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The maximum power efficiency $1 - \sqrt{\tau}$: Research, education, and bibliometric relevance

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Abstract. The well-known efficiency at maximum power for a cyclic system working between hot T_h and low T_c temperatures given by the equation $1 - \sqrt{\tau}$ ($\tau = T_c/T_h$), has become a landmark result with regards to the thermodynamic optimization of a great variety of energy converters. Its wide applicability and sole dependence on the external heat bath temperatures (as the Carnot efficiency does) allows for an easy comparison with experimental efficiencies leading to a striking fair agreement. Reversible, finite-time, and linear-irreversible derivations are analyzed in order to show a broader perspective about its meaning from both researching and pedagogical point of views. Its scientific relevance and historical development are also analyzed in this work by means of some bibliometric data.

This article is supplemented with comments by Hong Qian and a final reply by the authors.

1 Introduction

The studies on the efficiency at maximum power for both cyclic and steady-state energy converters have nowadays a growing interest in all research fields involved with the optimal management of energetic resources and sustainable working fluids. Surely, many of these studies are inspired by an old and pedagogic-oriented result usually known as the Curzon-Ahlborn efficiency [1]. It accounts for the maximum power efficiency, $\widetilde{\eta}$, for a Carnot-like heat engine working between hot, T_h, and low, T_c, temperatures and it reads as

$$\widetilde{\eta} = 1 - \sqrt{\tau} \tag{1}$$

where $\tau = T_c/T_h$. However, recent findings show that it was previously worked out by Reitlinger [2], Yvon [3], Chambadal [4], and Novikov [5]. An update on the historical roots of this equation can be found in a recent publication [6].

The result given by Eq. (1) is becoming as paradigmatic with regards to thermodynamic optimization of energy converters. Its wide applicability and sole dependence on the external heat bath temperatures (as the Carnot efficiency does) allows for an easy and fair comparison with experimental efficiencies. Many different analytical derivations have been reported in order to assess its validity, applications, and shortcomings for a broad variety of thermal cycles and steady state systems based on Equilibrium [7–10], Finite-Time [11–20], Linear-Irreversible [21–28], Stochastic [29–35], and Quantum [35–41] thermodynamic frameworks. Its soundness has been also checked by means of molecular dynamics experiments when the external heat bath temperatures are not too different [42,43]. In the last years, several attempts were made trying to achieve similar bounds for refrigerators [44–50].

The $\widetilde{\eta}$-result has led to a broad field of research but it also has an undoubted interest from a pedagogical perspective. This paper focuses on these two complementary aspects and on its emanating bibliometric significance. We outline some derivations fully based on Classical Equilibrium Thermodynamics in Sect. II. In Sect. III, previous derivations are faced to some known finite-time and irreversible ones in order to explore differences and similarities in very different conceptual frameworks. This provides a broader perspective about the meaning of $\widetilde{\eta}$ from both researching and pedagogical point of views. Finally, the set of bibliometric data in Sect. IV accounts for its received cites, journals, years of publications and so on, in order to assess how it has evolved along the time and its (bibliometric) scientific relevance.

2 Reversible derivations

Result in Eq. (1) can be exactly obtained (or nearly approximated) in the realm of Classical Equilibrium Thermodynamics (CET). Here we show two reversible realizations. The first one has been reported long time ago [7,8] though it is illustrative to summarize again the two main results: a) reversible cycles with two adiabatic processes alternating with two other of the same nature (for instance, two constant volume (Otto) or two constant pressure (Joule-Brayton)) show an efficiency at maximum work that is exactly $\widetilde{\eta}$; and b) reversible cycles with two adiabatic steps alternating with two other of different nature (for instance Diesel and Atkinson cycles) the efficiency at maximum work conditions is working-fluid dependent, though it is very close (but not exactly the same) to $\widetilde{\eta}$.

We begin by analyzing the air-standard Otto cycle. This cycle can be described in terms of two independent variables: the compression ratio $r = V_{max}/V_{min} \geq 1$ and the ratio between the minimum and maximum temperatures, $\tau = T_{min}/T_{max} \leq 1$. In terms of these variables and by considering an ideal gas as the working fluid, the work output and the efficiency are given, respectively, as

$$|W_{\mathrm{rev}}| = C_V T_{max}\left[(1-\tau r^{\gamma-1}) - \left(r^{1-\gamma}-\tau\right)\right] \tag{2}$$

$$\eta_{\mathrm{rev}} = 1 - r^{1-\gamma} \tag{3}$$

where $\gamma = C_p/C_V$ and C_p and C_V are the pressure- and volume-constant heat capacities (An significant study on the influence of heat capacities with temperature dependence can be see in [10]). A pictorial scheme, not usually shown in textbooks, of $|W_{\mathrm{rev}}|$ and η_{rev} is given in Fig. 1. The work shows a positive parabolic shape between $r = 1$ and $r = \tau^{1/(1-\gamma)}$ with a maximum value at $\bar{r} = \tau^{1/2(1-\gamma)}$ while the efficiency increases from 0 to the maximum Carnot value $1-\tau$. The efficiency at

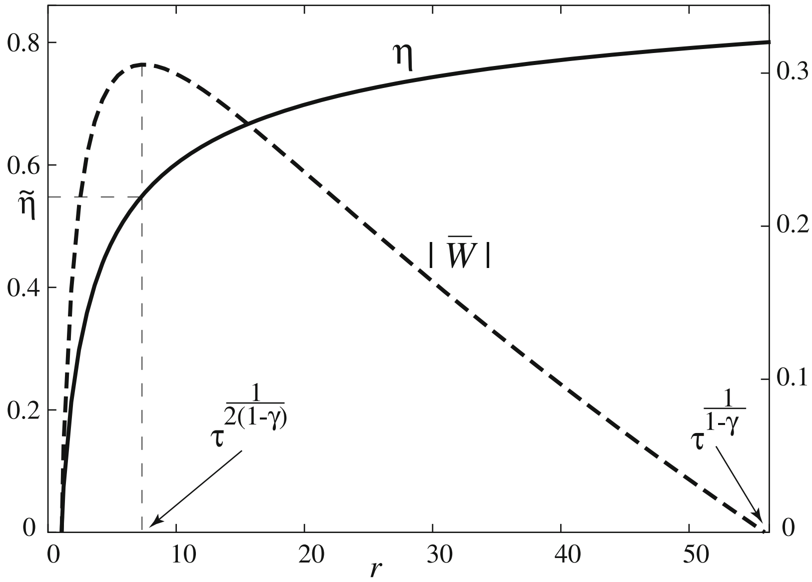

Fig. 1. Behavior of the efficiency, η, and dimensionless work output, $\overline{W} \equiv W/C_vT_{max}$, with the compression ratio, r, for an ideal Otto cycle.

