Coefﬁcient of performance for a low-dissipation Carnot-like refrigerator with nonadiabatic dissipation

Yong Hu,1 Feifei Wu,1 Yongli Ma,2 Jizhou He,1 Jianhui Wang,1,2,* A. Calvo Hernández,3 and J. M. M. Roco3

1Department of Physics, Nanchang University, Nanchang 330031, China
2State Key Laboratory of Surface Physics and Department of Physics, Fudan University, Shanghai 200433, China
3Departamento de Física Aplicada and Instituto Universitario de Física y Matemáticas (IUFFYM), Universidad de Salamanca, 37008 Salamanca, Spain

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We study the coefﬁcient of performance (COP) and its bounds for a Carnot-like refrigerator working between two heat reservoirs at constant temperatures $T_h$ and $T_c$, under two optimization criteria $\chi$ and $\Omega$. In view of the fact that an “adiabatic” process usually takes ﬁnite time and is nonisentropic, the nonadiabatic dissipation and the time required for the adiabatic processes are taken into account by assuming low dissipation. For given optimization criteria, we ﬁnd that the lower and upper bounds of the COP are the same as the corresponding ones obtained from the previous idealized models where any adiabatic process is undergone instantaneously with constant entropy. To describe some particular models with very fast adiabatic transitions, we also consider the inﬂuence of the nonadiabatic dissipation on the bounds of the COP, under the assumption that the irreversible entropy production in the adiabatic process is constant and independent of time. Our theoretical predictions match the observed COPs of real refrigerators more closely than the ones derived in the previous models, providing a strong argument in favor of our approach.

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I. INTRODUCTION

The issue of thermodynamic optimization of cyclic converters has attracted much attention because of sustainable development in relation to any energy converter operation. Concerning this issue, a number of different performance regimes [1–3] have been considered within different ﬁgures of merit to disclose possible universal and uniﬁed features, with special emphasis on the possible consistency between theoretical predictions and experimental data. If heat engines, or refrigerators, work between two heat reservoirs at constant temperatures $T_h$ and $T_c$, in practice they operate far from the ideal maximum Carnot efﬁciency ($\eta_{\text{max}} = \frac{\Delta U}{Q_h})$, or the maximum Carnot coefﬁcient of performance (COP) ($\chi_{\text{max}} = \frac{\Delta U}{Q_c})$, which requires inﬁnite time to complete a cycle. By contrast, the maximum output for heat engines, or the maximum cooling rate for refrigerators, can be achieved within a ﬁnite cycle time. In most studies of Carnot-like heat-engine models, the power output as a target function is always maximized to ﬁnd valuable and simple expressions for the optimized efﬁciency [4–13]. Without assuming any speciﬁc heat transfer law or the linear-response regime, Esposito et al. [11] proposed the low-dissipation assumption that the irreversible entropy production in a heat-exchange process is inversely proportional to the time spent on the corresponding process, and they rederived the paradigmic Curzon-Ahlborn value [14] $\eta_{\text{CA}} = 1 - \sqrt{T_c/T_h}$ in the limit of symmetric dissipation. In addition to the power output, the per-unit-time efﬁciency, a compromise between the efﬁciency and the speed of the whole heat-engine cycle, was considered as another criterion [15] of optimization.

It is more diﬃcult to adopt a suitable optimization criterion and determine its corresponding COP for refrigerators, in comparison with dealing with the issue of the efﬁciency at maximum power for heat engines. Various optimization criteria [16–22] have been proposed in analysis of optimization of a classical or quantum refrigeration cycle. Yan and Chen [16] introduced the function $\chi = \frac{\varepsilon Q_c}{\tau}$, the heat transported from the cold reservoir and $\tau$ the cycle time, as a target function within the context of ﬁnite-time thermodynamics. Velasco et al. [17] adopted the per-unit-time COP as a target function while Allahverdyan et al. [18] introduced $\varepsilon Q_c$ as the target function. de Tomás et al. [20] proved that the COP at maximum $\chi$ for symmetric low-dissipation refrigerators is $\varepsilon_{\text{C}} = \sqrt{\varepsilon_{\text{E}} + T_c} - 1$, where $\varepsilon_{\text{E}} = T_c/(T_h - T_c)$ is the Carnot COP. Based on the $\chi$ ﬁgure of merit, Wang et al. [21] obtained the lower and upper bounds of the COP and showed that these bounds can be achieved in extremely asymmetric dissipation limits. Very recently, de Tomás et al. [19] studied low-dissipation heat devices and obtained the bounds of the COP under general and symmetric conditions, by applying the uniﬁed $\Omega$ optimization criterion, which was ﬁrst proposed in [23] to consider a compromise between energy beneﬁts and losses for a speciﬁc job. This criterion, which takes advantage of being independent of environmental parameters and does not require explicit evaluation of the entropy generation, has been applied to performance optimization for a wide variety of energy converters [24–26].

