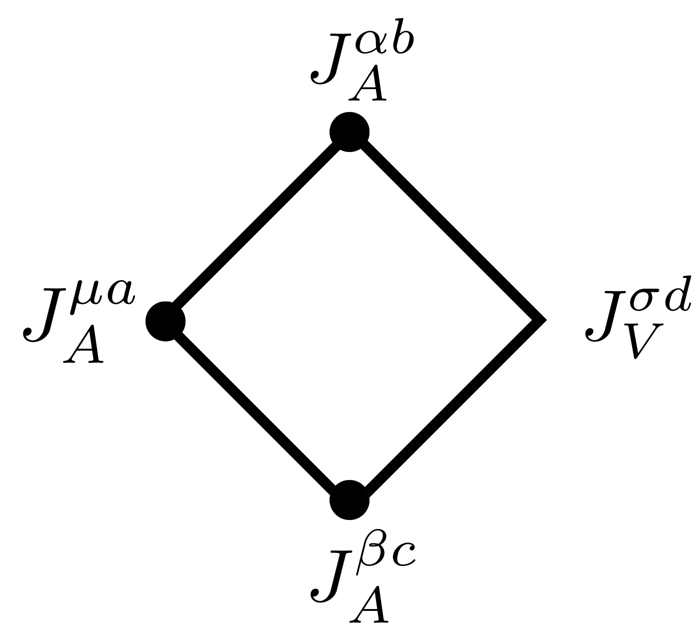


$$\text{Anomaly} = \langle (\partial_\mu J^{\mu a}_{\rm A} + f^{abc}\mathcal{V}^b_\mu J^{\mu c}_{\rm A} + f^{abc}\mathcal{A}^b_\mu J^{\mu c}_{\rm A})\rangle_{\mathcal{V},\mathcal{A}}$$

In the non-Abelian case, there are terms in the triangle with three gauge fields.

Their contribution combines with terms coming from the (logarithmically divergent) box diagrams