maximum work is $\eta(\bar{r}) = 1 - \tau^{1/2} \equiv \tilde{\eta}$, i.e., Eq. (1). The same result is found for a reversible Joule-Brayton cycle where r denotes now the pressure ratio P_{max}/P_{min}. For the Diesel and Atkinson cycles the maximum work efficiencies are γ-dependent but quite close to Eq. (1).

The second reversible realization of $\tilde{\eta}$ does not need to specify any particular cycle and it is based on a standard problem in Zemansky's thermodynamics textbooks since long time ago (see problem 9.21 in [51]). Consider a heat device working at maximum work between two finite bodies with finite and constant heat capacities, C_{p1} and C_{p2}, and initial temperatures $T_{i,h}$ and $T_{i,c}$ for the hot and cold bodies, respectively (see Fig. 2). After each cycle, $T_{i,h}$ will decrease and $T_{i,c}$ will increase to reach a common and final temperature, T_{eq}, at which the device will stop. The work output is given by

$$W = Q_h - Q_c = C_{p1}(T_{i,h} - T_{eq}) - C_{p2}(T_{eq} - T_{i,c}). \tag{4}$$

Assuming reversibility in all processes (zero universe entropy production, $\Delta S_u = 0$) the final equilibrium temperature and the maximum work are given, respectively, by

$$T_{eq} = T_{i,h}^{\frac{C_{p1}}{C_{p1}+C_{p2}}} T_{i,c}^{\frac{C_{p2}}{C_{p1}+C_{p2}}} \tag{5}$$

$$W_{max} = C_{p1}T_{i,h}\left[1 + \frac{C_{p2}}{C_{p1}}\tau - (1 + \frac{C_{p2}}{C_{p1}})\tau^{\frac{C_{p2}}{C_{p1}+C_{p2}}}\right] \tag{6}$$

where $\tau = T_{i,c}/T_{i,h}$. Finally, the efficiency at maximum work is given by

$$\eta = \frac{W_{max}}{Q_h} = 1 + \frac{C_{p2}}{C_{p1}} \frac{\tau - \tau^{\frac{C_{p2}}{C_{p1}+C_{p2}}}}{1 - \tau^{\frac{C_{p2}}{C_{p1}+C_{p2}}}}. \tag{7}$$

It is straightforward to show that if $C_{p1} = C_{p2}$, the final equilibrium temperature is the geometric mean, $T_{eq} = \sqrt{T_{i,h}T_{i,c}}$, and the efficiency is again given by Eq. (1).

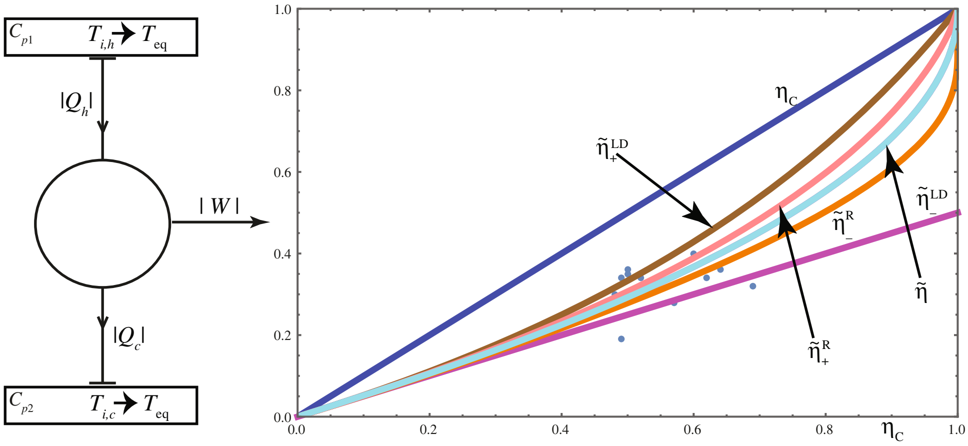

Fig. 2. Left: sketch of a thermodynamic cycle coupled to external heat baths with finite heat capacities $C_{p,1}$ and $C_{p,2}$ and initial temperatures $T_{i,h}$ and $T_{i,c}$. T_{eq} is the final common equilibrium temperature. Right: comparison among observed efficiencies [19] (solid points) with $\eta_C = 1 - \tau$, $\widetilde{\eta} = 1 - \sqrt{\tau}$, and the upper and lower bounds obtained with the reversible model, Eqs. (8) and (9), and the low dissipation model, Eqs. (10) and (11). See text for details.

The equation above is amenable to evaluate the role played by extremely asymmetric conditions $C_{p,2}/C_{p,1} \to 0$ and $C_{p,2}/C_{p,1} \to \infty$ which give lower, $\widetilde{\eta}_-^R$, and upper, $\widetilde{\eta}_+^R$, bounds for the optimized efficiency as

$$\widetilde{\eta}_-^R = 1 + \frac{\eta_C}{Ln(1-\eta_C)} \tag{8}$$

$$\widetilde{\eta}_+^R = 1 + \frac{(1-\eta_C)Ln(1-\eta_C)}{\eta_C} \tag{9}$$

in terms of the Carnot efficiency $\eta_C = 1 - \tau$. In Fig. 2 can be checked the behavior of $\widetilde{\eta}_-^R$ and upper $\widetilde{\eta}_+^R$ together with some observed efficiencies. These bound have also been reported by Yan and Guo [52] for a time-dependent Carnot-like model where the time is introduced by means of Newton's laws in the limit case of long contact time.This is not a surprise assuming that reversibility and infinite-time processes are synonymous concepts.