Most of the previous studies about the performance in ﬁnite time of heat devices did not take into account nonadiabatic dissipation for the cyclic converter, assuming that the adiabatic steps run instantaneously with constant entropy, although the importance of nonadiabatic dissipation in an adiabatic process was suggested by Novikov [27]. The inﬂuence on the performance of a classical or quantum heat engine, induced by internally dissipative dissipation (such as inner friction and internal dynamics, etc.), has been discussed in several papers [28–35]. To the best of our knowledge, so far little attention has been paid to the eﬀects of nonadiabatic dissipation on the
performance characteristics of refrigerators. It is therefore of
significance to consider a more generalized refrigerator model
by involving the nonadiabatic dissipation and the time spent
on adiabatic processes.

In the present paper, we consider a low-dissipation Carnot-
like refrigeration cycle of two irreversible isothermal and
two irreversible adiabatic processes, and analyze its COP
under the $\chi$ and $\Omega$ conditions, respectively. Assuming that
the nonadiabatic dissipation can be described using the
low-dissipation assumption, we show that the inclusion of
nonadiabatic dissipation does not lead to any change in the
bounds of the COP at a given figure of merit. When the
dissipations of the two isothermal and two adiabatic processes
are symmetric, we find that our results agree well with the
data for real refrigerators, thereby indicating that inclusion
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where we have defined $\Delta S \equiv \Delta S_t = S_2 - S_1$. It is clear from Eq. (5) that the variation of entropy $\Delta S (=\Delta S_t)$ during the cold isothermal process is quite different from that $\Delta S_h$ during the hot isothermal process, when the entropy production during any adiabatic process is nonvanishing.

To continue our analysis, we denote by $\Delta S_{\kappa}^{irr} \equiv L_{\kappa}(t_{\kappa})$ [11,21,29,34] with $\kappa = h,c,a,b$ the irreversible entropy production for the optimized protocols. As emphasized, the irreversible entropy production in any adiabatic process $[L_a(t_a) \text{ or } L_b(t_b)]$ cannot be included within the irreversible entropy production in any isothermal process $[L_c(t_c) \text{ or } L_h(t_h)]$, because the irreversible entropy production $L_{\kappa}(t_{\kappa})$ as a function of the time $t_{\kappa}$ depends on the time taken for the corresponding process $\kappa$. Here $L_{\kappa}(t_{\kappa})$ are process variables depending on the detailed protocols.

Considering Eqs. (1), (2), (3), (4), and (5), the heat values $Q_c$ and $Q_h$ are obtained,

$$Q_c = T_c(\Delta S - L_c)$$

and

$$Q_h = T_h(\Delta S + L_a + L_b + L_h).$$

From Eqs. (6) and (7), we find the relation

$$Q_h/T_h - Q_c/T_c = L_a + L_b + L_h + L_c.$$

As a consequence, the work consumed by the system after a single cycle ($W$) and the COP of the refrigeration cycle ($\varepsilon$) are derived as

$$W = Q_h - Q_c = (T_h - T_c)\Delta S + T_h L_h + T_c L_c$$

+ $T_h(L_a + L_b)$

and

$$\varepsilon = \frac{Q_c}{Q_h - Q_c} = \frac{T_c(\Delta S - L_c)}{(T_h - T_c)\Delta S + T_h(L_a + L_b + L_h) + T_c L_c}.\tag{10}$$

The last term in Eq. (9) represents the additional work consumed by the system because of the dissipation in the two adiabatic processes. This additional work to overcome the internally nonadiabatic dissipation is represented by the two blue areas in Fig. 1.

### III. OPTIMIZATION ANALYSIS

In this section we present an optimization analysis of a refrigerator with internal dissipation which accounts for the irreversible entropy production during a nonisentropic adiabatic process (more details about nonisentropic adiabatic processes can be found in Ref. [29]). If the adiabatic processes are assumed to proceed instantaneously with constant entropy, we recall that [19,21] (i) the bounds of the COP under the $\Omega$ criterion, between which there are small differences, are in agreement with the real experimental data within a range of temperatures of the working substance; (ii) under the $\chi$ criterion, the upper bound of the COP fits well with the experimental data, but the COP in the symmetric limit ($\chi_{\kappa} = 0$) seems to be considerably larger than the experimental data. In what follows, our theoretical predictions are expected to agree well with the experimental data. In particular, for the $\chi$ criterion, our theoretical data in the symmetric limit should match more closely with the experimental data than the ones obtained from the previous models without consideration of nonadiabatic dissipation [19].

