Introduction to Anomalies in QFT

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The axial anomaly

The symmetries of QED: a reminder

The QED action

$$S_{\mathrm{QED}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi} A\!\!\!/ \psi \right]$$

is invariant under global U(1)$_V$ transformations of the fermion field

$$\psi(x) \longrightarrow e^{i\alpha}\psi(x), \qquad \overline{\psi}(x) \longrightarrow e^{-i\alpha}\overline{\psi}(x), \qquad \text{with} \qquad \alpha \in \mathbb{R}$$

leading to the conservation equation

$$J^\mu_{\mathrm{V}} = \overline{\psi}\gamma^\mu\psi \qquad \Longrightarrow \qquad \partial_\mu J^\mu_{\mathrm{V}} = 0$$

This symmetry can be promoted to U(1) gauge invariance

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x), \qquad A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu\alpha(x)$$

We can also allow a second type of axial global transformations of the fermion field:

$$\psi(x) \longrightarrow e^{i\beta\gamma_5}\psi(x), \quad \overline{\psi}(x) \longrightarrow \overline{\psi}(x)e^{i\beta\gamma_5}, \qquad \text{with} \qquad \beta \in \mathbb{R}$$

where

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$$

This is not a symmetry of the action, due to the mass term. If we define the axial-vector current

$$J^\mu_{\rm A} = \overline{\psi}\gamma^\mu\gamma_5\psi$$

it satisfies

$$\partial_\mu J^\mu_A = 2im\overline{\psi}\gamma_5\psi.$$

(pseudovector-pseudoscalar equivalence)

Axial global symmetry is recovered in the massless limit $m \longrightarrow 0$

At the level of the amplitudes, conservations equations give rise to **Ward identities**.

In the case of QED, a general amplitude in momentum space has the structure

$$\begin{aligned}\mathcal{A}(p_1,\ldots,p_n;q_1,\ldots,q_m) &= \varepsilon_{\mu_1}(p_1)\ldots\varepsilon_{\mu_n}(p_n)\varepsilon_{\nu_1}(q_1)^*\ldots\varepsilon_{\nu_n}(q_m)^* \\ &\times \Gamma^{\mu_1\ldots\mu_n\nu_1\ldots\nu_m}(p_1,\ldots,p_n;q_1,\ldots,q_m)\end{aligned}$$

Invariance under gauge transformations

$$\varepsilon_\mu(p)\longrightarrow\varepsilon_\mu(p)+\lambda p_\mu$$

leads to the **gauge Ward identity**

$$p_{\mu_i}\Gamma^{\ldots\mu_i\ldots\nu_1\ldots\nu_m}(p_k;q_\ell)=0=q_{\nu_i}\Gamma^{\mu_1\ldots\mu_m\ldots\nu_i\ldots}(p_k;q_\ell).$$

Or more generally, $\langle\partial_\mu J^\mu_{\rm V}(y)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle=0$ with $\mathcal{O}_i(x)$ gauge invariant operators.

What about the axial-vector current?

We study a Dirac fermion coupled to an external gauge field $\mathscr{A}_\mu(x)$

$$S_{\rm int} = -e\int d^4x\, J^\mu_{\rm V}(x)\mathscr{A}_\mu(x)$$

and compute

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{\displaystyle\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\; J^\mu_{\rm A}(x)e^{i\int d^4x\left[(i\partial\!\!\!/ -m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}{\displaystyle\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\; e^{i\int d^4x\left[\overline{\psi}(i\partial\!\!\!/ -m)\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right]}}.$$

Expanding in perturbation theory in the coupling constant,

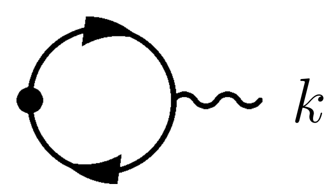
$$\begin{aligned}\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} &= -ie\int d^4y\,\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y)]|0\rangle\mathscr{A}_\alpha(y) \\ &- \frac{e^2}{2}\int d^4y_1 d^4y_2\,\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y_1)J^\beta_{\rm V}(y_2)]|0\rangle\mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2) + \ldots\end{aligned}$$

We look at the first term

$$e\langle 0|T[J_{\rm A}^{\mu}(0)J_{\rm V}^{\alpha}(y-x)]|0\rangle = \int \frac{d^4p}{(2\pi)^4}\Gamma^{\mu\alpha}(k)e^{ik\cdot(x-y)}$$

and diagrammatically:

$$i\Gamma^{\mu\nu}(k) \quad = $$



$$= \quad e\int \frac{d^4\ell}{(2\pi)^4}\text{Tr}\left(\gamma^{\mu}\gamma_5\frac{i}{\not{\ell}-m+i\epsilon}\gamma^{\nu}\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\right)$$

Our aim is to compute its contribution to the axial-vector Ward identity

$$\langle\partial_{\mu}J_{\rm A}^{\mu}(x)\rangle_{\mathscr{A}} = \,? \quad \longrightarrow \quad k_{\mu}i\Gamma^{\mu\nu}(k) = \,?$$

To compute the integral

$$i\Gamma^{\mu\nu}(k) \;\;=\;\; e\int \frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{\ell}-m+i\epsilon}\gamma^\nu\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\right)$$

we use some Diracology

$$\mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu)=\mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha)=0, \quad \mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)=-4i\epsilon^{\mu\nu\alpha\beta}$$

to find

$$i\Gamma^{\mu\nu}(k) \;\;=\;\; -4ie\,\epsilon^{\mu\alpha\nu\beta}k_\beta\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell_\alpha}{(\ell^2-m^2+i\epsilon)[(\ell-k)^2-m^2+i\epsilon]}.$$

Due to the antisymmetry of $\epsilon_{\mu\nu\alpha\beta}$ the amplitude satisfy both the vector and axial-vector Ward identities

$$k_\mu i\Gamma^{\mu\nu}(k)=0=k_\nu i\Gamma^{\mu\nu}(k).$$

Moreover, by Lorentz invariance $i\Gamma^{\mu\nu}=0$

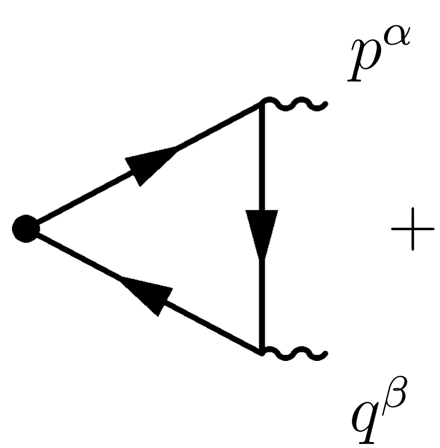
To find any anomaly, we have to go to the next order. Going to momentum space

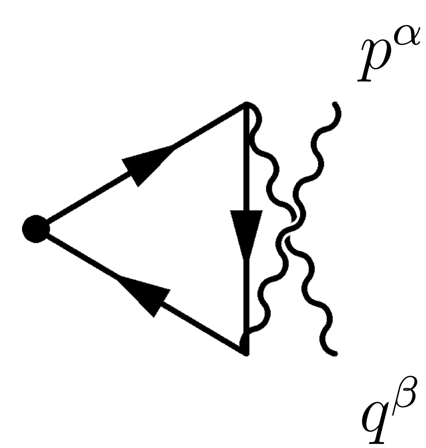
$$e^2\langle 0|T[J^\mu_{\rm A}(0)J^\alpha_{\rm V}(x_1)J^\beta_{\rm V}(x_2)]|0\rangle = \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}i\Gamma^{\mu\alpha\beta}(p,q)e^{ip\cdot x_1+iq\cdot x_2},$$

the conservation equation is

$$\begin{aligned}\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} &= \frac{i}{2}\int d^4y_1 d^4y_2 \mathscr{A}^\alpha(y_1)\mathscr{A}^\beta(y_2) \\ &\times \int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q)e^{ip\cdot(y_1-x)+iq\cdot(y_2-x)}.\end{aligned}$$

The calculation involves now two **triangle diagrams**:

$i\Gamma_{\mu\alpha\beta}(p,q) = (p+q)^\mu$ 

$+\ (p+q)^\mu$ 

Applying the Feynman rules of QED, we have

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha\right)\\
&+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).
\end{aligned}
$$

so we only need to compute the integrals…

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&+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).
\end{aligned}
$$

so we only need to compute the integrals…

These integrals are linearly divergent!!

Interlude: linearly divergent integrals

Let us begin with the simplest one-dimensional case:

$$I(\xi)=\int_{-\infty}^{\infty}dx\Big[f(x+\xi)-f(x)\Big].$$

If the function $f(x)$ is integrable on $\mathbb{R}$ we conclude that $I(\xi)=0$.

Let us however assume that for large $|x|$

$$f(x)\sim\frac{1}{x}$$ (logarithmically divergent integral)

$$f(x)\sim\text{constant}$$ (linearly divergent integral)

expanding the integrand around x

$$I(\xi)=\int_{-\infty}^{\infty}dx\left[f'(x)\xi+\frac{1}{2}f''(x)\xi^2+\ldots\right]$$

we arrive at: $$I(\xi)=\xi\int_{-\infty}^{\infty}dx\,f'(x)=\xi\Big[f(\infty)-f(-\infty)\Big].$$

Thus, for linearly divergent integrals:

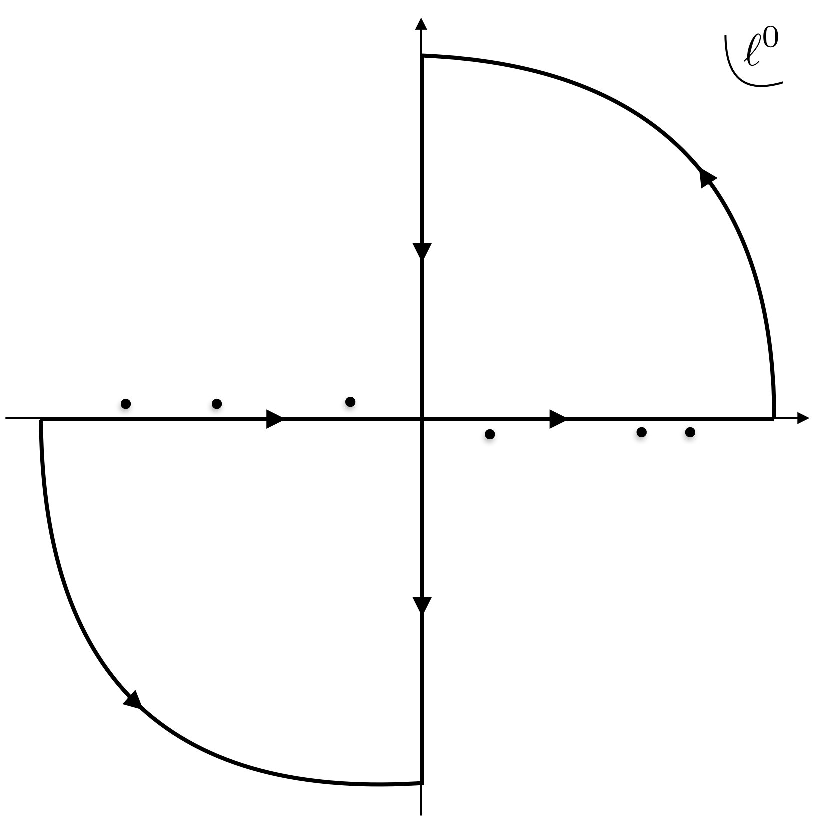
$$\int_{-\infty}^{\infty} dx\Big[f(x+\xi)-f(x)\Big] = f(\infty)-f(-\infty)\neq 0.$$

Shifting the integration variable changes the value of a linearly divergent integral!

Something similar happens in four dimensions

$$I_4^{\mu}(\xi) = \int \frac{d^4\ell}{(2\pi)^4}\Big[f^{\mu}(\ell+\xi)-f^{\mu}(\ell)\Big].$$

To make sense of the integral, we perform a Wick rotation into Euclidean space



$$I_4^{\mu}(\xi) = i\int \frac{d^4\ell_E}{(2\pi)^4}\Big[f^{\mu}(\ell_E+\xi)-f^{\mu}(\ell_E)\Big].$$

If the integral is linearly divergent

$$f^\mu(\ell_E) \sim C\frac{\ell_E^\mu}{\ell_E^4} \qquad \textbf{as} \qquad |\ell_E| \longrightarrow \infty$$

Expanding the integrand

$$I_4^\mu(\xi) = i\int \frac{d^4\ell_E}{(2\pi)^4}\Big[f^\mu(\ell_E+\xi) - f^\mu(\ell_E)\Big]$$

$$= i\int \frac{d^4\ell_E}{(2\pi)^4}\left[\xi^\alpha \left.\frac{\partial f^\mu}{\partial \ell_E^\alpha}\right|_{\xi=0} + \frac{1}{2}\xi^\alpha\xi^\beta \left.\frac{\partial^2 f^\mu}{\partial\ell_E^\alpha \partial\ell_E^\beta}\right|_{\xi=0} + \ldots\right]$$

Again, only the first term contributes. Applying Gauß' theorem

$$I_4^\mu(\xi) \;=\; \frac{i}{16\pi^4}\int_{S^3_\infty} d\Sigma_\alpha \xi^\alpha f^\mu(\ell_E) = \frac{iC}{16\pi^4}\xi_\alpha \int d\Omega_3 \frac{\ell_E^\mu \ell_E^\alpha}{\ell_E^2}$$

The remaining integral can be done using asymptotic rotational invariance

$$\int d\Omega_3 \frac{\ell_E^\mu \ell_E^\alpha}{\ell_E^2} = \frac{1}{4}\delta^{\mu\alpha}\mathrm{Vol}(S^3) = \frac{\pi^2}{2}\delta^{\mu\alpha}$$

With this, we got

$$\int \frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi) - f^\mu(\ell)\Big] = \frac{iC}{32\pi^2}\xi^\mu.$$

Very important: remember the origin of the constant C

$$f^\mu(\ell_E) \sim C\frac{\ell_E^\mu}{\ell_E^4} \qquad \text{as} \qquad |\ell_E| \longrightarrow \infty$$

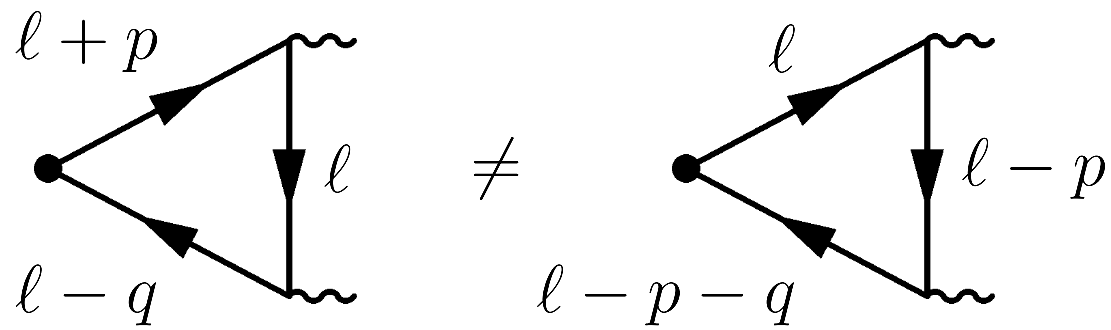
Thus, the ambiguity only depends on the large momentum behavior of the integrand (i.e., it doesn't depend on the masses of the particles running in the loop!)