Previous examples show that in the reversible framework the heat capacities of the working system and/or the (finite) heat capacities of the auxiliary thermal baths seem to play a key role in obtaining a maximum work efficiency given exactly (or approximately) by $\widetilde{\eta}$. This issue has been addressed in detail by Angulo-Brown and co-workers [9,10]. These authors analyzed reversible cycles running two adiabatic steps alternating with two heat transfer branches, where the heat capacities are assumed to behave as aT^n in the heat-input step and bT^n in the output (see Sect. II in [9]). The intermediate variable temperatures of the working fluid are linked by the global reversibility condition, so that only one degree of freedom is again available to be optimized. As expected, for the values $n = 0, a = b$ (constant and equal heat capacities) $\widetilde{\eta}$ is recovered. The noticeable point is that some other combinations of a, b, n values give the $\widetilde{\eta}$-result either for $a = b$ or for $a \neq b$. Even more, these authors note that when $n = -1/2$, $\widetilde{\eta}$ is exactly obtained independently of any symmetry condition linked to the coefficients a, b [9].

In short, we note that $\widetilde{\eta}$ can be deduced in CET with or without symmetry conditions; the only common characteristic of the above reversible scenarios is the existence of one degree of freedom for fixed external temperatures.

3 Some irreversible derivations

3.1 Finite-Time models

The most popular derivation based on the finite-time (FT) models is indeed the one by Curzon-Ahlborn [1](not repeated here) in an oriented pedagogical journal (in their own words: "We have found it instructive in our classes on thermodynamics to consider another fundamental limitation on efficiency which is caused by the rate at which heat can be exchanged between the working material and the heat reservoirs"). Here the two key points are the following: a) irreversibilities of the Carnot-like cycle are limited to the coupling of the working medium with two external heat reservoirs (endo-reversible or exo-irreversible hypothesis), and b) the heat transfer laws between the isothermal steps of the working system and the reservoirs are linear with the temperature difference and each one is characterized by a finite time duration while the time to complete the two adiabatic processes remains proportional to the time required for the isothermal processes. On these assumptions, and maximizing the power output with respect to the two intermediate temperatures of the working system, Eq. (1) emerges. Thus, this derivation needs a 2D-optimization space since the times spent along the internal isothermal and the two intermediate temperatures are linked by the Clausius relation.

The original proof by Curzon-Ahlborn was further simplified in 1985 by De Vos in the same pedagogical journal [12]. This author eliminated the different times duration of the heat transfer processes and the energetic magnitudes were referred as fluxes (per cycle time). Additionally, the two intermediate temperatures were linked by the Clausius equality, so that the optimization of power only needs one degree of freedom. In this way, the FT-derivation by de Vos [12] shares with the CET-derivations in Sect. I two common characteristics: a reversible cyclic system with two adiabatic branches alternating with two other of the same nature and a 1D-optimization space (compression ratio for the Otto cycle, pressure ratio for the Brayton cycle, and T_{eq} for the cycle with finite heat capacities). However some subtle differences there exist in regard to the irreversible coupling of the inner part with the two external heat reservoirs via (finite-time) linear heat transfer across finite temperature differences. Indeed, this is a superfluous ingredient in the infinite-time framework where global reversibility ($\Delta S_u = 0$) is assumed (i.e., the existence of infinite external heat reservoirs). The core of above FT-derivations, the endoreversibility assumption, has been matter of controversy [13–16], but it is amenable to further simplifications especially oriented to undergraduate students [17,18].

A complementary FT-derivation, but conceptually different, was reported by Esposito et al. [19]. It does not need the endoreversible assumption, or any specific heat transfer law between the cyclic system and external heat baths but incorporates some symmetry properties which determine upper and lower bounds for the efficiency. Let us consider a Carnot-like reversible cycle coupled to external heat baths at temperatures T_h and T_c ($T_h > T_c$). Irreversibilities are taken into account by considering that isothermal heat transfers proceed in finite times, t_h and t_c, during the upper absorption and low rejection processes, respectively. Thus, entropy generation in these processes are Σ_h/t_h and Σ_c/t_c, where Σ_h and Σ_c are coefficients that globally contain all the information on the corresponding irreversibilities. Indeed, under infinite time limits the model recovers reversible behavior without entropy generation. The heats involved are given as $Q_h = T_h \left(\Delta S - \frac{\Sigma_h}{t_h} \right)$ and $Q_c = T_c \left(-\Delta S - \frac{\Sigma_c}{t_c} \right)$, where ΔS ($-\Delta S$) is the entropy increase (decrease) of the working system while in contact with the hot (cold) reservoir. The power output, $\dot{W} = \frac{Q_h+Q_c}{t_h+t_c}$, is maximized with respect to the time durations t_h and t_c and the optimized efficiency is Σ-dependent (see [19] for details). When $\Sigma_h = \Sigma_c$ (symmetric dissipation) $\widetilde{\eta}$ is recovered while under strong asymmetric limits $\Sigma_c/\Sigma_h \to 0$ or $\Sigma_c/\Sigma_h \to \infty$ the following lower and upper bounds are obtained:

$$\widetilde{\eta}_{-}^{LD} = \eta_C/2 \tag{10}$$
$$\widetilde{\eta}_{+}^{LD} = \eta_C/(2-\eta_C). \tag{11}$$

Note the similarity of this model with the one by Curzon-Alborn relative to the optimization with two independent variables and the similarity with the second reversible model in Sect. II in regards the role play by the ratios Σ_c/Σ_h and $C_{p,2}/C_{p,1}$ to obtain lower and upper bounds. In Fig. 2 we also show these (finite-time) bounds for the sake of comparison with observed efficiencies and the (infinite-time) reversible bounds given by Eqs. (8) and (9).