#### A. COP at maximum $\chi$ figure of merit

In the following, we make the variable transformation $x_\kappa = 1/\tau_\kappa$ ($\kappa = a,b,c,h$) by taking the inverse of time instead of the time itself as a variable, and then we can write the total cycle time $\tau_{cycle}$ as $\tau_{cycle} = 1/x_a + 1/x_b + 1/x_c + 1/x_h$. Substitution of Eqs. (6) and (10) into the $\chi$ figure of merit as the target function leads to

$$\chi = \varepsilon Q_c/\tau_{cycle} = \frac{T_c^2(\Delta S - L_c)^2}{((T_h - T_c)\Delta S + T_h(L_a + L_b + L_h) + T_c L_c)(1/x_a + 1/x_b + 1/x_c + 1/x_h)}.$$

We optimize the target function $\chi$ over the time variables $x_\kappa$ to specify the time spent on any thermodynamic process and also to maximize this figure of merit. Considering $\partial\chi/\partial x_\kappa = 0$ ($\kappa = a,b,c,h$), we find the four following relations:

$$\begin{align*}
(Q_h - Q_c)x_a x_b x_h &= T_h L'_a x_a (2Q_h/Q_c - 1)(x_a x_b x_c + x_a x_b x_h + x_b x_c x_h + x_a x_c x_h), \tag{12} \\
(Q_h - Q_c)x_b x_c x_h &= T_h L'_a x_a (x_a x_b x_c + x_a x_b x_h + x_b x_c x_h + x_a x_c x_h), \tag{13} \\
(Q_h - Q_c)x_a x_c x_h &= T_h L'_h x_a (x_a x_b x_c + x_a x_b x_h + x_b x_c x_h + x_a x_c x_h), \tag{14} \\
(Q_h - Q_c)x_a x_b x_c &= T_h L'_h x_a (x_a x_b x_c + x_a x_b x_h + x_b x_c x_h + x_a x_c x_h). \tag{15}
\end{align*}$$

Here and hereafter we define $L'_\kappa \equiv \frac{\partial L_{\kappa}}{\partial x_\kappa}$ ($\kappa = a,b,c,h$). Dividing Eq. (12) by Eq. (13), Eq. (14), and Eq. (15), respectively, we obtain

$$\begin{align*}
\varepsilon \tau_h T_h L'_a x_a^2 &= (\varepsilon^* + 2)T_h L'_a x_a^2, \tag{16} \\
\varepsilon \tau_h T_h L'_b x_b^2 &= (\varepsilon^* + 2)T_h L'_b x_b^2, \tag{17} \\
\varepsilon \tau_h T_h L'_c x_c^2 &= (\varepsilon^* + 2)T_h L'_c x_c^2, \tag{18}
\end{align*}$$

where $\varepsilon^*$ is the COP under the maximum $\chi$ condition. From Eqs. (16), (17), and (18), we find that the times spent on the four thermodynamic processes are optimally distributed...
describe the nonadiabatic dissipation, assuming $\alpha$ assumes $0 = \Delta x_{k} = \Delta x_{b} = \Delta x_{c}$ with $\Sigma_{c}$ and $\Sigma_{b}$ being the dissipation constants. This assumption is quite plausible for isothermal processes, since the larger the time for completing the isothermal processes, the closer these processes are to quasistatic processes taking infinitely long times. Usually, for a Carnot (or Carnot-like) cycle that works with a classical gas, the time required for completing an adiabatic process should be very long in order for work to be produced during the adiabatic process (for a quantum adiabatic process, the time must be long enough such that the quantum adiabatic theorem can apply [29]). It is therefore indicated that the irreversible entropy production decreases as the time for the process increases, and it tends to be vanishing when the time becomes long enough.