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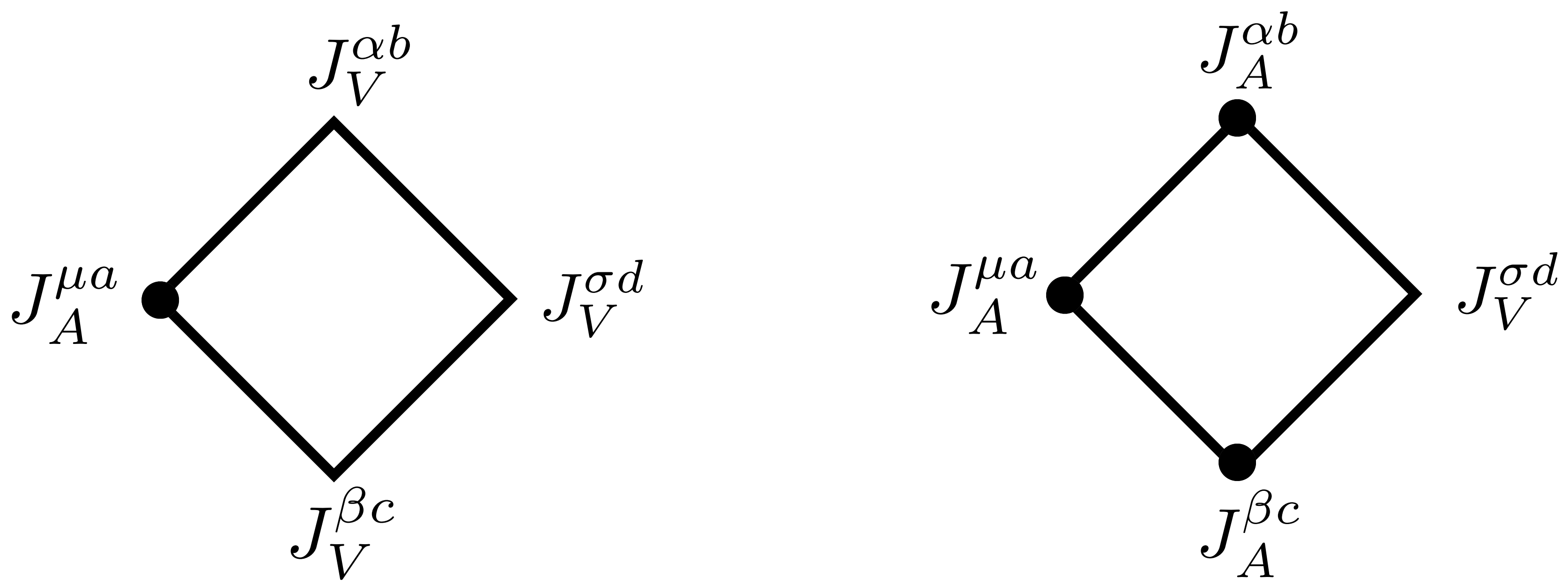
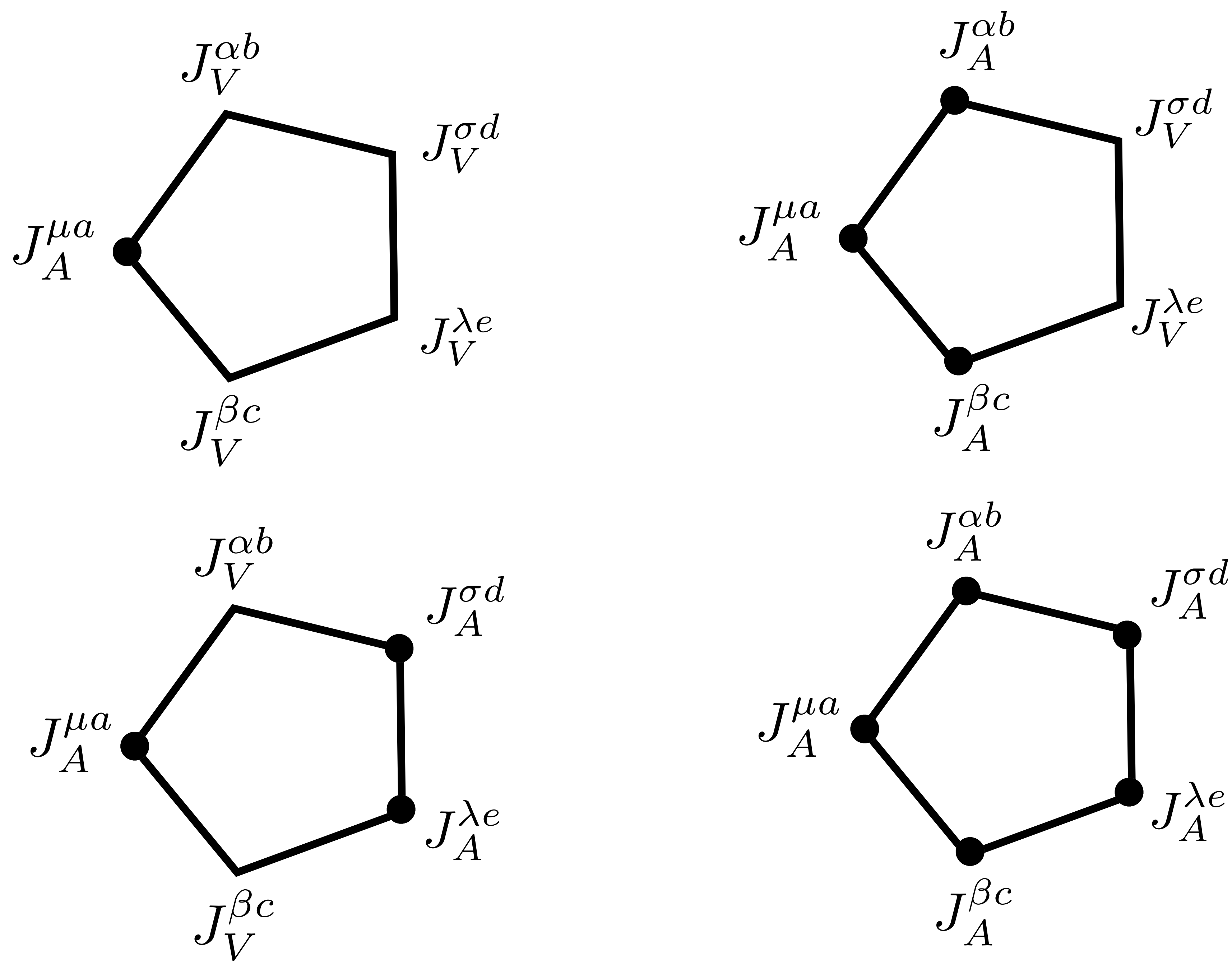