3.2 Linear irreversible models

An obvious criticism to some FT-derivations is their lack of generality because of their specific-model dependence. An essential step in this way was reported by Van den Broeck [21] showing that Eq. (1) is a result which can be obtained in the well-founded formalism of linear irreversible thermodynamics (LIT) thus supporting their validity and generality. LIT is based on the assumption of local thermodynamic equilibrium (accordingly, the local and instantaneous relations among thermodynamic quantities in a system out of equilibrium are the same as for an equilibrium system) and on the linear dependence of each flux, J_i, on the thermodynamic forces (or affinities), X_j, through some coupling parameter, $L_{ij} = \partial J_i/\partial X_j$, that globally contains the local information of the system. The resulting positive entropy production is found to be $\dot{\sigma} = \Sigma_i J_i X_i = \Sigma_{i,j} L_{ij} X_j \geq 0$, in accordance to the second law.

In this framework the main steps of the proposal by Van den Broeck [21] are as follows. Let us consider a cyclic system connected to two external thermal baths with temperatures $T+\Delta T$ and T, so that the power output of the working system against an external force F (mechanical, electric, or so on) is $\dot{W} = -F\dot{x}$, where $\dot{x}$ is the conjugate variable to F. To get this, the system absorbs a heat flux $|\dot{Q}|$ from the hot thermal bath and delivers a heat flux $|\dot{Q}|-|\dot{W}|$ to the cold thermal bath. The entropy generation is given by

$$\dot{\sigma} = -\frac{|\dot{Q}|}{T+\Delta T} + \frac{|\dot{Q}|-|\dot{W}|}{T} \equiv -\frac{F\dot{x}}{T} + |\dot{Q}| \left(\frac{1}{T} - \frac{1}{T+\Delta T} \right) \cdot \tag{12}$$

From this equation it is straightforward to identify the fluxes $J_1 \equiv \dot{x}$ and $J_2 \equiv \dot{Q}$ as well as the the corresponding thermodynamic forces (or affinities) $X_1 \equiv \frac{F}{T}$ and $X_2 \equiv \left(\frac{1}{T} - \frac{1}{T+\Delta T} \right) \approx \frac{\Delta T}{T^2}$. The power output can be written as

$$\dot{W} = -F\dot{x} = -J_1 X_1 T = -(L_{11}X_1^2 + L_{12}X_1X_2)T \tag{13}$$

and it shows a maximum under the condition $(\partial\dot{W}/\partial X_1)_{X_1=\overline{X}_1} = 0$. The efficiency of conversion, $\eta = \dot{W}/J_2$, evaluated at $\overline{X_1} = -L_{12}X_2/2L_{11}$ reads as

$$\eta(\overline{X}_1) \equiv \eta_{max\dot{W}} = \frac{1}{2}\frac{\Delta T}{T}\frac{q^2}{2-q^2} \tag{14}$$

where $q = L_{12}/\sqrt{L_{11}L_{22}}$ is a coupling parameter bounded by $-1 \leq q \leq +1$. In the limit of perfect coupling, $q = 1$, the efficiency is $\eta_{max\dot{W}} = \frac{1}{2}\frac{\Delta T}{T}$, which is the first term in the Taylor expansion of $\widetilde{\eta} = 1 - \sqrt{\frac{T}{T+\Delta T}}$.

For the extension beyond the linear approximation, Van den Broeck [21] uses a cascade construction of infinite cycles, each working between infinitesimal temperature differences, which under perfect coupling conditions allows to recover the exact CA-efficiency. An interesting and striking property of the cascade construction is that the overall performance regimes of the whole system may be different to those showed by each particular device [22,23]. This derivation, as the one by Esposito et al. [19], does not need any explicit assumption for heat transfers processes or the endoreversible hypothesis. It emerges as a straightforward consequence of the appropriate selection of the fluxes and forces from the entropy generation (see below, however). Note again that the LIT-deduction is one degree of freedom problem under tight-coupling conditions ($q = 1$).

An extension of the LIT-framework was proposed by Izumida and Okuda [24] incorporating extended Onsager relations with a nonlinear term accounting for power dissipation in the external heat baths. This minimally non-linear model allows to reproduce all the results of the low dissipation model (in particular $\widetilde{\eta}$) under tight-coupling and equal dissipations in the two heat baths. Besides, it allows an analytical calculation of the corresponding Onsager coefficients of the low dissipation models. Further progress on the connection between FT- and LIT-models have been reported recently by Sheng and Tu [27,28]. These authors pose two valuable questions: the election of appropriate fluxes and of the corresponding thermodynamic forces in heat devices where both thermal and mechanical fluxes coexist, and the necessity or not of the local equilibrium assumption. By using the novel concepts of weighted thermal flux and weighted reciprocal of temperature, these authors derive an expression of the entropy generation expressed in a canonical form and without the assumption of the local equilibrium hypothesis. In particular, these authors map the original derivation and obtain $\widetilde{\eta}$ up to quadratic order, independently of symmetry conditions for the external couplings, as Curzon and Ahlborn did.

Indeed, the above two key questions concerning to local equilibrium assumption and the appropriate election of the fluxes and the associated thermodynamic forces could be a key point, not only in the research field, but also from a pedagogical perspective, because of these two points are the very core of the LIT formalism. Useful information on the extension to mesoscopic non equilibrium steady state thermodynamics can be found in [53,54].