As for isothermal processes, we adopt the low-dissipation assumption for such an adiabatic process [28, 29, 32–34] to describe the nonadiabatic dissipation, assuming $L_{c} = \Sigma_{a}$ and $L_{b} = \Sigma_{b}$ with $\Sigma_{a}$ and $\Sigma_{b}$ being constants of time. This is physically reasonable since the irreversible entropy production ($\Delta x'_{c}$ or $\Delta x'_{b}$) becomes much smaller and is vanishing in the long-time limit. In this case, since $\alpha = 1$, Eq. (23) becomes

$$\varepsilon_{\chi} = \frac{\varepsilon_{C}(\alpha_{1} - 3\alpha_{2} + (\alpha_{1} - 1)\varepsilon_{C} + \sqrt{[\alpha_{1} - 1 - 3\alpha_{2} + (\alpha_{1} - 1)\varepsilon_{C}]^{2} + 8\alpha_{2}(\alpha_{1} - \alpha_{2} + \alpha_{1}\varepsilon_{C})}}}{2(\alpha_{1} - \alpha_{2} + \alpha_{1}\varepsilon_{C})},$$

(24)

where we have used $\alpha_{1} = \frac{L_{c}x_{a} + L_{c}x_{b} + L_{c}x_{c} + L_{c}x_{a}}{L_{c} + L_{c} + L_{c} + L_{c}}$ and $\alpha_{2} = \frac{L_{c}x_{b}}{L_{c} + L_{c}}$.

Now we turn to the low-dissipation case [11] where one assumes $L_{c} = \Sigma_{a}$ and $L_{b} = \Sigma_{b}$ with $\Sigma_{a}$ and $\Sigma_{b}$ being the dissipation constants of time. This is physically reasonable since the irreversible entropy production ($\Delta x_{a}$ or $\Delta x_{b}$) becomes much smaller and is vanishing in the long-time limit. In such a case, since $\alpha = 1$, Eq. (23) becomes

$$\varepsilon_{\chi} = \frac{\varepsilon_{C}(\alpha_{1} - 3\alpha_{2} + (\alpha_{1} - 1)\varepsilon_{C} + \sqrt{[\alpha_{1} - 1 - 3\alpha_{2} + (\alpha_{1} - 1)\varepsilon_{C}]^{2} + 8\alpha_{2}(\alpha_{1} - \alpha_{2} + \alpha_{1}\varepsilon_{C})}}}{2(\alpha_{1} - \alpha_{2} + \alpha_{1}\varepsilon_{C})},$$

(24)

with

$$\alpha_{2} = \frac{\Sigma_{a}x_{b}}{\Sigma_{c}x_{a} + \Sigma_{b}x_{b} + \Sigma_{c}x_{c} + \Sigma_{b}x_{b}}.$$

(25)

The expression for $\alpha_{2}$ is derived from the more general model in which the nonadiabatic dissipation and the time spent on any adiabatic process are involved. Since $0 \leq \alpha_{2} \leq 1$ and $\varepsilon_{C} > 0$, $\varepsilon_{\chi}$ increases monotonically with $\alpha_{2}$, and vice versa. As a result, we rederive the bounds of the COP at the maximum $\chi$ figure of merit [21,36],

$$0 = \varepsilon_{\chi} \leq \varepsilon_{\chi}^{*} \leq \varepsilon_{\chi}^{+} = (\sqrt{9} + 8\varepsilon_{C} - 3)/2,$$

(26)

whether or not the nonadiabatic dissipation in any adiabatic process is considered. It is thus clear that the inclusion of the nonadiabatic dissipation as well as the time taken for the adiabatic process does not change the upper and lower bounds of the COP at the maximum $\chi$ figure of merit. These lower and upper bounds of $\varepsilon_{\chi}^{+}$ are achieved when $\alpha_{2} \to 0$ and $\alpha_{2} \to 1$, respectively. Combination of Eqs. (20) and (25) yields

$$\alpha_{2} = \frac{1}{\sqrt{m}\left(\frac{\varepsilon_{C}}{\Sigma_{a}} + \frac{\varepsilon_{C}}{\Sigma_{b}} + \frac{\varepsilon_{C}}{\Sigma_{c}}\right) + 1},$$

(27)

where $m$ was defined in Eq. (20) and $x_{c} = 1/\tau_{c}$ ($k = a, b, c, h$) has been used. The complete asymmetric limits $\Sigma_{c}/\Sigma_{a} \to 0$ and $\Sigma_{b}/\Sigma_{c} \to \infty$, where $k$ represents $h, a, b$ but not $c$, cause the COP at the maximum $\chi$ merit of figure to approach its upper and lower bounds, $\varepsilon_{\chi}^{+} = 0$ and $\varepsilon_{\chi}^{+} = (\sqrt{9} + 8\varepsilon_{C})/2$.