Back to the axial anomaly…

Applying the Feynman rules of QED, we have

$$i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right) + \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).$$

What is the relevance of the previous discussion?



The value of the triangle diagram depends on how we parametrize the loop momentum!

How do we deal with this ambiguity?

The amplitude $i\Gamma_{\mu\alpha\beta}(p,q)$ should satisfy a number of condition:

- **Parity**: begin parity odd, it should contain an $\epsilon_{\mu\nu\alpha\beta}$ tensor
- **Poincaré invariance**: it should be a rank-three tensor depending only on p and q

This forces the following structure for the amplitude

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\ &+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda \\ &+ f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda + f_7\epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda + f_8\epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda \end{aligned}$$

where f_i are scalar functions of the momenta. Moreover, using

$$\epsilon_{\alpha\beta\sigma\lambda}w_\mu + \epsilon_{\beta\sigma\lambda\mu}w_\alpha + \epsilon_{\sigma\lambda\mu\alpha}w_\beta + \epsilon_{\lambda\mu\alpha\beta}w_\sigma + \epsilon_{\mu\alpha\beta\sigma}w_\lambda = 0,$$

we can absorb f_7 and f_8 into the other f's.

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\
&+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda + f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda
\end{aligned}
$$

- **Bose symmetry**: it should satisfy $i\Gamma_{\mu\alpha\beta}(p,q) = i\Gamma_{\mu\beta\alpha}(q,p)$

This imposes the following conditions on the coefficients

$$
f_1(p,q) = -f_2(q,p), \qquad f_3(p,q) = -f_6(q,p), \qquad f_4(p,q) = -f_5(q,p).
$$

Let's do a bit of dimensional analysis:

$$
[i\Gamma_{\mu\alpha\beta}] = E \quad \longrightarrow \quad \begin{cases} [f_1] = [f_2] = E^0 \\ [f_3] = \ldots = [f_6] = E^{-2} \end{cases}
$$

Hence f_1 and f_2 are **logarithmically divergent** integrals while f_3 ,…, f_6 are **convergent integrals.**

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^\sigma + f_2\epsilon_{\mu\alpha\beta\sigma}q^\sigma + f_3\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda \\
&+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda + f_5\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda + f_6\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda
\end{aligned}
$$

All **ambiguities** in the amplitude are confined to the coefficients f_1 and f_2.

Next we look at the contractions

$$
\begin{aligned}
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(f_2 - p^2 f_5 - p\cdot q f_6\Big)\epsilon_{\mu\beta\alpha\sigma}q^\alpha p^\sigma, \\
q^\beta i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(f_1 - q^2 f_4 - p\cdot q f_3\Big)\epsilon_{\mu\alpha\beta\sigma}q^\beta p^\sigma, \\
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= \Big(-f_1 + f_2\Big)\epsilon_{\alpha\beta\sigma\lambda}q^\sigma p^\lambda.
\end{aligned}
$$

Imposing the vector (gauge) Ward identities

$$
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = 0 = q^\beta i\Gamma_{\mu\alpha\beta}(p,q)
$$

completely fixes the ambiguous integrals in terms of finite ones

$$
f_1(p,q) = q^2 f_4(p,q) - p\cdot q f_3(p,q)
$$

$$
f_2(p,q) = p^2 f_5(p,q) - p\cdot q f_6(p,q)
$$

Using these identities, the axial anomaly is given by

$$(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) = \Big[p^2 f_5 - q^2 f_4 + p\cdot q(-f_3+f_6)\Big]\epsilon_{\alpha\beta\sigma\lambda}q^{\sigma}p^{\lambda}$$

With these general considerations we learn a number of things:

- All **ambiguities** in the triangle diagram are codified in **logarithmically divergent integrals**.

- These are **completely fixed** by requiring the **conservation of the gauge current**.

- Once this is done, the **axial anomaly** is given by **finite** integrals (i.e., free of UV ambiguities).

The (actual) calculation

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) &= e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\frac{i}{\not{\ell}-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right) \\
&+ \left(\begin{array}{c} p\leftrightarrow q \\ \alpha\leftrightarrow\beta\end{array}\right).
\end{aligned}
$$

We start with the computation of

$$
I_{\alpha\beta}(\ell,p,q) = \mathrm{Tr}\,\left[\frac{i}{\not{\ell}-m+i\epsilon}(\not{p}+\not{q})\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}\gamma_\beta\frac{i}{\not{\ell}-\not{p}-m+i\epsilon}\gamma_\alpha\right].
$$

To reduce the expression, we use the trivial identity

$$
\not{p}+\not{q} = (\not{\ell}-m)-(\not{\ell}-\not{p}-\not{q}+m)+2m
$$

and write

$$
\begin{aligned}
\frac{i}{\not{\ell}-m+i\epsilon}(\not{p}+\not{q})\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m} &= i\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}+i\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5 \\
&+ 2m\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{p}-\not{q}-m+i\epsilon}
\end{aligned}
$$

The integrand takes the form

$$\begin{aligned} I_{\alpha\beta}(\ell,p,q) &= i\mathrm{Tr}\left(\gamma_5 \frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta \frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha\right) \\ &- i\mathrm{Tr}\left(\gamma_5 \frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha \frac{i}{\not\ell-m+i\epsilon}\gamma_\beta\right) \\ &+ 2m\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_5 \frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta \frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha\right) \end{aligned}$$

and integrate the result over the loop momentum

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\alpha\beta}(\ell,p,q) + e^2\int\frac{d^4\ell}{(2\pi)^4}I_{\beta\alpha}(\ell,q,p).$$

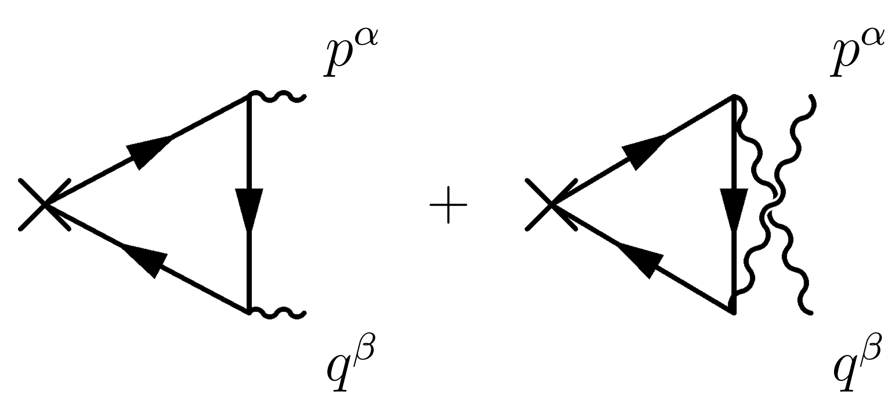
The integrand takes the form

$$\begin{aligned} I_{\alpha\beta}(\ell,p,q) &= i\mathrm{Tr}\left(\gamma_5 \frac{i}{\not\ell - \not p - \not q - m + i\epsilon}\gamma_\beta \frac{i}{\not\ell - \not p - m + i\epsilon}\gamma_\alpha\right) \\ &- i\mathrm{Tr}\left(\gamma_5 \frac{i}{\not\ell - \not p - m + i\epsilon}\gamma_\alpha \frac{i}{\not\ell - m + i\epsilon}\gamma_\beta\right) \\ &+ 2m\mathrm{Tr}\left(\frac{i}{\not\ell - m + i\epsilon}\gamma_5 \frac{i}{\not\ell - \not p - \not q - m + i\epsilon}\gamma_\beta \frac{i}{\not\ell - \not p - m + i\epsilon}\gamma_\alpha\right) \end{aligned}$$

and integrate the result over the loop momentum

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = e^2 \int \frac{d^4\ell}{(2\pi)^4} I_{\alpha\beta}(\ell,p,q) + e^2 \int \frac{d^4\ell}{(2\pi)^4} I_{\beta\alpha}(\ell,q,p).$$

The last term is the one-loop contribution to $2im\langle\overline{\psi}\gamma_5\gamma\rangle_{\mathscr{A}}$

$i\Gamma_{\mu\nu}(p,q) \equiv$ 

$\times \equiv 2m\gamma_5$

$$
\begin{aligned}
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha\right.\\
&\quad\left.-\ \gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\gamma_\alpha\right)\\
&- ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\gamma_5\frac{i}{\not\ell-\not p-m+i\epsilon}\gamma_\alpha\frac{i}{\not\ell-m+i\epsilon}\gamma_\beta\right.\\
&\quad\left.-\ \gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\alpha\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)+2mi\Gamma_{\alpha\beta}(p,q)
\end{aligned}
$$

and reducing the propagators:

$$
\begin{aligned}
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-p)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-p)^2-m^2+i\epsilon]}\right.\\
&\quad\left.-\ \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-q)^\sigma\ell^\nu}{[(\ell-q)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right\}\\
&+\ 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p)^\sigma\ell^\nu}{[(\ell-p)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right.\\
&\quad\left.-\ \frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-q)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-q)^2-m^2+i\epsilon]}\right\}+2mi\Gamma_{\alpha\beta}(p,q).
\end{aligned}
$$

$$
\begin{aligned}
(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) &= 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-p)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-p)^2-m^2+i\epsilon]}\right. \\
&\left.-\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-q)^\sigma\ell^\nu}{[(\ell-q)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right\} \\
&+ 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p)^\sigma\ell^\nu}{[(\ell-p)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right. \\
&\left.-\frac{\epsilon_{\alpha\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-q)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-q)^2-m^2+i\epsilon]}\right\}+2mi\Gamma_{\alpha\beta}(p,q).
\end{aligned}
$$

The two integrals are linearly divergent and have the structure

$$
\int\frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi)-f^\mu(\ell)\Big]=\frac{iC}{32\pi^2}\xi^\mu.
$$

with

$$
\xi^\mu = -p^\mu
$$

$$
\xi^\mu = q^\mu
$$

respectively.

We find the result

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = \frac{ie^2}{4\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma_{\alpha\beta}(p,q).$$

The axial Ward identity is violated in the limit $m \to 0$.

↓

The axial-vector symmetry is anomalous!

But not so fast… what happens with the vector current?

$$\begin{aligned} p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell - m + i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\not p\right) \\ &+ e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell - m + i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\not p\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)\end{aligned}$$

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\not p\right)$$

$$+\ e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\not p\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)$$

Using the identities

$$\not p = (\not\ell-m)-(\not\ell-\not p-m), \qquad \not p = -(\not\ell-\not p-\not q-m)+(\not\ell-\not q-m)$$

we have

$$\begin{aligned}p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\right.\\ &\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right)\\ &-\ ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right.\\ &\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right).\end{aligned}$$

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\not p\right)$$

$$+\ e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-m+i\epsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\not p\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\right)$$

Using the identities

$$\not p = (\not\ell-m)-(\not\ell-\not p-m), \qquad \not p = -(\not\ell-\not p-\not q-m)+(\not\ell-\not q-m)$$

we have

$$\begin{aligned}p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\right.\\ &\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right)\\ &-\ ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right.\\ &\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right).\end{aligned}$$

$= 0$

(no shift required)

The remaining integral

$$
\begin{aligned}
p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) &= ie^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\,\left(\gamma_\mu\gamma_5\frac{i}{\not\ell-\not p-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-\not p-m+i\epsilon}\right.\\
&\quad\left.-\ \gamma_\mu\gamma_5\frac{i}{\not\ell-\not q-m+i\epsilon}\gamma_\beta\frac{i}{\not\ell-m+i\epsilon}\right)\\
&= 4e^2\int\frac{d^4\ell}{(2\pi)^4}\left\{\frac{\epsilon_{\mu\beta\sigma\nu}(\ell-p-q)^\sigma(\ell-p)^\nu}{[(\ell-p-q)^2-m^2+i\epsilon][(\ell-p)^2-m^2+i\epsilon]}\right.\\
&\quad\left.-\ \frac{\epsilon_{\mu\beta\sigma\nu}(\ell-q)^\sigma\ell^\nu}{[(\ell-q)^2-m^2+i\epsilon](\ell^2-m^2+i\epsilon)}\right\}
\end{aligned}
$$

has again the structure

$$
\int\frac{d^4\ell}{(2\pi)^4}\Big[f^\mu(\ell+\xi)-f^\mu(\ell)\Big]=\frac{iC}{32\pi^2}\xi^\mu.
$$

with

$$
\xi^\mu=-p^\mu
$$

The computation shows that the gauge Ward identity is violated!

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) = -\frac{ie^2}{8\pi^2}\epsilon_{\mu\beta\sigma\nu}p^\sigma q^\nu$$

But remember the ambiguity in parametrizing the loop momentum. It seems we made the wrong choice…

Changing the parametrization

$$\ell^\mu \longrightarrow \ell^\mu + \alpha p^\mu + \beta q^\mu$$

introduces a change in the amplitude

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \Delta_{\mu\alpha\beta}(\alpha,\beta)$$

Can we select α and β so the vector Ward identity is enforced?

Luckily, we don't have to redo the whole computation! Imposing:

- **Parity**
- **Lorentz invariance**
- **Bose symmetry**

and remembering that the ambiguity **does not depend on masses**, we only have one possibility for the change in the amplitude

$$i\Gamma_{\mu\alpha\beta}(p,q) \longrightarrow i\Gamma_{\mu\alpha\beta}(p,q) + \frac{ie^2}{8\pi^2} a\epsilon_{\mu\alpha\beta\sigma}(p-q)^\sigma$$

where $a = a(\alpha, \beta)$

Using now our results for the triangle diagrams

$$
\begin{aligned}
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= \frac{ie^2}{4\pi^2}(1-a)\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu} + 2mi\Gamma_{\alpha\beta}(p,q), \\
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= -\frac{ie^2}{8\pi^2}(1+a)\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu}.
\end{aligned}
$$

Thus, the **physically correct choice** is to take $a=-1$ for which

$$
\begin{aligned}
p^{\alpha} i\Gamma_{\mu\alpha\beta}(p,q) &= 0, \\
(p+q)^{\mu} i\Gamma_{\mu\alpha\beta}(p,q) &= \frac{ie^2}{2\pi^2}\epsilon_{\alpha\beta\sigma\nu}p^{\sigma}q^{\nu} + 2mi\Gamma_{\alpha\beta}(p,q)
\end{aligned}
$$

The axial-vector current is anomalous!