4 Some bibliometric data

This section analyze bibliometric results in order to explore the scientific relevance of the $\tilde{\eta}$-efficiency. However, caution is required because, as Newman [55] notes, the number of citations that a given paper receives is influenced by diverse factors such as the orientation or scope of the journal where the paper is pusblished, publishing seasons and the author name recognition among others. Since $\tilde{\eta}$ it is mostly known as

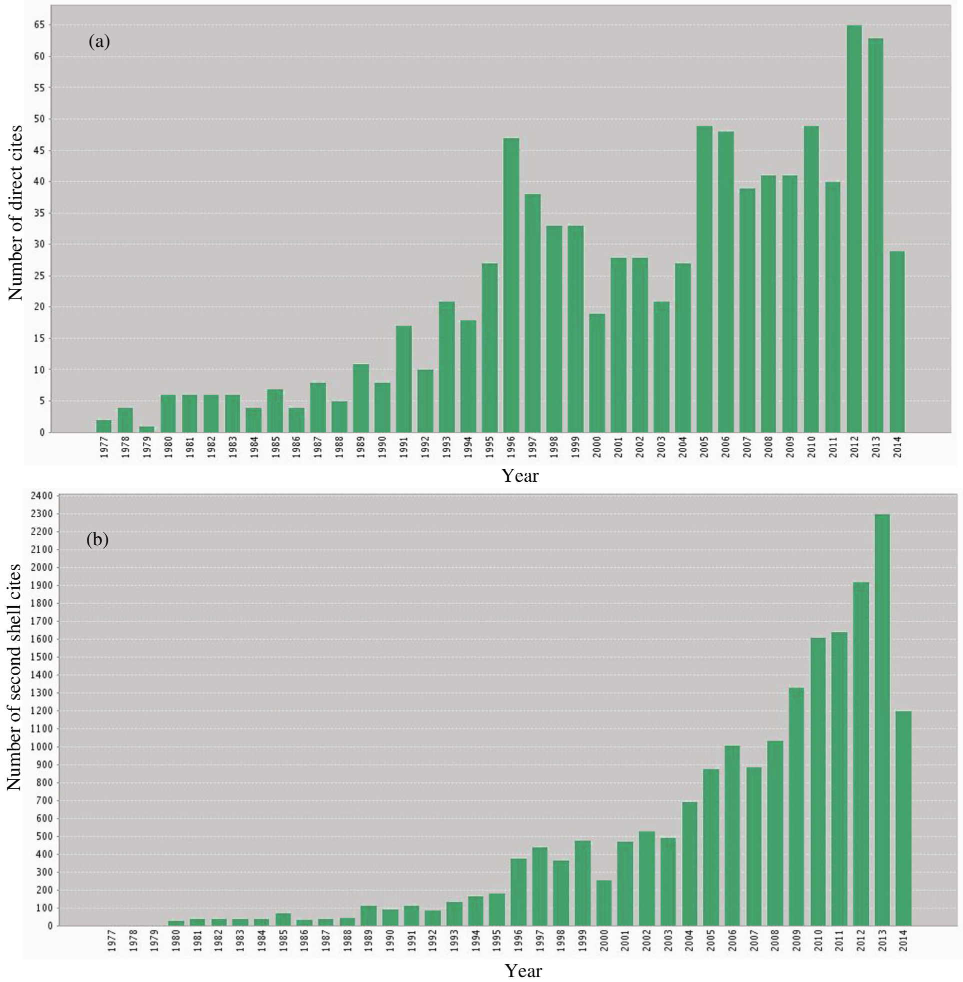

Fig. 3. Snapshot from Web of Science for the evolution with time of the direct cites (left panel) and second shell cites of the Curzon-Ahlborn paper [1] (right panel). See text for details.

the Curzon-Ahlborn effciency, we focused the bibliometric results on the title of the 1975 paper *Efficiency of a Carnot engine at maximum power output*. The data were downloaded on September 9th, 2104 from Web of Science Core Collection database. No exhaustive analysis has been pursued and then comparison with data from other well-known database has not been realized.

The citation report of this paper consists of 909 outputs (from 1977). Each paper is identified in terms of authors, title, date, journal and an unique identifier. Figure 3a shows the distribution of citations by years. Probably its number of cites is relatively modest (about 25 per year) but it should be remainded that this old paper was conceived and published as a pedagogical paper in an educational-oriented journal. The number of cites grows slowly at the beginning but show a clear increase from the mid-1990s which even is higher in the next years.

Other concrete results (not shown) are the following: a) the two particular years with most cites are 2012 and 2013 with 7.16% and 6.93%, respectively and by countries, the distribution clearly shows that China (with 254 outputs) and USA (246) are

(a) Field: Web of Science Categories	Record Count	% of 909	Bar Chart
ENERGY FUELS	265	29.153 %	
THERMODYNAMICS	240	26.403 %	
PHYSICS MULTIDISCIPLINARY	174	19.142 %	
MECHANICS	155	17.052 %	
PHYSICS APPLIED	119	13.091 %	
ENGINEERING MECHANICAL	116	12.761 %	
PHYSICS MATHEMATICAL	100	11.001 %	
PHYSICS FLUIDS PLASMAS	68	7.481 %	
PHYSICS NUCLEAR	66	7.261 %	
ENGINEERING CHEMICAL	47	5.171 %	
PHYSICS ATOMIC MOLECULAR CHEMICAL	41	4.510 %	
EDUCATION SCIENTIFIC DISCIPLINES	35	3.850 %	
CHEMISTRY PHYSICAL	22	2.420 %	
ENGINEERING MULTIDISCIPLINARY	20	2.200 %	
PHYSICS CONDENSED MATTER	18	1.980 %	
NUCLEAR SCIENCE TECHNOLOGY	18	1.980 %	
MATHEMATICS APPLIED	18	1.980 %	
ENGINEERING ENVIRONMENTAL	12	1.320 %	
COMPUTER SCIENCE INFORMATION SYSTEMS	12	1.320 %	
STATISTICS PROBABILITY	11	1.210 %	
MULTIDISCIPLINARY SCIENCES	11	1.210 %	
MATERIALS SCIENCE MULTIDISCIPLINARY	11	1.210 %	
OPTICS	10	1.100 %	
ENGINEERING ELECTRICAL ELECTRONIC	10	1.100 %	
AUTOMATION CONTROL SYSTEMS	10	1.100 %	