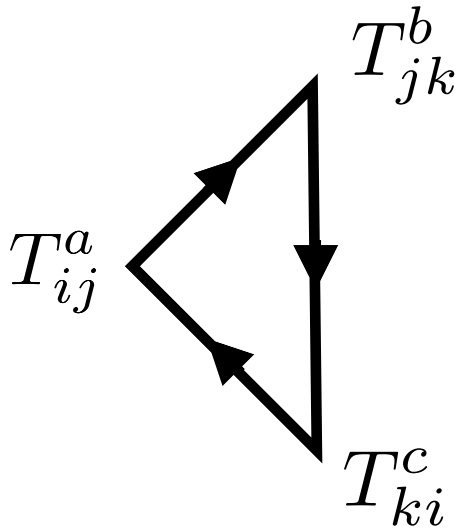
$$\text{Anomaly} = \langle (\partial_\mu J_{\rm A}^{\mu a} + f^{abc}\mathcal{V}_\mu^b J_{\rm A}^{\mu c} + f^{abc}\mathcal{A}_\mu^b J_{\rm A}^{\mu c})\rangle_{\mathcal{V},\mathcal{A}}$$

Finally, there are also contributions to the anomaly from the (UV finite) pentagon diagrams:



What about the group theory factors?

For triangle we have (AVV and AAA):

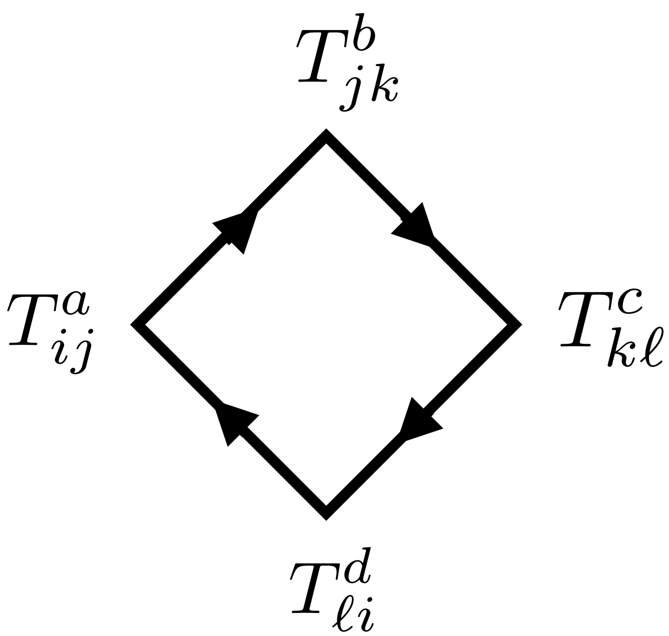


+Bose symmetry

$$\sim \epsilon_{\mu\nu\alpha\beta} p^\mu q^\nu \mathscr{A}^\alpha(p) \mathscr{A}^\beta(q)$$

$$\sim \mathrm{Tr}\left[T^a\{T^b, T^c\}\right]$$

whereas the result for the box is (AVVV and AAAV):