It is important that there is **no value** of a for which **both** Ward identities are satisfied **simultaneously**.

In our calculation we did not commit to any particular regularization method (in fact, we didn't have to), only to the preservation of gauge invariance.

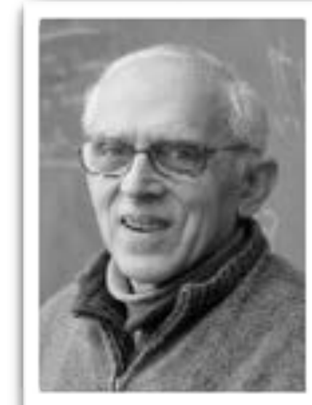
The result for the axial anomaly is obtained computing the triangle diagram using any regularization method that preserves gauge invariance: e.g.

- Pauli-Villars (see Bertlmann)
- Dimensional regularization, but beware of γ_5 (see Peskin & Schroeder)
- Point-splitting
- Dispersion relations (see Bertlmann)

⋮

Transforming the result back to position space,

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{i}{2}\int d^4y_1 d^4y_2 \mathscr{A}^\alpha(y_1)\mathscr{A}^\beta(y_2)$$
$$\times \int \frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q)e^{ip\cdot(y_1-x)+iq\cdot(y_2-x)}.$$



Steven Adler
(b. 1939)

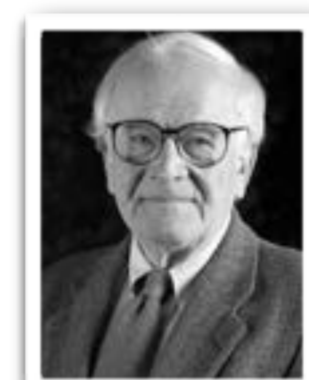
we arrive at the celebrated **Adler-Bell-Jackiw anomaly**

Jack Steinberger
(b. 1921)

Julian Schwinger
(1918-1994)

$$\partial_\mu \langle J^\mu_A(x)\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta} + 2im\langle\overline{\psi}(x)\gamma_5\psi(x)\rangle_{\mathscr{A}}$$

John S. Bell
(1928-1990)



Roman Jackiw
(b. 1939)

An example of the “wrong” regularization

Our analysis of the ambiguities in the triangle diagram might seem a bit formal…

The ambiguity, however, can be reobtained using a point-splitting regularization of the axial-vector current composite operator.

$$J^\mu_{\rm A}(x)_{\rm reg} = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma_\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right]$$

where $\alpha\in\mathbb{R}$ and ϵ^μ satisfies $\epsilon^0>0$

Under a gauge transformation

$$\left.\begin{aligned}\psi\left(x-\frac{\epsilon}{2}\right) &\longrightarrow e^{i\alpha\left(x-\frac{\epsilon}{2}\right)}\psi\left(x-\frac{\epsilon}{2}\right)\\ \overline{\psi}\left(x+\frac{\epsilon}{2}\right) &\longrightarrow e^{-i\alpha\left(x+\frac{\epsilon}{2}\right)}\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\\ \mathscr{A}_\mu(y) &\longrightarrow \mathscr{A}_\mu(y)+\frac{1}{e}\partial_\mu\epsilon(y)\end{aligned}\right\} \quad J^\mu_{\rm A}(x)_{\rm reg}\longrightarrow e^{i(a-1)\left[\alpha\left(x+\frac{\epsilon}{2}\right)-\alpha\left(x-\frac{\epsilon}{2}\right)\right]}J^\mu_{\rm A}(x)_{\rm reg}$$

The regularization is gauge invariant only for $a=1$

$$J^\mu_{\rm A}(x)_{\rm reg} = \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

We compute now its divergence

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = \partial_\mu\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

$$+\,\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\partial_\mu\psi\left(x-\frac{\epsilon}{2}\right)\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

$$+\,iea\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\left[\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

$$\times\,\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

and use the fermion EOM

$$i\gamma^\mu\partial_\mu\psi = m\psi - e\mathscr{A}_\mu\gamma^\mu\psi \qquad\qquad -i\partial_\mu\overline{\psi}\gamma^\mu = m\overline{\psi} - e\overline{\psi}\gamma^\mu\mathscr{A}_\mu$$

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - ie\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)$$

$$\times\ \left[\mathscr{A}_\mu\left(x+\frac{\epsilon}{2}\right) - \mathscr{A}_\mu\left(x-\frac{\epsilon}{2}\right) - \partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2} dy^\alpha \mathscr{A}_\alpha(y)\right]$$

$$\times\ \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_\alpha \mathscr{A}^\alpha(y)\right]$$

Identifying $J^\mu_{\rm A}(x)_{\rm reg}$ and expanding to first order in ϵ^μ we have

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - iJ^\mu_{\rm A}(x)\epsilon^\alpha\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

and now compute its vacuum expectation value

$$\partial_\mu\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - ie\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^\mu\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)$$

$$\times\left[\mathscr{A}_\mu\left(x+\frac{\epsilon}{2}\right)-\mathscr{A}_\mu\left(x-\frac{\epsilon}{2}\right)-\partial_\mu\int_{x-\epsilon/2}^{x+\epsilon/2}dy^\alpha\mathscr{A}_\alpha(y)\right]$$

$$\times\ \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

Identifying $J^\mu_{\rm A}(x)_{\rm reg}$ and expanding to first order in ϵ^μ we have

$$\partial_\mu J^\mu_{\rm A}(x)_{\rm reg} = 2im[\overline{\psi}\gamma_5\psi]_{\rm reg} - iJ^\mu_{\rm A}(x)\epsilon^\alpha\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

and now compute its vacuum expectation value

$$\partial_\mu\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

Next, we evaluate $\langle J_{\mathrm{A}}^{\mu}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}$

$$\langle J_{\mathrm{A}}^{\mu}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}} = \left\langle \overline{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^{\mu}\gamma_5\psi\left(x-\frac{\epsilon}{2}\right)\right\rangle_{\mathscr{A}} \exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2} dy_{\alpha}\mathscr{A}^{\alpha}(y)\right]$$

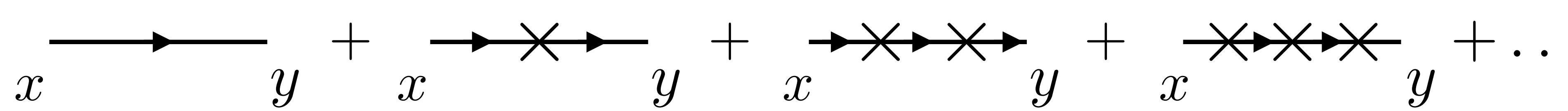
$\epsilon^0 > 0$

$$\gamma^{\mu}_{ab}\gamma_{5\,bc}\left\langle T\left[\overline{\psi}_a\left(x+\frac{\epsilon}{2}\right)\psi_c\left(x-\frac{\epsilon}{2}\right)\right]\right\rangle_{\mathscr{A}}$$

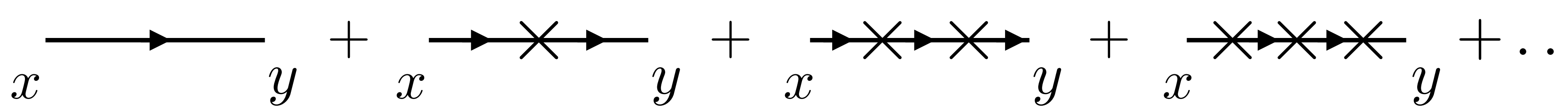
$$= -\mathrm{Tr}\left\{\gamma^{\mu}\gamma_5\left\langle T\left[\psi\left(x-\frac{\epsilon}{2}\right)\overline{\psi}\left(x+\frac{\epsilon}{2}\right)\right]\right\rangle_{\mathscr{A}}\right\}$$

$$= -\mathrm{Tr}\left[\gamma^{\mu}\gamma_5 G\left(x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]$$

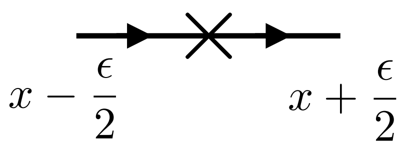
where the propagator can be computed diagrammatically as:

$G(x,y)_{\mathscr{A}} =$


$$G(x,y)_{\mathscr{A}} =$$



We look at the term linear in the gauge field:



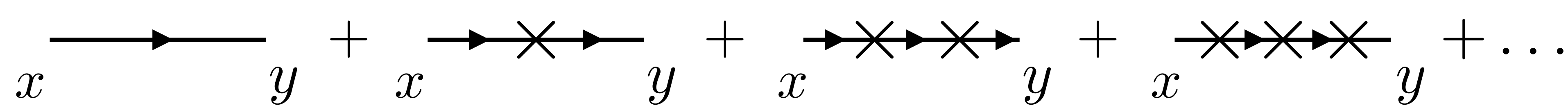
$$= ie \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \left(\frac{i}{\not{p} + \frac{1}{2}\not{q} - m} \gamma^{\mu} \frac{i}{\not{p} - \frac{1}{2}\not{q} - m} \right) e^{-iq\cdot x} e^{ip\cdot\epsilon} \mathscr{A}_{\mu}(q)$$

With this we go back to

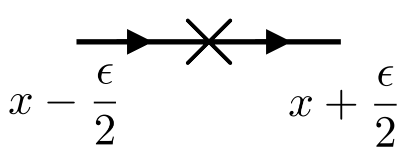
$$\partial_{\mu} \langle J_{\mathrm{A}}^{\mu}(x)_{\mathrm{reg}} \rangle_{\mathscr{A}} = 2im \langle [\overline{\psi}\gamma_5\psi]_{\mathrm{reg}} \rangle_{\mathscr{A}} - ie\epsilon^{\alpha} \langle J_{\mathrm{A}}^{\mu}(x) \rangle_{\mathscr{A}} \Big(\partial_{\alpha}\mathscr{A}_{\mu} - a\partial_{\mu}\mathscr{A}_{\alpha} + \ldots \Big)$$

and

$$\langle J_{\mathrm{A}}^{\mu}(x) \rangle_{\mathscr{A}} = -\mathrm{Tr}\, \left[\gamma^{\mu}\gamma_5 G\left(x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}\right)_{\mathscr{A}} \right] \exp\left[iea \int_{x-\epsilon/2}^{x+\epsilon/2} dy_{\alpha} \mathscr{A}^{\alpha}(y) \right]$$

$$G(x,y)_{\mathscr{A}} =$$ 

We look at the term linear in the gauge field:



$$= ie\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\left(\frac{i}{\not p+\frac{1}{2}\not q-m}\gamma^\mu\frac{i}{\not p-\frac{1}{2}\not q-m}\right)e^{-iq\cdot x}e^{ip\cdot\epsilon}\mathscr{A}_\mu(q)$$

With this we go back to

$$\partial_\mu\langle J^\mu_{\rm A}(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

and

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -{\rm Tr}\left[\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]\exp\left[iea\int_{x-\epsilon/2}^{x+\epsilon/2}dy_\alpha\mathscr{A}^\alpha(y)\right]$$

and use

$$\epsilon^\alpha e^{ip\cdot\epsilon} = -i\frac{\partial}{\partial p_\alpha}e^{ip\cdot\epsilon}$$ ⟶ integration by parts

$$-\mathrm{Tr}\,\left[\epsilon^{\alpha}\gamma^{\mu}\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=e\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\mathscr{A}_{\nu}(q)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot\epsilon}\frac{\partial}{\partial p_{\alpha}}\mathrm{Tr}\,\left(\gamma^{\mu}\gamma_5\frac{i}{\not{p}+\frac{1}{2}\not{q}-m}\gamma^{\nu}\frac{i}{\not{p}-\frac{1}{2}\not{q}-m}\right)$$

$$\epsilon^{\mu}\longrightarrow 0$$

$$\lim_{\epsilon\to 0}\epsilon^{\alpha}\langle J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=-\lim_{\epsilon\to 0}\mathrm{Tr}\,\left[\epsilon^{\alpha}\gamma^{\mu}\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=\frac{ie}{16\pi^2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\nu\sigma}(x)$$

With this result we return to the regularized anomaly

$$\partial_{\mu}\langle J^{\mu}_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=2im\langle[\overline{\psi}\gamma_5\psi]_{\mathrm{reg}}\rangle_{\mathscr{A}}-ie\epsilon^{\alpha}\langle J^{\mu}_{\mathrm{A}}(x)\rangle_{\mathscr{A}}\Big(\partial_{\alpha}\mathscr{A}_{\mu}-a\partial_{\mu}\mathscr{A}_{\alpha}+\ldots\Big)$$

$$-\mathrm{Tr}\left[\epsilon^\alpha\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=e\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\mathscr{A}_\nu(q)\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot\epsilon}\frac{\partial}{\partial p_\alpha}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{p}+\frac{1}{2}\not{q}-m}\gamma^\nu\frac{i}{\not{p}-\frac{1}{2}\not{q}-m}\right)$$

$$\epsilon^\mu\longrightarrow 0$$

$$\lim_{\epsilon\to 0}\epsilon^\alpha\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=-\lim_{\epsilon\to 0}\mathrm{Tr}\left[\epsilon^\alpha\gamma^\mu\gamma_5 G\left(x-\frac{\epsilon}{2},x+\frac{\epsilon}{2}\right)_{\mathscr{A}}\right]=$$

$$=\frac{ie}{16\pi^2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\nu\sigma}(x)$$

With this result we return to the regularized anomaly

$$\partial_\mu\langle J^\mu_{\mathrm{A}}(x)_{\mathrm{reg}}\rangle_{\mathscr{A}}=2im\langle[\overline{\psi}\gamma_5\psi]_{\mathrm{reg}}\rangle_{\mathscr{A}}-ie\epsilon^\alpha\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu-a\partial_\mu\mathscr{A}_\alpha+\ldots\Big)$$

$$\partial_\mu \langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} - ie\epsilon^\alpha\langle J_A^\mu(x)\rangle_{\mathscr{A}}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha + \ldots\Big)$$

$$\lim_{\epsilon\to 0}\epsilon^\alpha\langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = \frac{ie}{16\pi^2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\nu\sigma}(x)$$