(b) Field: Source Titles	Record Count	% of 909	Bar Chart
PHYSICAL REVIEW E	67	7.371 %	
ENERGY CONVERSION AND MANAGEMENT	65	7.151 %	
JOURNAL OF PHYSICS D APPLIED PHYSICS	52	5.721 %	
JOURNAL OF APPLIED PHYSICS	52	5.721 %	
ENERGY	36	3.960 %	
APPLIED ENERGY	32	3.520 %	
REVISTA MEXICANA DE FISICA	28	3.080 %	
AMERICAN JOURNAL OF PHYSICS	27	2.970 %	
JOURNAL OF NON EQUILIBRIUM THERMODYNAMICS	26	2.860 %	
JOURNAL OF CHEMICAL PHYSICS	26	2.860 %	
JOURNAL OF THE ENERGY INSTITUTE	22	2.420 %	
PHYSICAL REVIEW LETTERS	20	2.200 %	
INTERNATIONAL JOURNAL OF ENERGY RESEARCH	17	1.870 %	
APPLIED THERMAL ENGINEERING	17	1.870 %	
INTERNATIONAL JOURNAL OF HEAT AND MASS TRANSFER	16	1.760 %	
INTERNATIONAL JOURNAL OF THERMAL SCIENCES	14	1.540 %	
EPL	13	1.430 %	
OPEN SYSTEMS INFORMATION DYNAMICS	11	1.210 %	
ENTROPY	11	1.210 %	
RENEWABLE ENERGY	10	1.100 %	

Fig. 4. Snapshot from Web of Science showing the distribution by categories (a) and source titles (b) for the direct cites of the Curzon-Ahlborn paper [1].

the two most active in this search with 28% and 27%, respectively. Mexico (72) and India (54) are the following but at remarkable distance (8% and 6%, respectively). It is surprising that this thermodynamic result has been cited in a great variety of fields, mainly in Physics and Engineering (as expected) but also in Computer Science Information, Ecology, and Biology.

In Fig. 4a we collect the number of cites according to Web of Science categories. The output data are distributed over 60 categories, mainly distributed in the fields of Energy Fuels (265 cites), followed by Thermodynamics (240) and Physics Multidisciplinary (174). In the opposite side, residual cites appear in fields as polymer science and construction building technology. Accordingly, the research outputs were published in more than 200 different journals. The ones with more than 10 articles are shown in Fig. 4b where it can be checked that those of multidisciplinary, applied, and energy character are the most productive. We emphasize once more the pedagogic character of the original $\tilde{\eta}$-derivation. Among the web of science categories appears education scientific disciplines with 33 cites mainly published in two journals (Am. J. Phys. (27) and Eur. J. Phys. (6)). This shows that this topic has been and continues being a relevant issue for teachers and students. By considering the 909 papers which directly cite the CA-paper, we search for the cites among them and get the citation graph with 910 nodes and 5290 links (see Fig. 5).

In order to evaluate the impact of the CA-paper beyond the articles which directly cited it, we expand our dataset to the second generation of cites (those works that cite papers citing the seminal one). This new dataset consists in about 19000 papers and the citations among them and to the papers in the first generation. The distribution of these data along the years are showed in Fig. 3b. The immediate conclusion at sight of this figure is the permanent and fast increasing of the research works associated to the original paper. A detailed inspection of the whole citation with 19910 nodes reveals interesting features of the network topology, see Fig. 6. First, we notice the presence of triangles which are related to the interaction of authors when citing previous works. We also warn about the presence of community structure (defined as groups of nodes in a given network that are more densely linked internally than with the rest of the network) according to the criteria imposed by the algorithm of Blondel et al. [56]. The results are also displayed in Fig. 6, where the color of the nodes represents a group of nodes which are densely connected. An interactive version of this Figure can

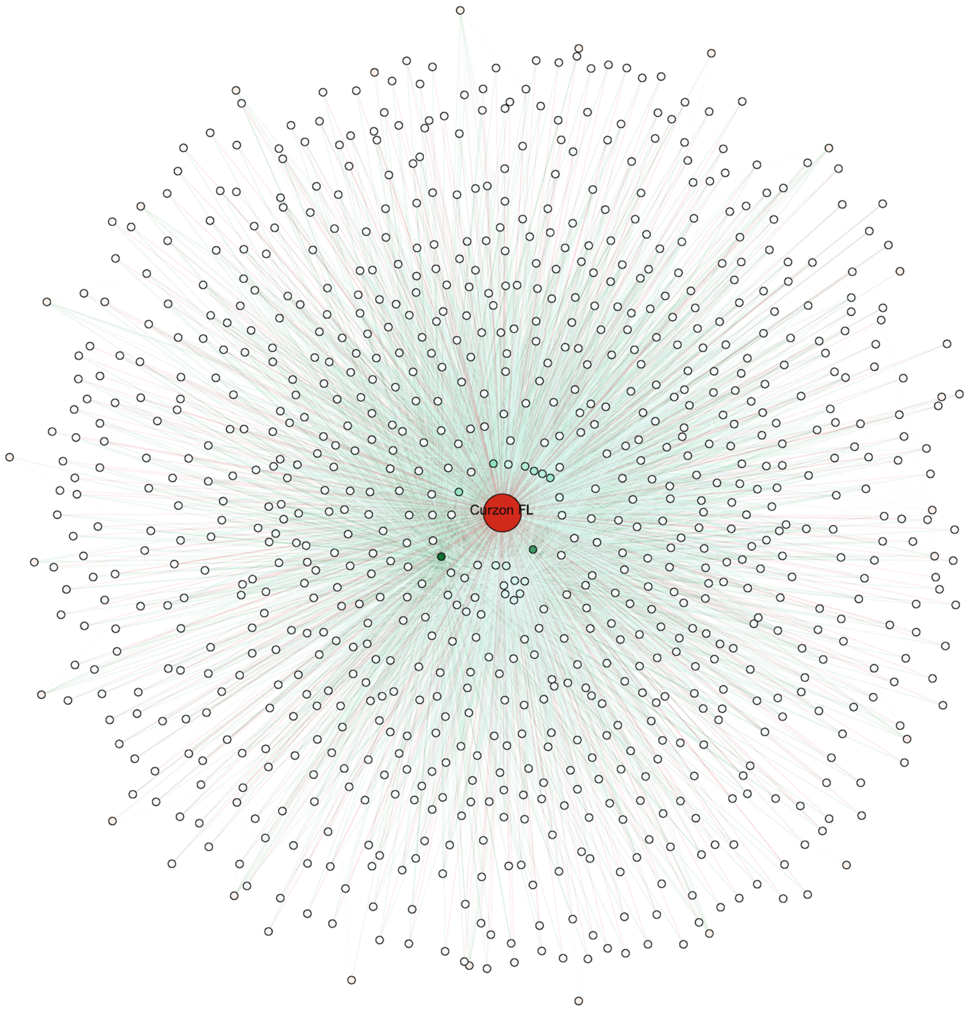

Fig. 5. Citation network of the Curzon-Ahlborn (1975) paper. The graph consists in 910 nodes and 5290 edges. The red node located at the center represents the CA-paper.