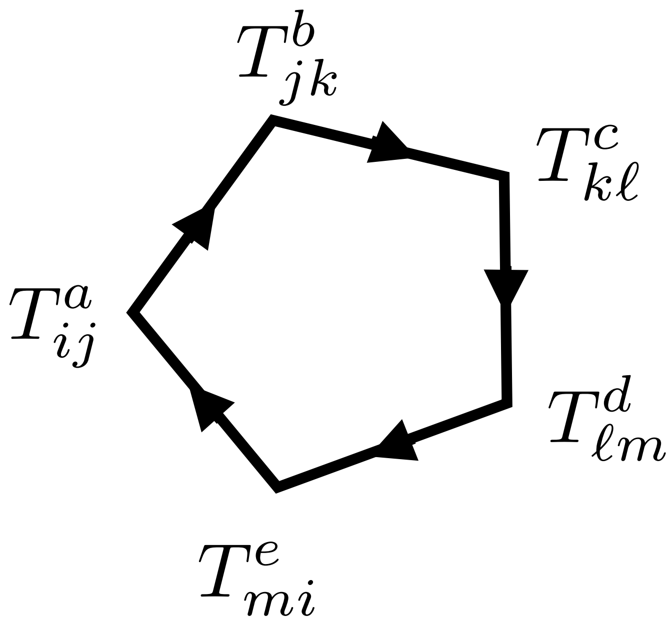
+Bose symmetry

$$\sim \epsilon_{\mu\nu\alpha\beta} p_i^\mu \mathscr{A}^\nu(p_1) \mathscr{A}^\alpha(p_2) \mathscr{A}^\beta(p_3)$$

$$\sim \mathrm{Tr}\left[T^a\{T^b, [T^c, T^d]\}\right]$$

$$= \; i f^{cde} \mathrm{Tr}\left[T^a\{T^b, T^e\}\right]$$

Finally, we deal with the pentagon (AVVVV, AVVAA, and AAAA):



$$\sim \epsilon_{\mu\nu\alpha\beta}\mathscr{A}^\mu(p_1)\mathscr{A}^\nu(p_2)\mathscr{A}^\alpha(p_3)\mathscr{A}^\beta(p_4)$$

+Bose symmetry → $\sim \mathrm{Tr}\left[T^aT^{[b}T^cT^dT^{e]}\right]$

$$\sim f^{r[bc}f^{de]s}\mathrm{Tr}\left[T^a\{T^r,T^s\}\right]$$

- The box and pentagon diagrams only contribute to non-Abelian case.

- The cancellation condition for the triangle diagram

$$\mathrm{Tr}\left[T^a\{T^b,T^c\}\right]=0$$

automatically implies the **cancellation of the box and the pentagon** as well.

Therefore, to cancel the gauge anomaly we only have to care about the triangle!

Computing all these diagrams and imposing vector current conservation

$$\langle(\mathcal{D}_\mu J^\mu_{\rm V})^a\rangle_{\mathcal{V},\mathcal{A}} = 0$$

one arrives at the expression of the **Bardeen anomaly**

$$\langle(\mathcal{D}_\mu J^\mu_{\rm A})^a\rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left\{T^a\left[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta}\right.\right.$$
$$\left.\left. - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right]\right\}$$

where

$$\mathcal{V}_{\mu\nu} = \partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu - i[\mathcal{V}_\mu,\mathcal{V}_\nu] - i[\mathcal{A}_\mu,\mathcal{A}_\nu]$$

$$\mathcal{A}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{V}_\mu,\mathcal{A}_\nu] - i[\mathcal{A}_\mu,\mathcal{V}_\nu]$$

The result is **covariant** under vector gauge transformations (it depends on the vector field strength $\mathcal{V}_{\mu\nu}$ alone).

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,\Big\{T^a\Big[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta} - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\Big]\Big\}$$

This expression can be used as a "master formula" for different situations.

- **QED axial anomaly**: Abelian case, $\mathcal{A}_\mu = 0, \mathcal{V}_\mu = e\mathscr{A}_\mu$

$$\partial_\mu\langle J^\mu_{\rm A}\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