Using the simple identity

$$\epsilon^{\mu\alpha\nu\sigma}\Big(\partial_\alpha\mathscr{A}_\mu - a\partial_\mu\mathscr{A}_\alpha\Big) = (1+a)\epsilon^{\mu\alpha\nu\sigma}\partial_\alpha\mathscr{A}_\mu = \frac{1+a}{2}\epsilon^{\mu\alpha\nu\sigma}\mathscr{F}_{\alpha\mu}$$

we arrive at the result

$$\lim_{\epsilon\to 0}\partial_\mu\langle J_A^\mu(x)_{\rm reg}\rangle_{\mathscr{A}} = 2im\lim_{\epsilon\to 0}\langle[\overline{\psi}\gamma_5\psi]_{\rm reg}\rangle_{\mathscr{A}} + \frac{e^2}{32\pi^2}(1+a)\epsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}$$

For $a = 1$

We recover the ABJ **anomaly** and J_A^μ is **gauge invariant**

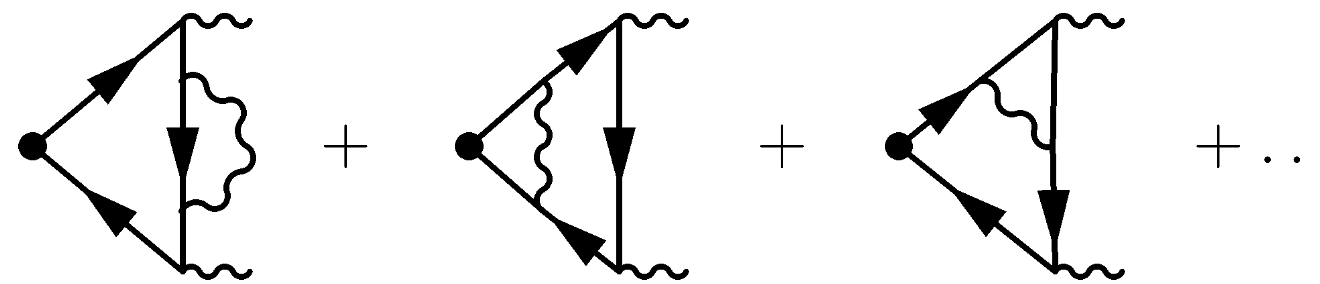
For $a = -1$

J_A^μ is **conserved** but is **not gauge invariant**

Quantum corrections

What about higher loops?

The ABJ anomaly is a one-loop result. Is it corrected by higher loop diagrams? E.g.



These diagrams contain **five** fermion propagator. The integration over the fermion loop momentum

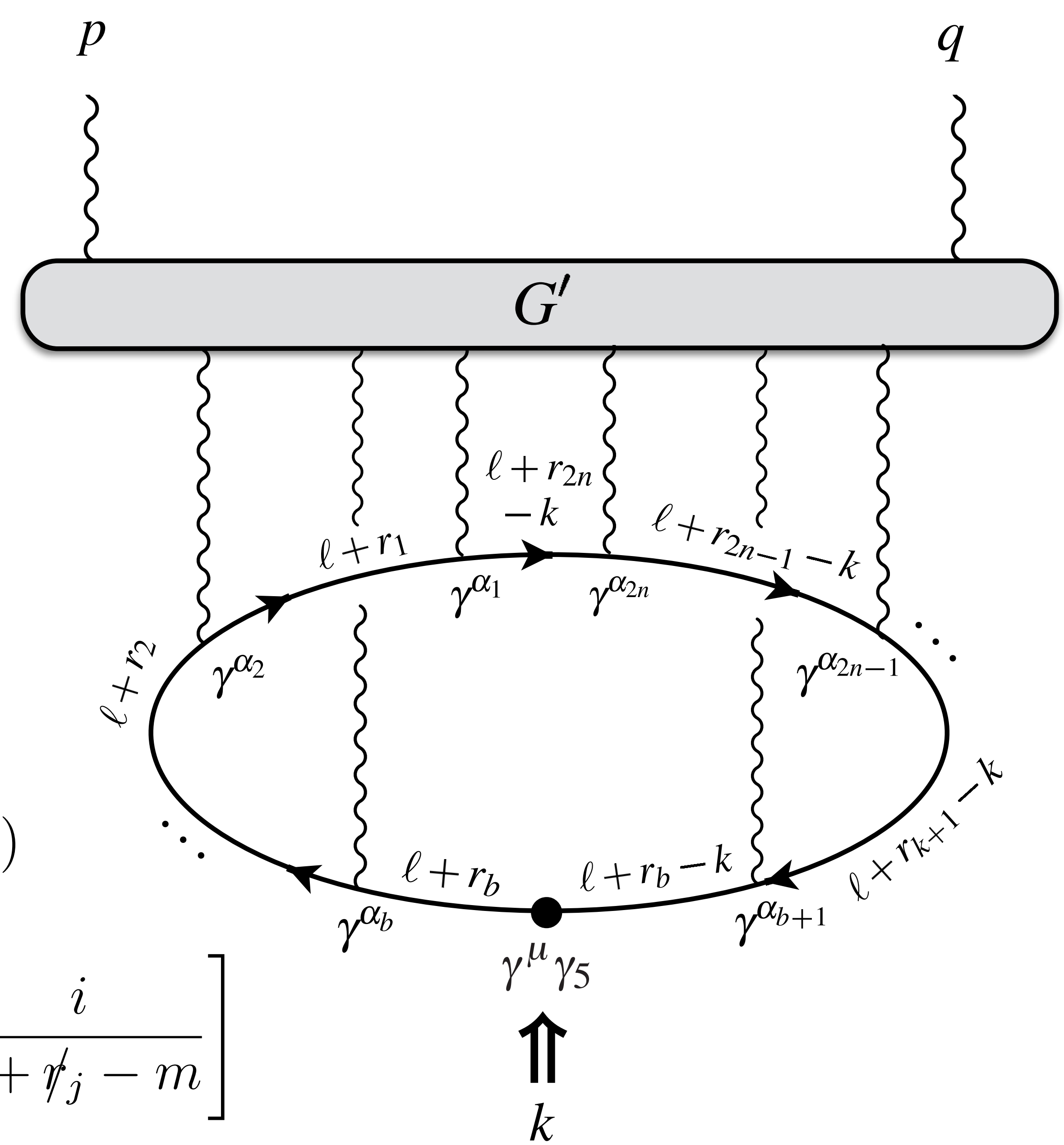
$$\cdots \int \frac{d^4\ell}{(2\pi)^4} \prod_{i=1}^{5} \frac{i}{\not{\ell} + \not{\Delta}_i + i\varepsilon} \cdots$$

is convergent. The remaining loops can be handled using a gauge invariant regulator, for example

$$\Delta S = \frac{1}{\Lambda^2} \int d^4x F_{\mu\nu} \Box F^{\mu\nu} \quad \Longrightarrow \quad G_{\mu\nu}(p) \sim \frac{\Lambda^2}{p^4}$$

This heuristic argument can be made more precise.

Consider a generic topology contributing to the divergence of the axial-vector current:



$$
\begin{aligned}
k^{\mu} i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} &= \int \prod_{a=1}^{L-1} \frac{d^4\ell_a}{(2\pi)^4} \Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q) \\
&\times \int \frac{d^4\ell}{(2\pi)^4} \sum_{b=1}^{2n} \mathrm{Tr}\left\{ \left[\prod_{j=1}^{b-1} (-ie\gamma^{\alpha_j}) \frac{i}{\not{\ell} + \not{r}_j - m} \right] \right. \\
&\quad \times (-ie\gamma^{\alpha_b}) \frac{i}{\not{\ell} + \not{r}_b - m} i k_\mu \gamma^\mu \gamma_5 \frac{i}{\not{\ell} + \not{r}_b - k - m} \\
&\quad \times \left. \left[\prod_{j=b+1}^{2n} (-ie\gamma^{\alpha_j}) \frac{i}{\not{\ell} + \not{r}_j - \not{k} - m} \right] \right\}.
\end{aligned}
$$

$$
\begin{aligned}
k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} &= \int \prod_{a=1}^{L-1} \frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\int \frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\,\left\{\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]\right. \\
&\left.\times\,(-ie\gamma^{\alpha_b})\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m}\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}.
\end{aligned}
$$

We simplify this expression using,

$$
\not k\gamma_5 = (\not\ell+\not r_b-m)\gamma_5+\gamma_5(\not\ell+\not r_b-\not k-m)+2m\gamma_5
$$

to write

$$
\begin{aligned}
\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m} &= \frac{i}{\not\ell+\not r_b-m}(2im\gamma_5)\frac{i}{\not\ell+\not r_b-k-m} \\
&- \frac{i}{\not\ell+\not r_b-m}\gamma_5-\gamma_5\frac{i}{\not\ell+\not r_b-\not k-m}.
\end{aligned}
$$

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} = \int \prod_{a=1}^{L-1} \frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\int\frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\left\{\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]\right.$$

$$\left.\times\ (-ie\gamma^{\alpha_b})\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m}\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}.$$

We simplify this expression using,

$$\not k\gamma_5 = (\not\ell+\not r_b-m)\gamma_5+\gamma_5(\not\ell+\not r_b-\not k-m)+2m\gamma_5$$

to write

$$\frac{i}{\not\ell+\not r_b-m}ik_\mu\gamma^\mu\gamma_5\frac{i}{\not\ell+\not r_b-k-m} = \frac{i}{\not\ell+\not r_b-m}(2im\gamma_5)\frac{i}{\not\ell+\not r_b-k-m}$$

$$-\ \frac{i}{\not\ell+\not r_b-m}\gamma_5-\gamma_5\frac{i}{\not\ell+\not r_b-\not k-m}.$$

Thus, the result has the structure:

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} \;=\; 2mi\Gamma_{\alpha\beta}(p,q)_{L\text{-loop}} + \Delta_{\alpha\beta}(p,q).$$

The relevant term contributing to $\Delta_{\alpha\beta}(p,q)$ is

$$-\sum_{b=1}^{2n}\mathrm{tr}\,\Bigg\{\left[\prod_{j=1}^{b}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]i\gamma_5\left[\prod_{j=b+1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]$$
$$-\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]i\gamma_5\left[\prod_{j=b}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\Bigg\}.$$

and most terms cancel

$$-\mathrm{tr}\,\Bigg\{(-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]-i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]$$
$$+\left[\prod_{j=1}^{2}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]i\gamma_5\left[\prod_{j=3}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]$$
$$-(-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]+\ldots\Bigg\}$$

Thus, the result has the structure:

$$k^\mu i\Gamma_{\mu\alpha\beta}(p,q)_{L\text{-loop}} = 2mi\Gamma_{\alpha\beta}(p,q)_{L\text{-loop}} + \Delta_{\alpha\beta}(p,q).$$

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$$\left.-\left[\prod_{j=1}^{b-1}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]i\gamma_5\left[\prod_{j=b}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right\}.$$

and most terms cancel

$$-\mathrm{tr}\left\{(-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]-i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]\right.$$
$$+\left[\prod_{j=1}^{2}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-m}\right]i\gamma_5\left[\prod_{j=3}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]$$
$$\left.-(-ie\gamma^{\alpha_1})\frac{i}{\not{\ell}+\not{r}_1-m}i\gamma_5\left[\prod_{j=2}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not{\ell}+\not{r}_j-\not{k}-m}\right]+\ldots\right\}$$

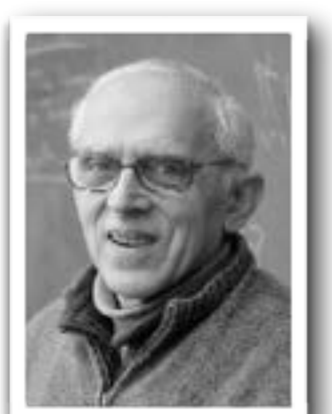
The only surviving terms are

$$-\mathrm{tr}\left\{\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}\right]i\gamma_5+i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell-\not r_j-\not k-m}\right]\right\}.$$

Hence, the final result for the anomalous piece is:

$$\begin{aligned}\Delta_{\alpha\beta}(p,q) &= -\int\prod_{a=1}^{L-1}\frac{d^4\ell_a}{(2\pi)^4}\Gamma^{(G')}_{\alpha\beta}(r_1,\ldots,r_{2n};p,q)\\ &\times\int\frac{d^4\ell}{(2\pi)^4}\sum_{b=1}^{2n}\mathrm{Tr}\left\{i\gamma_5\left[\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-m}-\prod_{j=1}^{2n}(-ie\gamma^{\alpha_j})\frac{i}{\not\ell+\not r_j-\not k-m}\right]\right\}\end{aligned}$$

For $n>1$ we can shift the integration momentum and cancel the terms.



Steven Adler
(b. 1939)

The ABJ anomaly does not receive quantum corrections

(**Adler-Bardeen theorem**)

William A. Bardeen
(b. 1941)

UV or IR?

Although given by finite integrals, on general grounds the anomaly can be seen as a **fundamental incompatibility** between the classical symmetry and the regularization procedure.

The symmetry is anomalous because the breaking introduced by the regularization **cannot** be subtracted by a **local counterterm** added to the action.

From this point of view the anomaly can be regarded as a **UV effect**.

But there is **also an IR side**…

Let us look at the on-shell amplitude

$$\langle 0|J_{\rm A}^{\mu}(0)|p,q\rangle_{\mathscr{A}} = \Gamma^{\mu\alpha\beta}(p,q)\widetilde{\mathscr{A}_{\alpha}}(p)\widetilde{\mathscr{A}_{\beta}}(q)\Big|_{p^2=q^2=0}$$

where $p^{\mu}\widetilde{\mathscr{A}_{\mu}}(p) = 0$. We recall,

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q) &= f_1\epsilon_{\mu\alpha\beta\sigma}p^{\sigma} + f_2\epsilon_{\mu\alpha\beta\sigma}q^{\sigma} + f_3\epsilon_{\mu\alpha\sigma\lambda}p_{\beta}p^{\sigma}q^{\lambda} \\ &+ f_4\epsilon_{\mu\alpha\sigma\lambda}q_{\beta}p^{\sigma}q^{\lambda} + f_5\epsilon_{\mu\beta\sigma\lambda}p_{\alpha}p^{\sigma}q^{\lambda} + f_6\epsilon_{\mu\beta\sigma\lambda}q_{\alpha}p^{\sigma}q^{\lambda} \end{aligned}$$

and due to the on-shell condition

$$f_i(p,q) = f_i(p\cdot q)$$ (symmetric in p and q)

and from Bose symmetry $f_1 = -f_2$, $f_3 = -f_6$, and $f_4 = -f_5$.