be seen at `http://www.cslupiita.com.mx/curzon/`. Quantitative results about the structure and topology of this citation network is in progress.

5 Summary

Efficiency at maximum power, in general, and the $\widetilde{\eta}$ result, in particular, is a useful tool in the optimization of macroscopic, mesoscopic and microscopic energy converters, because of its straightforward relation with the depletion of energy resources and the concerns of sustainable development. Some different analytical derivations have been summarized, both in reversible and irreversible thermodynamic frameworks, making special emphasis on the interplay between educational and research point of views. From it, a broad field of research has emerged with high scientific relevance and influence in pedagogic-oriented journals and thermodynamic textbooks [57–62]. So, probably $\widetilde{\eta}$ could be also stated as a paradigmatic example of feedback between education and research.

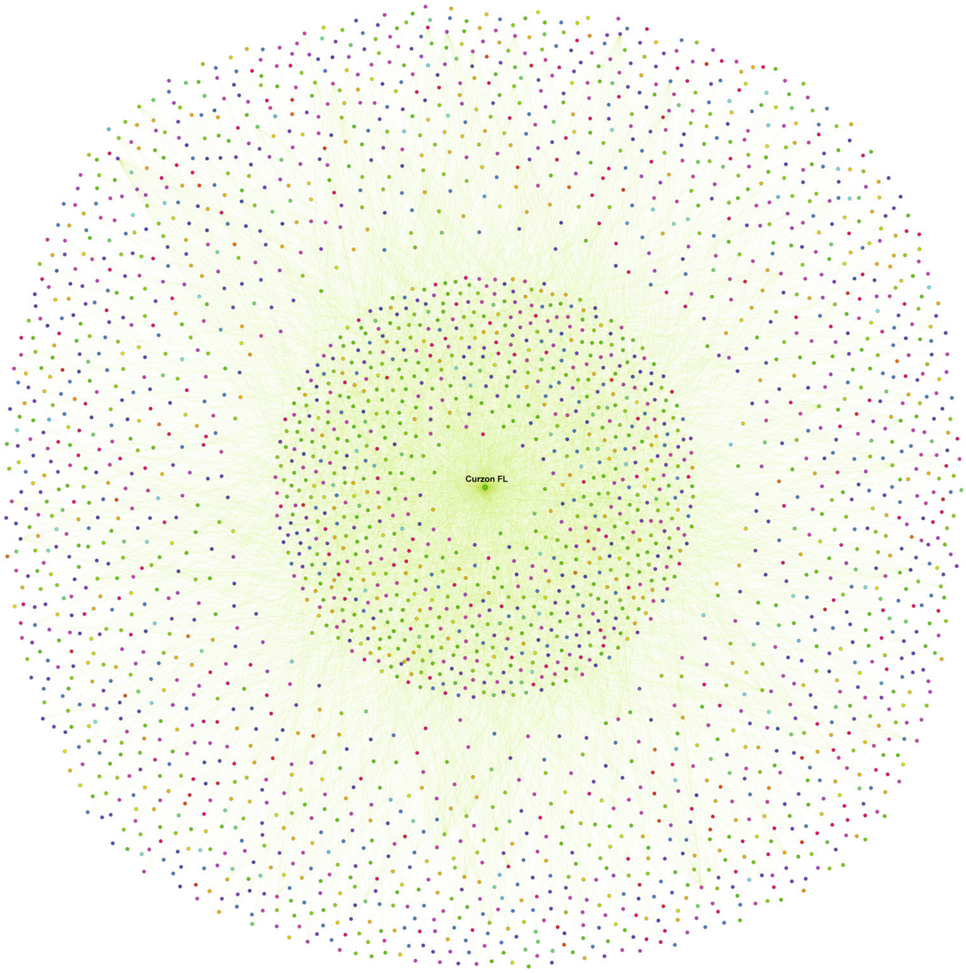

Fig. 6. Complete citation network of the Curzon-Ahlborn (1975) paper. The node at the center represents the CA-paper while the the first and second layer of nodes correspond to first and second generation of cites, respectively.

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Debate. The maximum power efficiency $1-\sqrt{\tau}$: Research, education, and bibliometric relevance by Calvo Hernandez, J.M.M. Roco, A. Medina, S. Velasco and L. Guzman-Vargas

Comments by H. Qian

American theoretical biophysicist T.L. Hill (1917–2014) has developed an extensive mesoscopic nonequilibrium steady state thermodynamics [1] in which the notion of kinetic cycle plays a prominent role. It was later shown mathematically that for any finite-state Markov dynamics with a master equation, the stionary entropy production rate can be expressed in terms of all the cycles [2]

$$\sigma^{ss} = k_B \sum_{\text{all cycle}C} \left(J_C^+ - J_C^-\right) \ln\left(\frac{J_C^+}{J_C^-}\right), \tag{15}$$

in which J_C^+ and J_C^- are the forward and backward fluxes on the cycle $C = \{i_0, i_1, \ldots, i_n, i_0\}$. The equation (15) is very significant since it clearly states that while the term $(J_C^+ - J_C^-)$ is a matter of kinematics of the stochastic dynamics, the essence of nonequilibrium thermodynamics is in the logarithmic term, which has been shown to be independent of the stationary probability:

$$\frac{J_C^+}{J_C^-} = \frac{k_{i_0i_1}k_{i_1i_2}\cdots k_{i_{n-1}i_n}k_{i_ni_0}}{k_{i_1i_0}k_{i_2i_1}\cdots k_{i_ni_{n-1}}k_{i_0i_n}}. \tag{16}$$

More specifically, for a given kinetic cycle C, its entropy production rate

$$\sigma_C = \left(J_C^+ - J_C^-\right)k_B \ln\left(\frac{J_C^+}{J_C^-}\right) \geq 0, \tag{17}$$

is the product of net number of cycles C in the positive direction per unit time, $(J_C^+ - J_C^-)$, and the entropy production per cycle, $k_B \ln(J_C^+/J_C^-)$ [3]. We shall always assume the direct of a cycle is take as $J_C^+ \geq J_C^-$.