- **Nonabelian singlet anomaly**: $T^a \longrightarrow \mathbb{I}, \mathcal{A}_\mu = 0, \mathcal{V}_\mu = g\mathscr{A}_\mu$

$$\partial_\mu\langle J^\mu_{\rm A}\rangle_{\mathscr{A}} = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,\Big(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\Big)$$

$$\langle (\mathcal{D}_\mu J^\mu_{\rm A})^a \rangle_{\mathcal{V},\mathcal{A}} = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left\{T^a\left[\mathcal{V}_{\mu\nu}\mathcal{V}_{\alpha\beta} + \frac{1}{3}\mathcal{A}_{\mu\nu}\mathcal{A}_{\alpha\beta} - \frac{8}{3}\Big(\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{V}_{\alpha\beta} + \mathcal{A}_\mu\mathcal{V}_{\nu\alpha}\mathcal{A}_\beta + \mathcal{V}_{\mu\nu}\mathcal{A}_\alpha\mathcal{A}_\beta\Big) + \frac{32}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\alpha\mathcal{A}_\beta\right]\right\}$$

This expression can be used as a "master formula" for different situations.

- **"Right-handed" QED**: Abelian case, $T^a \longrightarrow Q$

$$\mathcal{A}_\mu = -\mathcal{V}_\mu = -\frac{1}{2}\mathcal{L}_\mu$$

$$J^\mu_L = \frac{1}{2}\Big(J^\mu_{\rm V} - J^\mu_{\rm A}\Big)$$

$$\longrightarrow \left\{\begin{array}{l} \mathcal{V}_\mu = \dfrac{Q}{2}\mathscr{A}_\mu \\ \mathcal{A}_\mu = -\dfrac{Q}{2}\mathscr{A}_\mu \end{array}\right.$$

$$\Downarrow$$

$$\partial_\mu\langle J^\mu_L(x)\rangle_{\mathscr{A}} = -\frac{1}{96\pi^2}\left(\sum_{j=1}^N Q_j^3\right)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

We have seen that a theory of a chiral fermion is free of gauge anomalies whenever they transform in a representation $\mathbf{R}$ satisfying

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]=0$$

Let us do some group theory…

A Lie algebra representation is **real** or **pseudoreal** if there is an intertwining operator S satisfying

$$T^{a\,*}_{\mathbf{R}}=-ST^a_{\mathbf{R}}S^{-1} \qquad \begin{cases} S^T=S & \text{real} \\ S^T=-S & \text{pseudoreal} \end{cases}$$

Then

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]=\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]^T=\mathrm{Tr}\left[(T^a_{\mathbf{R}})^*\{(T^b_{\mathbf{R}})^*,(T^c_{\mathbf{R}})^*\}\right]$$

and for **real** and **pseudoreal** representations

$$\mathrm{Tr}\left[(T^a_{\mathbf{R}})^*\{(T^b_{\mathbf{R}})^*,(T^c_{\mathbf{R}})^*\}\right]=-\mathrm{Tr}\left[ST^a_{\mathbf{R}}S^{-1}\{ST^b_{\mathbf{R}}S^{-1},ST^c_{\mathbf{R}}S^{-1}\}\right]=-\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]$$

Thus, real and pseudoreal are anomaly-free representations

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\}\right]=0 \qquad \text{for } \mathbf{R} \text{ real or pseudoreal}$$

This happens for **all** representations of the following groups

- SU(2)
- SO(2N+1)
- SO(4N) for $N \geq 2$
- Sp(2N) for $N \geq 3$
- and the exceptional groups $G_2,\ F_4,\ E_7,\ E_8$

Other groups whose representations are neither real or pseudoreal but are still **safe** are

- SO(4N+2) for $N \geq 3$
- E_6

In addition, the **adjoint** representation of any group is real and therefore **safe**.

Potentially dangerous Lie group are

- U(1).
- SU(N) for $N \geq 3$.

In the case of non-safe groups, anomalies can also be cancelled summing among all chiral fields.

For example, if a theory contains a number of right- and left-handed fermions transforming in representations T^a_+ and T^a_- the anomaly cancellation condition reads:

$$\sum_{\text{right-handed}} \mathrm{Tr}\left[T^a_+\{T^b_+, T^c_+\}\right] - \sum_{\text{left-handed}} \mathrm{Tr}\left[T^a_-\{T^b_-, T^c_-\}\right] = 0$$

If the gauge group is a direct product, $G_1 \otimes \ldots \otimes G_n$, there might be **mixed gauge anomalies** associated with triangles with "different group factors" at each vertex