Vector current conservation further implies:

$$\begin{aligned} f_2 - p^2 f_5 - p\cdot q f_6 &= 0 \\ f_1 - q^2 f_4 - p\cdot q f_3 &= 0 \end{aligned} \quad\longrightarrow\quad f_1(p,q) = p\cdot q f_3(p,q)$$

The amplitude is then given only in terms of $f_3(p,q)$ and $f_4(p,q)$

$$\begin{aligned} i\Gamma_{\mu\alpha\beta}(p,q)\Big|_{p^2=q^2=0} &= f_3(p,q)\Big[p\cdot q\,\epsilon_{\mu\alpha\beta\sigma}(p^\sigma-q^\sigma)+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda-\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda\Big] \\ &+ f_4(p,q)\Big(\epsilon_{\mu\alpha\sigma\lambda}q_\beta-\epsilon_{\mu\beta\sigma\lambda}p_\alpha\Big)p^\sigma q^\lambda \end{aligned}$$

Due to $p^\mu\widetilde{\mathscr{A}_\mu}(p)=0$, the term with $f_4(p,q)$ does not contribute to the amplitude.

Using as well

$$\begin{aligned} -p\cdot q\epsilon_{\mu\alpha\beta\sigma}p^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda, \\ p\cdot q\epsilon_{\mu\alpha\beta\sigma}q^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda. \end{aligned}$$

the amplitude takes the form:

$$\langle 0|J^\mu_{\rm A}(0)|p,q\rangle_{\mathscr{A}}=i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda\widetilde{\mathscr{A}^\alpha}(p)\widetilde{\mathscr{A}^\beta}(q)$$

The amplitude is then given only in terms of $f_3(p,q)$ and $f_4(p,q)$

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q)\Big|_{p^2=q^2=0} &= f_3(p,q)\Big[p\cdot q\,\epsilon_{\mu\alpha\beta\sigma}(p^\sigma-q^\sigma)+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda-\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda\Big] \\
&+ f_4(p,q)\Big(\epsilon_{\mu\alpha\sigma\lambda}q_\beta-\epsilon_{\mu\beta\sigma\lambda}p_\alpha\Big)p^\sigma q^\lambda
\end{aligned}
$$

Due to $p^\mu\widetilde{\mathscr{A}_\mu}(p)=0$, the term with $f_4(p,q)$ does not contribute to the amplitude.

Using as well

$$\epsilon_{\alpha\beta\sigma\lambda}w_\mu+\epsilon_{\beta\sigma\lambda\mu}w_\alpha+\epsilon_{\sigma\lambda\mu\alpha}w_\beta+\epsilon_{\lambda\mu\alpha\beta}w_\sigma+\epsilon_{\mu\alpha\beta\sigma}w_\lambda=0$$

$$
\begin{aligned}
-p\cdot q\epsilon_{\mu\alpha\beta\sigma}p^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}p_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}p_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}p_\beta p^\sigma q^\lambda, \\
p\cdot q\epsilon_{\mu\alpha\beta\sigma}q^\sigma &= \epsilon_{\alpha\beta\sigma\lambda}q_\mu p^\sigma q^\lambda+\epsilon_{\mu\beta\sigma\lambda}q_\alpha p^\sigma q^\lambda+\epsilon_{\mu\alpha\sigma\lambda}q_\beta p^\sigma q^\lambda.
\end{aligned}
$$

the amplitude takes the form:

$$\langle 0|J_{\rm A}^\mu(0)|p,q\rangle_{\mathscr{A}}=i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda\widetilde{\mathscr{A}^\alpha}(p)\widetilde{\mathscr{A}^\beta}(q)$$

$$\langle 0|J^\mu_{\rm A}(0)|p,q\rangle_{\mathscr{A}} = i(p+q)^\mu f_3(p,q)\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q)$$

The function $f_3(p,q)$ can be computed from Feynman diagrams

$$f_3(p,q) = \frac{ie^2}{\pi^2}\int_0^1 dx\int_0^{1-x} dy\,\frac{xy}{2xyp\cdot q - m^2}$$

In the massless fermion limit, we have

$$\lim_{m\to 0} f_3(p,q) = \frac{ie^2}{2\pi^2}\frac{1}{(p+q)^2}$$

and we have

$$\lim_{m\to 0}\langle 0|J^\mu_A(0)|p,q\rangle_{\mathscr{A}} = -\frac{e^2}{2\pi^2}\frac{(p+q)^\mu}{(p+q)^2}\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q).$$

At the level of the **current**, the anomaly is signalled by a **massless pole**!

Thus, the anomaly has two faces:

- When looking at the **divergence of the current**, it comes associated with ambiguities in the **UV** regularization of the theory. Fixing them forces us to give up the axial-vector symmetry in favor of gauge invariance.

- When looking at the **current itself**, it is signaled by the appearance of a **massless pole** (i.e., an **IR effect**)

In fact, being careful, we should have written the result for the amplitude as

$$\lim_{m\to 0}\langle 0|J_A^\mu(0)|p,q\rangle_{\mathscr{A}} = -\frac{e^2}{2\pi^2}\frac{(p+q)^\mu}{(p+q)^2+i\epsilon}\epsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda \widetilde{\mathscr{A}}^\alpha(p)\widetilde{\mathscr{A}}^\beta(q).$$

The reason is that the integration over y in

$$f_3(p,q) = \frac{ie^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{xy}{2xyp\cdot q - m^2}$$

produces a logarithm and an imaginary part

$$\mathrm{Im}\, f_3(p,q) \neq 0 \qquad \text{for} \qquad (p+q)^2 > 4m^2$$

when $m\to 0$ the real part develops a pole and the imaginary part a delta-function singularity **whose coefficient is the anomaly**

$$\lim_{m\to 0}\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) = \frac{e^2}{2\pi}\epsilon^{\alpha\beta\sigma\lambda}p_\sigma q_\lambda (p+q)^\mu \delta\Big((p+q)^2\Big)$$

In fact, being careful, we should have written the result for the amplitude as

$$\lim_{m\to 0}\langle 0|J_A^{\mu}(0)|p,q\rangle_{\mathscr{A}} = -\frac{e^2}{2\pi^2}\frac{(p+q)^{\mu}}{(p+q)^2+i\epsilon}\epsilon_{\alpha\beta\sigma\lambda}p^{\sigma}q^{\lambda}\widetilde{\mathscr{A}}^{\alpha}(p)\widetilde{\mathscr{A}}^{\beta}(q).$$

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$$f_3(p,q)=\frac{ie^2}{\pi^2}\int_0^1 dx\int_0^{1-x}dy\,\frac{xy}{2xyp\cdot q-m^2}$$

produces a logarithm and an imaginary part

$$\frac{1}{x+i\epsilon}=\mathrm{PV}\frac{1}{x}-i\pi\delta(x)$$

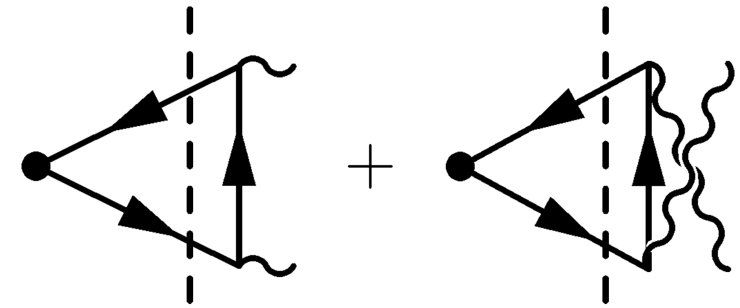
$\mathrm{Im}\, f_3(p,q)\neq 0$ for $(p+$

when $m\to 0$ the real part develops a pole and the imaginary part a delta-function singularity **whose coefficient is the anomaly**

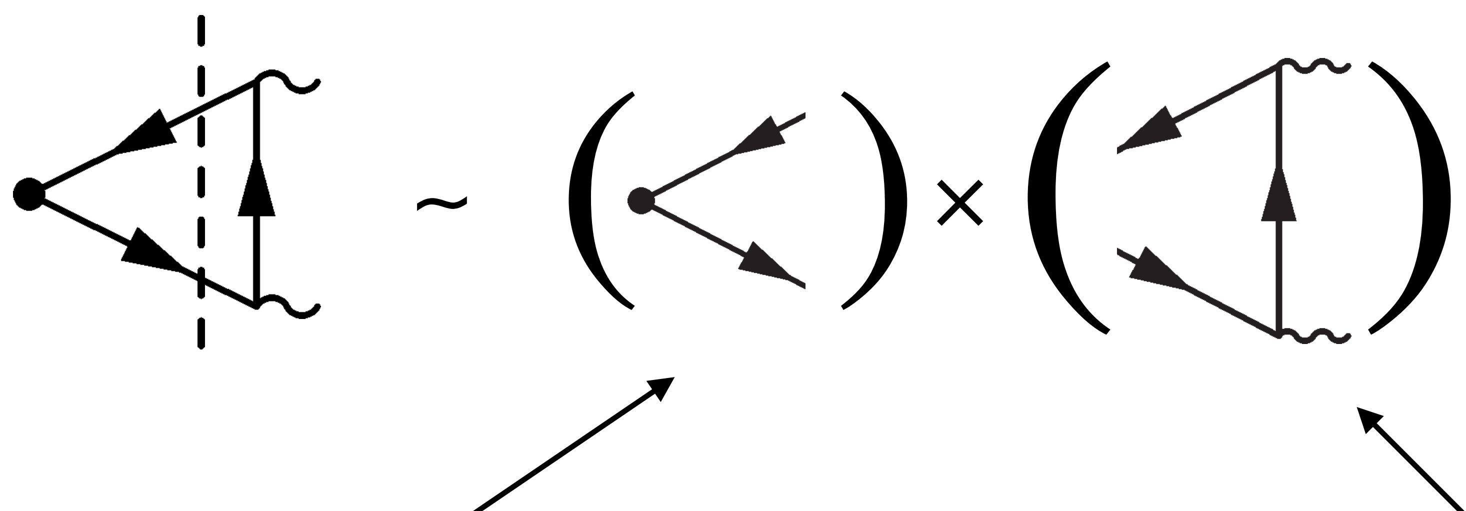
$$\lim_{m\to 0}\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)=\frac{e^2}{2\pi}\epsilon^{\alpha\beta\sigma\lambda}p_{\sigma}q_{\lambda}(p+q)^{\mu}\delta\Big((p+q)^2\Big)$$

This discontinuity in the imaginary part of the amplitude can be understood physically.

Let us use the Cutkosky rules:

$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) \sim$ 

where, e.g.



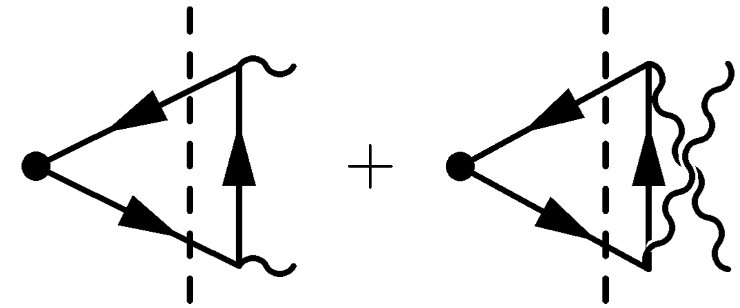
fermion-antifermion creation by J^{μ}_{A}

fermion-antifermion annihilation (2 diagrams)

This discontinuity in the imaginary part of the amplitude can be understood physically.

Let us use the Cutkosky rules:

$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q) \sim$$



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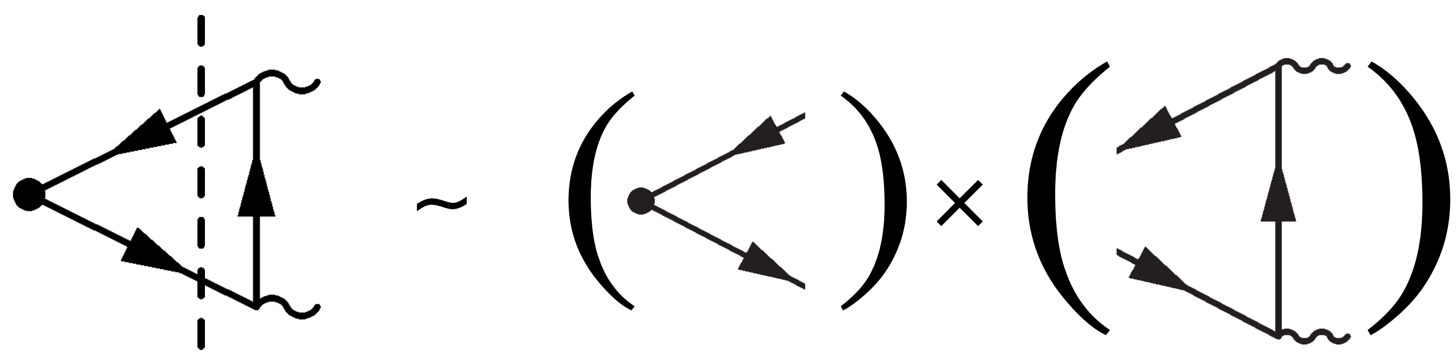


$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\epsilon_\alpha(\mathbf{p},\lambda_1)\epsilon_\beta(\mathbf{q},\lambda_2) \sim \sum_{\sigma_1,\sigma_2}\int d^3k_1\int d^3k_2\ {}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2\rangle_{\rm in}$$

$$\times\,{}_{\rm out}\langle \mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2|J_A^\mu(0)|0\rangle_{\rm in}$$

$$\mathrm{Im}\,\Gamma^{\mu\alpha\beta}(p,q)\epsilon_\alpha(\mathbf{p},\lambda_1)\epsilon_\beta(\mathbf{q},\lambda_2) \sim \sum_{\sigma_1,\sigma_2}\int d^3k_1\int d^3k_2\;{}_{\rm out}\langle\mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2\rangle_{\rm in}$$
$$\times{}_{\rm out}\langle\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2|J_A^\mu(0)|0\rangle_{\rm in}$$

The first important thing is to invoke the **Landau-Yang theorem**: no state of spin-one can decay into two on-shell photons.