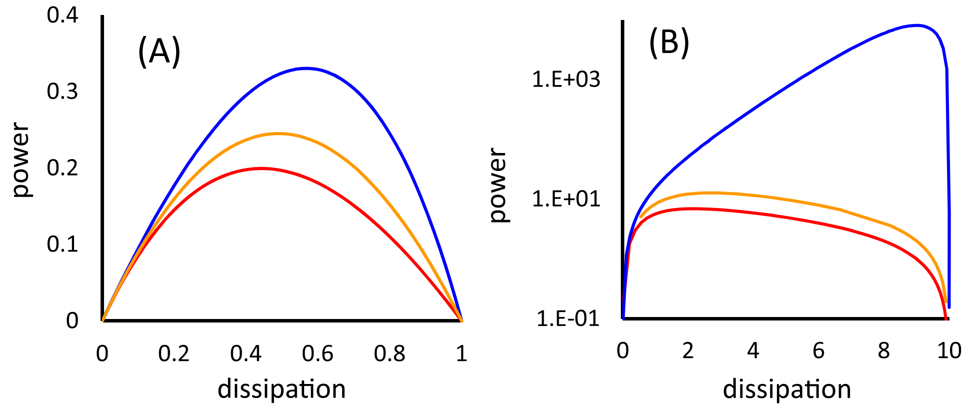

Fig. 1. Work output power P_W/k_BT given in (20) as functions of per cycle dissipation $\ln(J_C^+/J_C^-)$ with fixed $J_C^+ = 1$ and varying J_C^- (red), fixed $J_C^+ + J_C^- = 2$ (orange), and fixed $J_C^- = 1$ and varying J_C^+ (blue). Near zero dissipation, all curves have $J_C^+ \approx J_C^- \approx 1$: this is Onsager's linear irreversible regime. (A) $Q\eta_c/k_BT = 1$; (B) $Q\eta_c/k_BT = 10$.

Esposito and coworkers have applied the master equation stochastic thermodynamic approach to mesoscopic systems, e.g., quantum dots, with multiple heat baths [4]. Then one obaines

$$k_B \ln\left(\frac{J_C^+}{J_C^-}\right) = -\frac{Q}{T_h} + \frac{Q-W}{T_c}, \tag{18}$$

which yields a *per cycle* efficiency,

$$\frac{W}{Q} = 1 - \frac{T_c}{T_h} - k_BT_c \ln\left(\frac{J_C^+}{J_C^-}\right) \le 1 - \frac{T_c}{T_h}. \tag{19}$$

The right-hand-side is the Carnot efficiency η_c. This is reached when $\sigma_C = \ln(J_C^+/J_C^-) = 0$, which implies the output work per unit time, e.g., the power,

$$P_W = \left(J_C^+ - J_C^-\right)W = (J_C^+ - J_C^-)\left\{Q\eta_c - k_BT_c \ln\left(\frac{J_C^+}{J_C^-}\right)\right\}, \tag{20}$$

is actually zero: The Carnot efficiency is only achievable in a quasi-stationary process.

In principle, one can reach any efficiency $\eta_c - \epsilon$ with a power as great as one desires, if one can make J_C^- large: Let the $J_C^+/J_C^- = \exp(\epsilon/k_BT)$, then $J_C^+ - J_C^- = \left(e^{\epsilon/k_BT} - 1\right)J_C^-$. A large J_C^- is meaningful in mesoscopic dynamics, but it is not feasible in the macroscopic world.

The present paper is a nice exposition of the historical attempts to extend Carnot's theory of efficiency. To do so, one needs to introduce a "time scale", as clearly stated in [5]: "In general, the Curzon-Ahlborn engine can be optimized according to a variety of criteria, subject to the constraints of the reservoir temperatures and a fixed time τ for the period of the cycle."

However, Eq. (20) indicates that one can not choose $(J_C^+ - J_C^-)^{-1}$ as the τ: J_C^+ and J_C^- with fixed difference can have ratio as close to 1 as one desires, when both tend infinity. Figure 1 shows that if one holds J_C^+, or $(J_C^+ + J_C^-)$, or J_C^- constant, indeed, there will be an optimal with corresponding efficiency. For low value of $Q\eta_c/k_BT$, the efficiency at maximum power is around 0.5; for large value of $Q\eta_c/k_BT$, the limitation on J_C^+ made the optimal power is almost zero. Note the logarithmic scale in 1B. When one holds $J_C^- = 1$, optimal power has a very large per cycle dissipation.

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Response by A. Calvo Hernandez

We acknowledge your comments and enclosed note on our EPJ ST-paper. Indeed, we find your comments and suggestions very valuables and appropriate. In fact, all of them will be considered in a revised version of the paper.

On the other side, we have read your excellent paper making emphasis in the unified mathematical frame of the diffusion processes. Indeed, these processes play a central role in the stochastic thermodynamics as a suitable tool to study many different properties of mesoscopic systems. Our only minor comment points to the convenience of adding some ideas on the posible applications of these concepts in a more practical side as the optimal paths for efficiency of mesoscopic energy converters. Indeed, this topic is of great value now days. Maybe this could aid a very wide audience of readers involved in thermodynamic optimization of finite-time energy processes.