Thus, the fermion-antifermion system should have **zero spin**. This means that in the center of mass frame they have the **same helicities**

$$\sigma_1=\sigma_2\equiv\sigma$$

We begin with the pair creation by the axial-vector current:

$$ {}_{\rm out}\langle \mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma|J_A^\mu(0)|0\rangle_{\rm in} \sim \overline{v}(\mathbf{k}_1,\sigma)\gamma^\mu\gamma_5 u(\mathbf{k}_2,\sigma) $$

In the massless limit, the helicity turns into ± chirality

$$ \lim_{m\to 0} u(\mathbf{p},\pm\frac{1}{2}) = u_\pm(\mathbf{p}) \qquad\qquad \lim_{m\to 0} v(\mathbf{p},\pm\frac{1}{2}) = v_\mp(\mathbf{p}) $$

Thus, using that

$$ \overline{v}_\mp(\mathbf{k}_2)\gamma^\mu\gamma_5 u_\pm(\mathbf{k}_1) = 0 $$

we find

$$ \lim_{m\to 0} {}_{\rm out}\langle \mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma|J_A^\mu(0)|0\rangle_{\rm in} = 0 $$

We turn now to the annihilation of the two fermions

$$ {}_{\text{out}}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\text{in}} = -e^2\epsilon_\mu(\mathbf{p},\lambda_1)\epsilon_\nu(\mathbf{k},\lambda_2) $$
$$ \times\overline{v}(\mathbf{k}_2,\sigma)\left[\frac{\gamma^\mu(\not{k}_1-\not{p}+m)\gamma^\nu}{(k_1-p)^2-m^2}+\frac{\gamma^\nu(\not{k}_2-\not{q}+m)\gamma^\mu}{(k_2-q)^2-m^2}\right]u(\mathbf{k}_1,\sigma) $$

Using now that

$$ \overline{v}_\mp(\mathbf{k}_2)\gamma^\mu\gamma^\alpha\gamma^\nu u_\pm(\mathbf{k}) = 0 $$

we conclude that the second amplitude also vanish in the massless limit

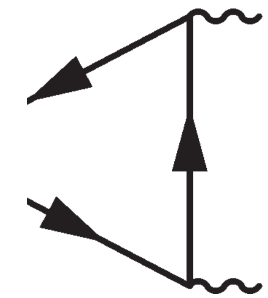
$$ \lim_{m\to 0} {}_{\text{out}}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\text{in}} = 0 $$

Thus, we would find that the amplitude approaches zero with the mass

$$ \operatorname{Im}\Gamma^{\mu\alpha\beta}(p,q)\sim 0 $$

$$
{}_{\rm out}\langle \mathbf{p},\lambda_1;\mathbf{q},\lambda_2|\mathbf{k}_1,\sigma;\mathbf{k}_2,\sigma\rangle_{\rm in} = -e^2\epsilon_\mu(\mathbf{p},\lambda_1)\epsilon_\nu(\mathbf{k},\lambda_2)
$$

$$
\times\overline{v}(\mathbf{k}_2,\sigma)\left[\frac{\gamma^\mu(\not{k}_1-\not{p}+m)\gamma^\nu}{(k_1-p)^2-m^2}+\frac{\gamma^\nu(\not{k}_2-\not{q}+m)\gamma^\mu}{(k_2-q)^2-m^2}\right]u(\mathbf{k}_1,\sigma)
$$



But not so fast…

In the massless limit, on-shell fermions can emit collinear on-shell photons, and the intermediate state can fall on-shell.

The denominator then vanishes and we have an indeterminate limit.

That is why, being more careful we obtained:

$$
\text{Im}\,\Gamma^{\mu\alpha\beta}(p,q) \sim (\text{anomaly}) \times \delta\Big((p+q)^2\Big)
$$

Thus, the anomaly appears as an **IR discontinuity** of the imaginary part of the amplitude.

Interestingly, this imaginary part in **unambiguous**.

A two-dimensional excursion: the Schwinger model

To keep things simple, we consider a **massless** Dirac fermion in 1+1 dimensions, and **compactify** the spatial direction to a circle of length L.

We consider the following representation of the Dirac matrices

$$\gamma^0 \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with the chirality matrix given by

$$\gamma_5 \equiv -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Decomposing the Dirac fermion into its Weyl components $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ the Dirac equation reads

$$(\partial_0 - \partial_1)u_+ = 0, \qquad (\partial_0 + \partial_1)u_- = 0$$

$$u_+ = \underbrace{u_+(x^0 + x^1)}_{\text{left-mover}}, \qquad u_- = \underbrace{u_-(x^0 - x^1)}_{\text{right-mover}}$$

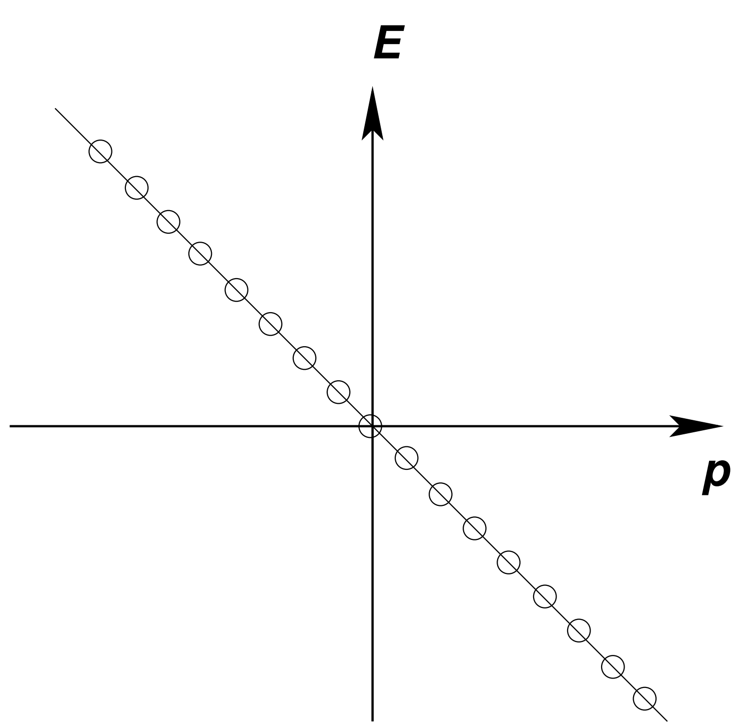
chirality is linked to the direction of motion the wave function for free fermions are

$$v_{\pm}^{(E)}(x^0 \pm x^1) = \frac{1}{\sqrt{L}} e^{-iE(x^0 \pm x^1)} \qquad \text{with} \qquad p = \mp E$$

and since the spatial direction is compatified, the momentum is **quantized**:

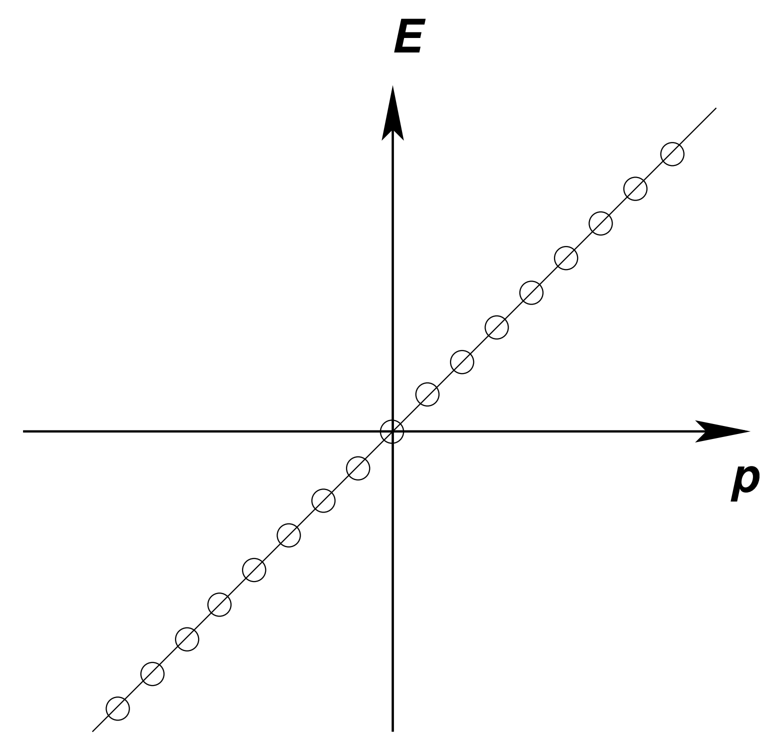
$$p = \frac{2\pi n}{L}, \qquad n \in \mathbb{Z}$$

the **spectrum** is:



v_+

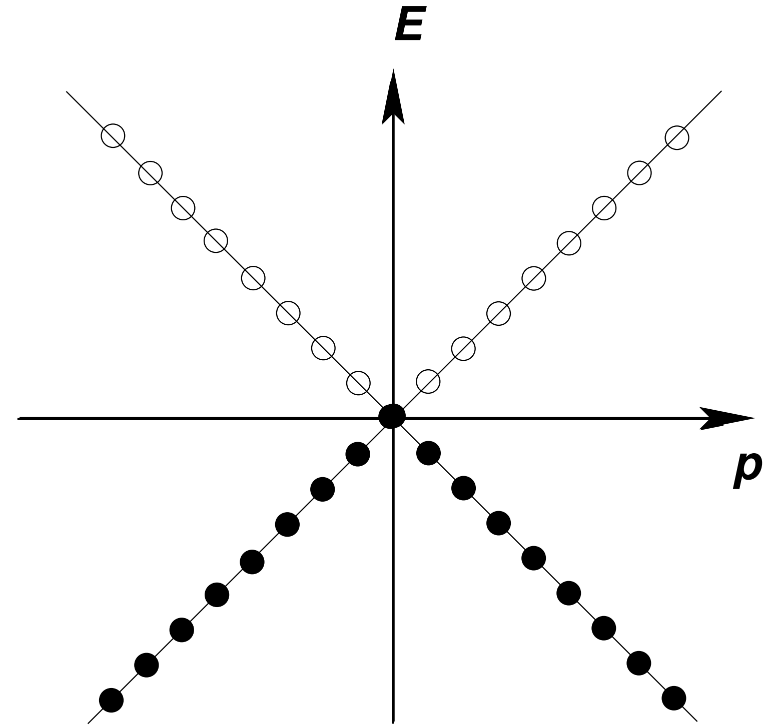
(positive chirality, left movers)



v_-

(negative chirality, right movers)

To quantize the Dirac fermion, we construct firts the **ground state** of the theory by filling all negative energy states (Dirac sea)



and expand:

$$u_{\pm}(x) = \sum_{E>0} \left[a_{\pm}(E) v_{\pm}^{(E)}(x) + b_{\pm}^{\dagger}(E) v_{\pm}^{(E)}(x)^{*} \right]$$

where,

$a_{\pm}(E)$: annihilates a **fermion** with $E > 0$ and $p = \pm E$

$b_{\pm}^{\dagger}(E)$: creates an **antifermion** with $E > 0$ and $p = \mp E$ → ($\mp$ chirality)

(i.e., annihilates a fermion with $E < 0$ and $p = \pm E$)

We look now at the **classical symmetries** of our theory

$$\mathcal{L} = iu_+^\dagger(\partial_0 + \partial_1)u_+ + iu_-^\dagger(\partial_0 - \partial_1)u_-$$

Vector U(1):

$$\psi \longrightarrow e^{i\alpha}\psi \qquad \Longrightarrow \qquad u_\pm \longrightarrow e^{i\alpha}u_\pm$$

whose associated Noether current is

$$J_{\mathrm{V}}^\mu = \overline{\psi}\gamma^\mu\psi \qquad \Longrightarrow \qquad J_{\mathrm{V}}^\mu = \begin{pmatrix} u_+^\dagger u_+ + u_-^\dagger u_- \\ -u_+^\dagger u_+ + u_-^\dagger u_- \end{pmatrix}$$

Axial U(1):

$$\psi \longrightarrow e^{i\beta\gamma_5}\psi \qquad \Longrightarrow \qquad u_\pm \longrightarrow e^{\pm i\beta}u_\pm$$

with

$$J_{\mathrm{A}}^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi \qquad \Longrightarrow \qquad J_{\mathrm{A}}^\mu = \begin{pmatrix} u_+^\dagger u_+ - u_-^\dagger u_- \\ -u_+^\dagger u_+ - u_-^\dagger u_- \end{pmatrix}$$

the corresponding conserved charges are

$$Q_{\rm V} \equiv \int_0^L dx^1\, J^0_{\rm V} = \int_0^L dx^1 \left(u_+^\dagger u_+ + u_-^\dagger u_-\right)$$

$$Q_{\rm A} \equiv \int_0^L dx^1\, J^0_{\rm A} = \int_0^L dx^1 \left(u_+^\dagger u_+ - u_-^\dagger u_-\right)$$

Using the orthogonality relations of the wave functions

$$\int_0^L dx^1\, v_\pm^{(E)}(x)^*\, v_\pm^{(E')}(x) = \delta_{E,E'}$$

we find

$$Q_{\rm V} \;=\; \sum_{E>0}\Big[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) + a_-^\dagger(E)a_-(E) - b_-^\dagger(E)b_-(E)\Big]$$

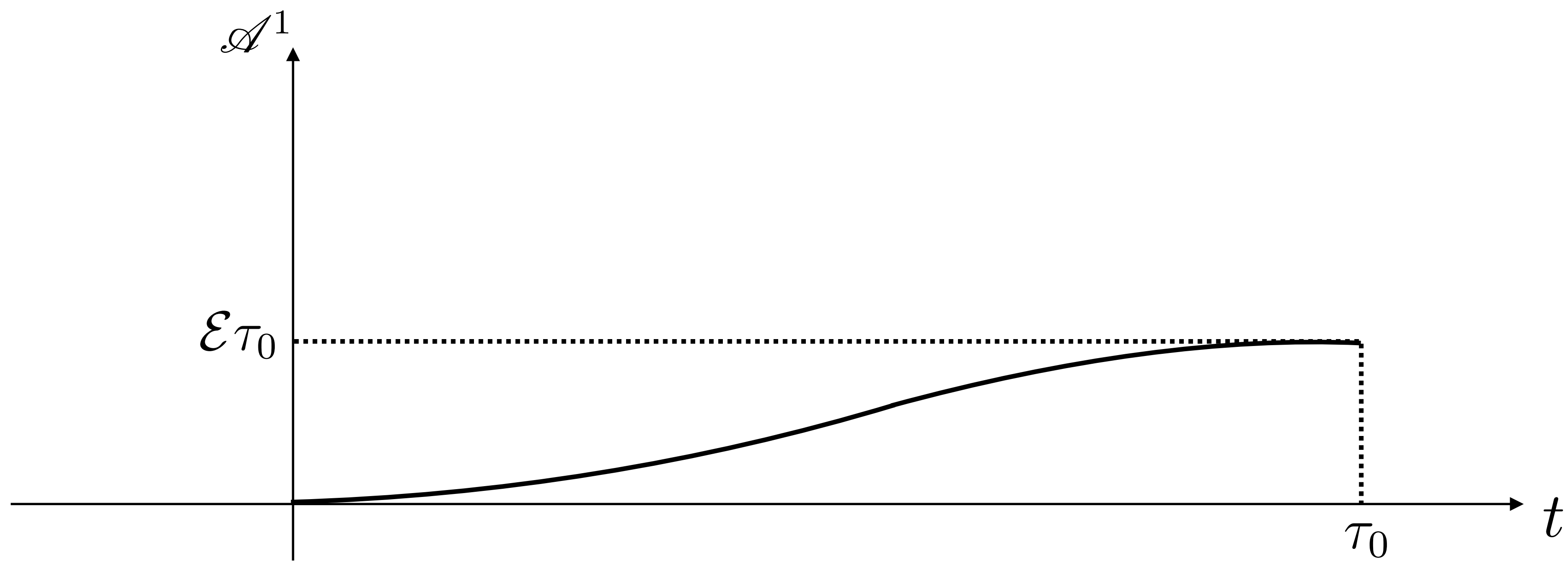
(fermions minus antifermions)

$$Q_{\rm A} \;=\; \sum_{E>0}\Big[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) - a_-^\dagger(E)a_-(E) + b_-^\dagger(E)b_-(E)\Big]$$

(“net” number of +’ve minus −’ve chirality states)

In the free theory, both charges are conserved… but what about switching an **external electrical field**?

We do it adiabatically. In the $\mathscr{A}^0 = 0$ gauge



The effect of this external field on the fermions is shifting the momentum by

$$p \longrightarrow p - e\mathscr{A}^1$$

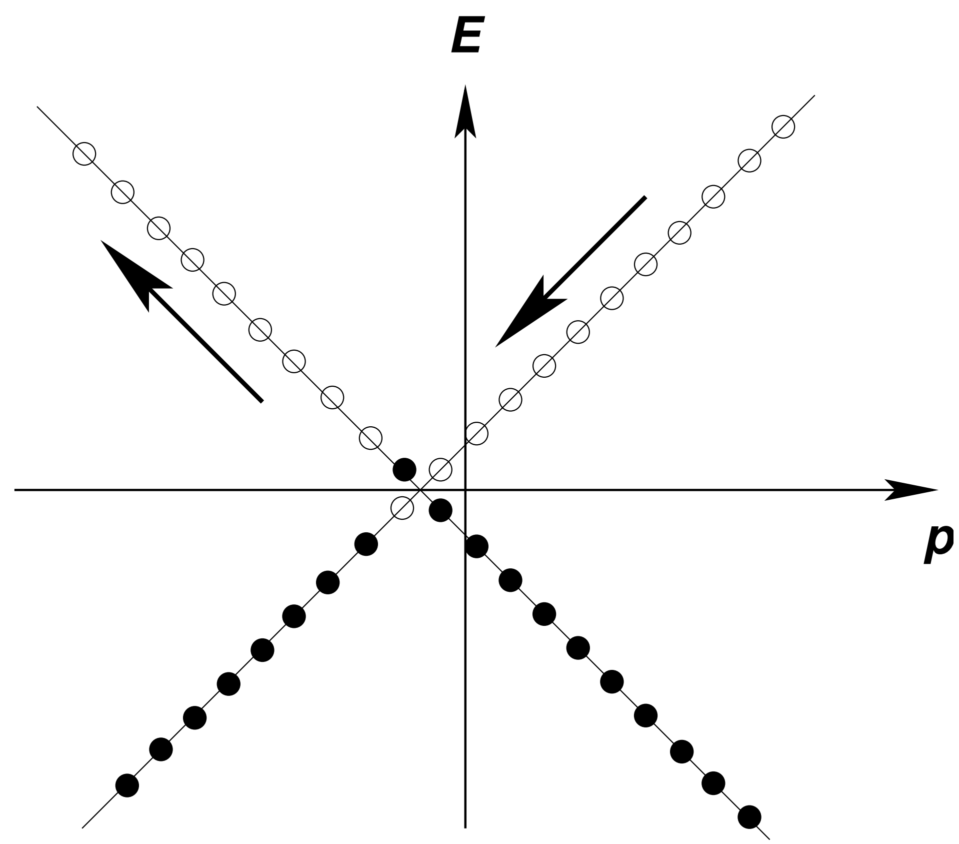
Adiabaticity allows to treat the system at each instant as "time independent".

$$p \longrightarrow p - e\mathscr{A}^1$$

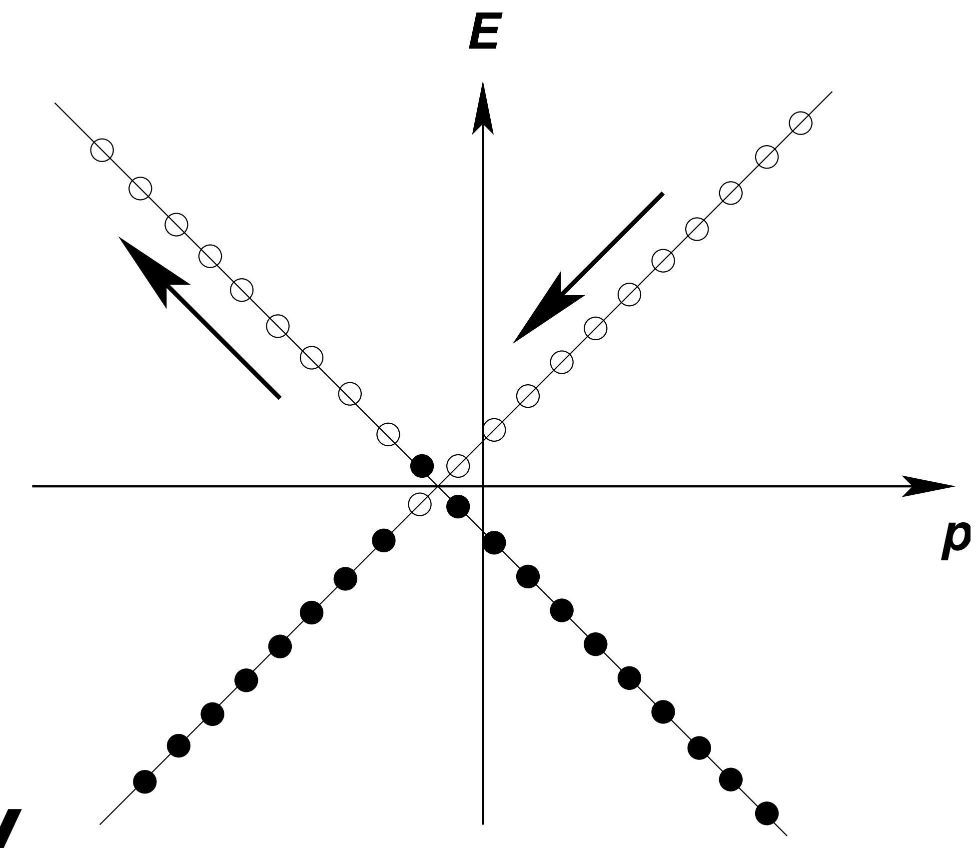
The shift have different effects on the states on each branch of the spectrum:

$E = p$ ⟶ $E = p - e\mathscr{A}^1$ (it "sinks")

$E = -p$ ⟶ $E = -p + e\mathscr{A}^1$ (it "lifts")



A number of negative chirality empty states become "holes" (negative chirality antifermions), while some occupied negative energy states with positive chirality get positive energy (positive chirality fermions)



The external field creates **pairs** of **+'ve chirality fermions** and **−'ve chirality antifermions**!

But, how many pairs?

$$N = \frac{\text{shift in the spectrum}}{\text{spectrum gap}} = \frac{e\mathscr{E}\tau_0}{2\pi/L} \quad \longrightarrow \quad N = \frac{L}{2\pi}e\mathscr{E}\tau_0$$

This preserves the vector charge:

$$Q_{\rm V}(\tau_0) \;=\; (N-0)+(0-N)=0$$

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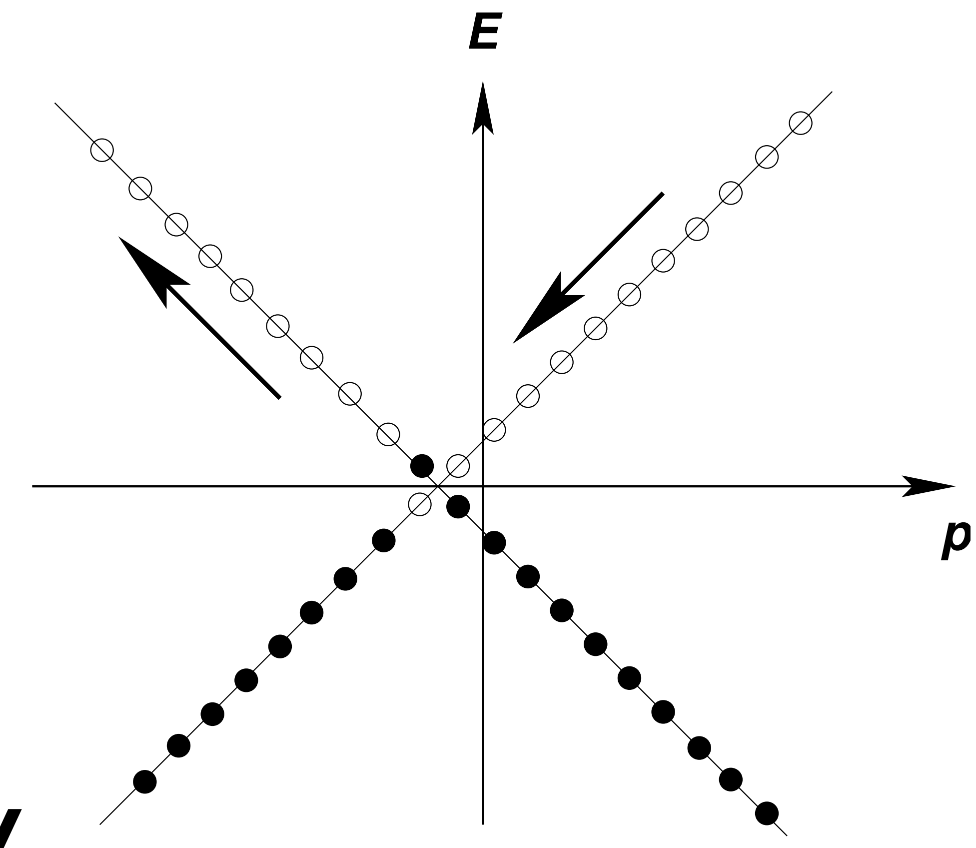

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But changes the axial charge:

$$Q_{\rm A}(\tau_0) = (N-0)-(0-N) = 2N$$

$$Q_{\mathrm{A}}(\tau_0) \;=\; (N-0)-(0-N)=2N \qquad\qquad N=\frac{L}{2\pi}e\mathscr{E}\tau_0$$

We have found that, in the presence of an external electric field, there is a **violation in the conservation of the axial current**.

Its rate of variation is

$$\dot{Q}_A=\frac{Q_A(\tau_0)}{\tau_0}=\frac{e}{\pi}L\mathscr{E}$$

This implies a violation in the conservation of the axial current

$$\partial_\mu J^\mu_{\mathrm{A}}=\frac{e}{\pi}\mathscr{E}$$

which gives the value of the axial anomaly in the Schwinger model:

$$\partial_\mu\langle J^\mu_{\mathrm{A}}(x)\rangle_{\mathscr{A}}=\frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x)$$

The anomaly in the massless Schwinger model has surprising consequences…

In fact, in two dimensions the vector and axial-vector currents are closely related.

$$\gamma_5 = -\gamma^0\gamma^1 \quad \longrightarrow \quad \gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu$$

Hence,

$$J_{\rm A}^\mu(x) = \epsilon^{\mu\nu} J_{{\rm V}\mu}(x)$$

Thus the anomaly can be recast in terms of the vector current as

$$\epsilon^{\mu\nu}\partial_\mu\langle J_{{\rm V}\nu}(x)\rangle_{\mathscr{A}} = \frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x) = \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu(x)$$

$$\epsilon^{\mu\nu}\partial_\mu\langle J_{V\nu}(x)\rangle_{\mathscr{A}} = \frac{e}{2\pi}\epsilon^{\mu\nu}\mathscr{F}_{\mu\nu}(x) = \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu(x)$$

In addition, the vector current has to satisfy the Maxwell equations

$$\partial_\mu\mathscr{F}^{\mu\nu}(x) = -e\langle J_V^\nu(x)\rangle_{\mathscr{A}} \quad \longrightarrow \quad \Box\mathscr{A}^\nu(x) - \partial^\nu\partial_\mu\mathscr{A}^\mu(x) = -e\langle J_V^\nu(x)\rangle_{\mathscr{A}}$$

Defining the pseudoscalar field $\mathscr{F}^* \equiv \frac{1}{2}\epsilon_{\mu\nu}\mathscr{F}^{\mu\nu} = \epsilon^{\mu\nu}\partial_\mu\mathscr{A}_\nu$ the two equations combine into:

$$\left(\Box + \frac{e^2}{\pi}\right)\mathscr{F}^* = 0$$

This means that the Schwinger model contains a propagating mode with mass

$$m^2 = \frac{e^2}{\pi}$$

What is this mode? Let's remember than in two dimensions, a vector can be decomposed as

$$A_\mu = \partial_\mu \eta + \epsilon_{\mu\nu}\partial^\nu \eta'$$

Due to the interaction with the fermions, the pseudoscalar mode acquires a mass.

The 2D Dirac fermion works like a "**technifermion**" which combine to produce a **massive photon**.

Unfortunately, this only works in 2D!

What is this mode? Let's remember than in two dimensions, a vector can be decomposed as

$$A_\mu = \partial_\mu \eta + \epsilon_{\mu\nu}\partial^\nu \eta'$$

(pure gauge: $\partial_\mu \eta$; pseudoscalar: η')

Due to the interaction with the fermions, the pseudoscalar mode acquires a mass.

The 2D Dirac fermion works like a "**technifermion**" which combine to produce a **massive photon**.

Unfortunately, this only works in 2D!

The Dirac-sea picture of the anomaly in the Schwinger model underlines its **IR character**

The anomaly it is determined by a number of states crossing the $E = 0$ Fermi level

Including non-Abelian fields: the singlet anomaly

Instead of QED, we consider now a fermion coupled (in a certain representation) to an external **non-Abelian** gauge theory

$$S=\int d^4x\Big(i\overline{\psi}\gamma^\mu\partial_\mu\psi-m\overline{\psi}\psi+g\overline{\psi}T^a_{\mathbf{R}}\gamma^\mu\psi\mathscr{A}^a_\mu\Big)$$

Classically, the gauge current $J^{\mu a}_{\rm V}=\overline{\psi}\gamma^\mu T^a_{\mathbf{R}}\psi$ satisfies the conservation equation

$$(D_\mu J^\mu_{\rm V})^a=0 \quad\longrightarrow\quad \partial_\mu J^{\mu a}_{\rm V}+gf^{abc}\mathscr{A}^b_\mu J^{\mu c}_{\rm V}=0$$

In addition we also have global axial transformations

$$\psi\longrightarrow e^{i\beta\gamma_5}\psi \qquad\qquad \overline{\psi}\longrightarrow\overline{\psi}e^{i\beta\gamma_5}$$

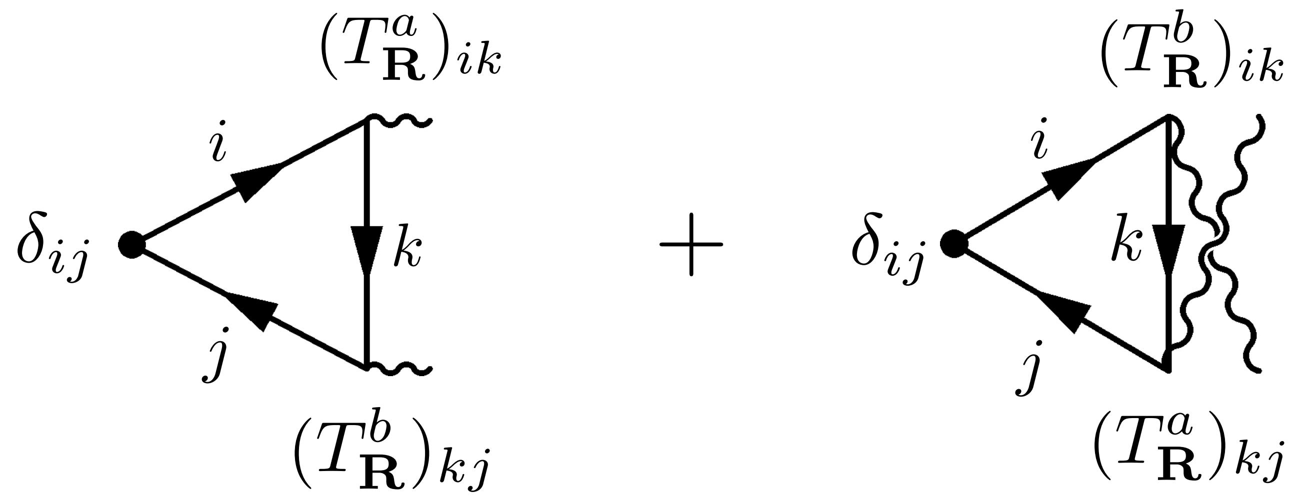
while its associated **singlet** axial current $J^\mu_{\rm A}=\overline{\psi}\gamma^\mu\gamma_5\psi$ satisfies the identity

$$\partial_\mu J^\mu_{\rm A}=2im\overline{\psi}\psi$$

Similarly to QED, the calculation of the axial anomaly boils down to computing

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_\mu \langle 0|T[J^\mu_{\rm A}(x)J^{\alpha a}_{\rm V}(y_1)J^{\beta b}_{\rm V}(y_2)]|0\rangle \mathscr{A}^a_\alpha(y_1)\mathscr{A}^b_\beta(y_2)+\ldots$$

Diagrammatically, we have again two triangle diagrams, these time with gauge group generators on the "vector" vertices



The two diagrams share the same color factor

$$\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}}) = \mathrm{Tr}\,(T^b_{\mathbf{R}}T^a_{\mathbf{R}})$$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = -\frac{g^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_\mu \langle 0|T[J^\mu_{\rm A}(x)J^{\alpha a}_{\rm V}(y_1)J^{\beta b}_{\rm V}(y_2)]|0\rangle \mathscr{A}^a_\alpha(y_1)\mathscr{A}^b_\beta(y_2) + \ldots$$

The rest of the calculation is identical to the case of QED. Using a gauge invariant regulator, we get in momentum space

$$(p+q)^\mu i\Gamma^{ab}_{\mu\alpha\beta}(p,q) = \frac{ig^2}{2\pi^2}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\epsilon_{\alpha\beta\sigma\nu}p^\sigma q^\nu + 2mi\Gamma^{ab}_{\alpha\beta}(p,q)$$

Adding the external gauge fields and Fourier transforming back to position space, this leads to

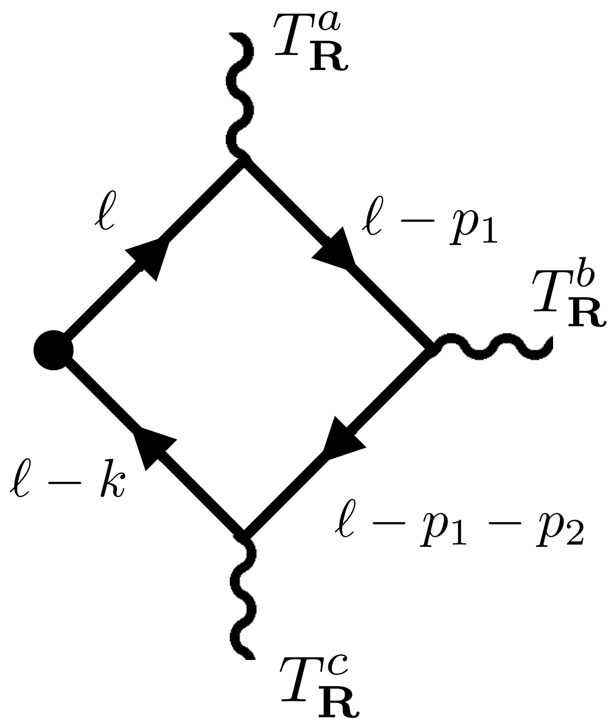
$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\mathscr{A}^a_\nu\partial_\alpha\mathscr{A}^b_\beta = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}})\partial_\mu\Big(\mathscr{A}^a_\nu\partial_\alpha\mathscr{A}^b_\beta\Big)$$

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu\mathrm{Tr}\,\Big(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta\Big)$$

The problem with this result is that it is **not gauge invariant**!

In fact, in the case of the singlet anomaly the triangle diagram is not enough.

We need to compute the **box diagrams** as well:



$+$ permutations

whose contributions are of the form

$$i\Gamma^{\mu\alpha\beta\gamma}(k,p_1,p_2) = ig^3 \mathrm{Tr}\,(T^a_{\mathbf{R}}T^b_{\mathbf{R}}T^c_{\mathbf{R}})$$

$$\times \int \frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\gamma^\mu\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\gamma^\alpha\frac{i}{\not{\ell}-\not{p}_1-\not{p}_2-m+i\epsilon}\gamma^\beta\frac{i}{\not{\ell}-\not{p}_1-m+i\epsilon}\gamma^\sigma\frac{i}{\not{\ell}-m+i\epsilon}\right)$$

$$+ \text{ permutations}$$

In computing the axial-vector Ward identity $k_\mu i\Gamma^{\mu\alpha\beta\gamma}(k,p_1,p_2)$ we encounter the trace

$$\mathrm{Tr}\,\left(\not{k}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}\gamma^\alpha\frac{i}{\not{\ell}-\not{p}_1-\not{p}_2-m+i\epsilon}\gamma^\beta\frac{i}{\not{\ell}-\not{p}_1-m+i\epsilon}\gamma^\sigma\frac{i}{\not{\ell}-m+i\epsilon}\right)$$

that we rewrite using

$$\not{k}\gamma_5=\gamma_5(\not{\ell}-\not{k}-m)+(\not{\ell}-m)\gamma_5+2m\gamma_5$$

The first two terms cancel one propagator each, while the last one effectively replaces the axial-vector current by the pseudoscalar bilinear.

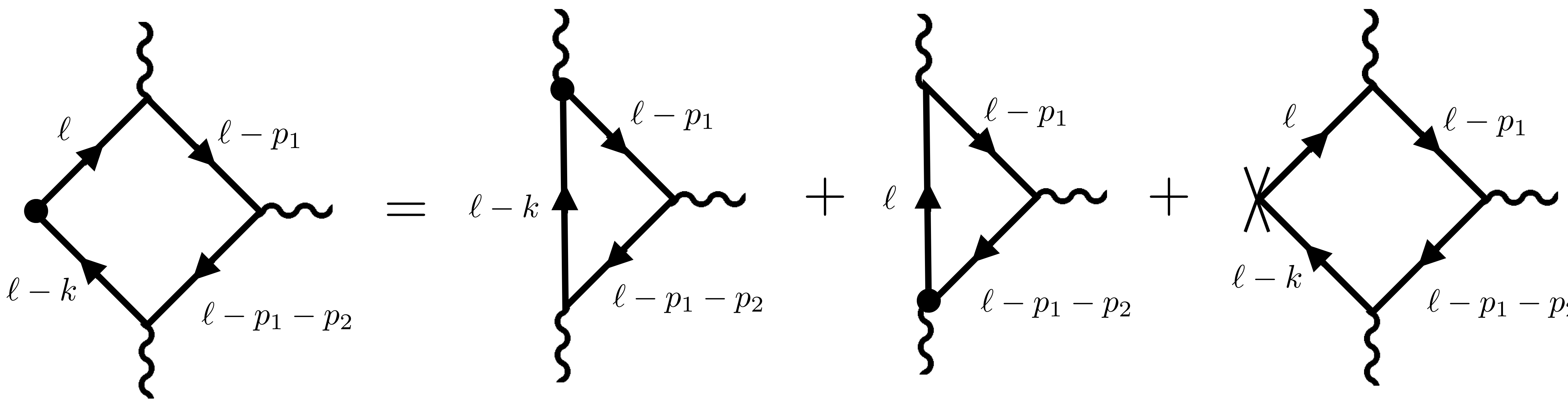
$$\frac{i}{\not{\ell}-m+i\epsilon}\not{k}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}$$

$$=\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5+\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}+2m\frac{i}{\not{\ell}-m+i\epsilon}\gamma_5\frac{i}{\not{\ell}-\not{k}-m+i\epsilon}$$

$$\frac{i}{\not\ell - m + i\epsilon}\not k\gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon}$$

$$= \frac{i}{\not\ell - m + i\epsilon}\gamma_5 + \gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon} + 2m\frac{i}{\not\ell - m + i\epsilon}\gamma_5\frac{i}{\not\ell - \not k - m + i\epsilon}$$

Diagrammatically,



The last term contributes to $2im\langle\overline{\psi}\psi\rangle_{\mathscr{A}}$, whereas the first two "triangles" give corrections to the anomaly **cubic** in the external field.

This combines with the triangle diagram to give the **singlet anomaly**:

$$\partial_\mu\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{4\pi^2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu {\rm Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right)$$

$$\frac{i}{\not\ell - m + i\epsilon}\not k\gamma_5 \frac{i}{\not\ell - \not k - m + i\epsilon}$$

$$= \frac{i}{\not\ell - m + i\epsilon}\gamma_5 + \gamma_5 \frac{i}{\not\ell - \not k - m + i\epsilon} + 2m\frac{i}{\not\ell - m + i\epsilon}\gamma_5 \frac{i}{\not\ell - \not k - m + i\epsilon}$$

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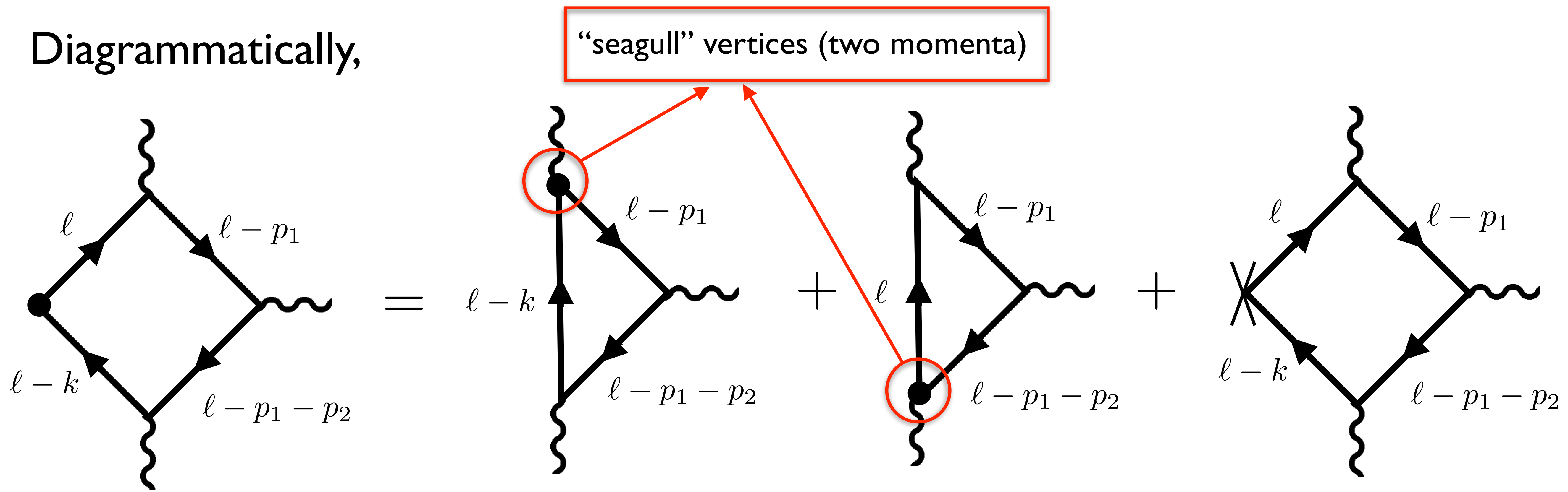

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Here we identify the Chern-Simons form,

$$\epsilon^{\mu\nu\alpha\beta}\partial_\mu \mathrm{Tr}\left(\mathscr{A}_\nu\partial_\alpha\mathscr{A}_\beta + \frac{2}{3}\mathscr{A}_\nu\mathscr{A}_\alpha\mathscr{A}_\beta\right) = \frac{1}{4}\mathrm{Tr}\left(\mathscr{F}^{\mu\nu}\widetilde{\mathscr{F}}_{\mu\nu}\right)$$

so the singlet anomaly can be written as

$$\partial_\mu \langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}\left(\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}\right)$$

which is **gauge invariant**.

It is important to stress that although there is contribution to the anomaly from the box diagram, its coefficient is determined by the **triangle diagram**