When the dissipations in the two adiabatic and two isothermal processes are respectively symmetric, we have $\Sigma_{a} = \Sigma_{b} = \Sigma_{c}$ with $\Sigma_{a}$ being the ratio. In such a case we consider three special situations: (i) $r \to 0$. The nonadiabatic dissipations for the two adiabatic processes vanish, while the dissipations during the two isothermal processes are symmetric. Making use of Eq. (24), the Curzon-Ahlborn (CA) COP is recovered, $\varepsilon_{\chi}^{+} = \varepsilon_{CA} = \sqrt{1 + \varepsilon_{C} - 1}$, which is also the upper bound of the COP in this case. (ii) $r \to \infty$. The lower bound of the COP is achieved, $\varepsilon_{\chi}^{+} = 0$. (iii) $r = 1$. The dissipations in the four thermodynamic processes are symmetric. Here $\varepsilon_{\chi}^{+} = \varepsilon_{\chi}^{+}$ ($r = 1$) is defined for convenience, and its value can be obtained numerically based on Eqs. (24) and (27) for any given value of $T_{b}/T_{c}$ (i.e., the value of $\varepsilon_{C}$). At this supersymmetric limit we obtain readily from Eqs. (19) and (20) that the time ratios of $\tau_{k}/\tau_{c}$ ($k = a, b, h$) are $\tau_{c}/\tau_{c} = \sqrt{T_{h}/(mT_{c})}$ with $m = (\varepsilon_{\chi}^{+} + 2)/\varepsilon_{\chi}^{+}$, and that the time allocations for the remaining three processes are equal ($\tau_{a} = \tau_{b} = \tau_{h}$). In Fig. 2(a) we plot the COP $\varepsilon_{\chi}^{+}$ as a function of $\varepsilon_{C}$, comparing $\varepsilon_{CA}$ with the upper bound $\varepsilon_{\chi}^{+}$ of the Carnot-like refrigeration cycle.

For an adiabatic process of some heat devices [12], in the sudden limit there is no time for relaxation and heat...
losses to the environment can be minimized. To describe such an adiabatic process for these special systems, we assume that the entropy production in an adiabatic process with vanishing time (τc = 1/xa → 0 and τh = 1/xb → 0) is constant and independent of time, while the low-dissipation assumption holds well for an isothermal process. That is, we set \( L_a = \Sigma_a \) and \( L_b = \Sigma_b \) with \( \Sigma_a \) and \( \Sigma_b \) being constants independent of time, and we have \( L_a' = L_b' = 0 \). Then the expressions for \( \alpha_1 \) and \( \alpha_2 \) below Eq. (23) become \( \alpha_1 = \Sigma_x/x_a + \Sigma_{x,x} + \Sigma_{x,x+x_b} \) and \( \alpha_2 = \Sigma_x/x_a + \Sigma_{x,x} + \Sigma_{x,x+x_b} \). For the strong-nonadiabatic-dissipation limit, \( \Sigma_a \rightarrow \infty \) or \( \Sigma_b \rightarrow \infty \), we find from Eq. (22) that the COP at the maximum figure of merit, \( \epsilon_x^* \), becomes vanishing, since \( \alpha_1 \) as well as \( \alpha_2 \) tends to zero. On the contrary, when \( \Sigma_a = 0 \) and \( \Sigma_b = 0 \), our result is reduced to the idealized nonadiabatic dissipation model, as discussed above. Therefore, if the entropy production in the adiabatic process is constant and independent of time, the lower and upper bounds of \( \epsilon_x^* \) are also given by Eq. (26) which was derived under the assumption that \( L_x = \Sigma_x x_x \) with \( \kappa = a,b,c,h \).

**B. COP at maximum \( \Omega \) figure of merit**

The \( \Omega \) criterion, a trade-off between maximum cooling and lost cooling loads, is defined as \( \Omega = (2\epsilon - \epsilon_{\text{max}})W \) [23]. The target function \( \Omega = (2\epsilon - \epsilon_{\text{max}})W \) can be expressed as

\[
\Omega = [2Q_c - \epsilon_c(Q_h - Q_c)] \frac{x_a x_b x_c}{x_h},
\]

where we have made the variable transformation \( x_h = 1/\tau_c \) (\( \kappa = h,c,a,b \)). Setting the derivatives of \( \Omega \) with respect to \( x_x \) (\( \kappa = h,c,a,b \)) equal to zero, we derive the optimal equations

\[
[2Q_c - \epsilon_c(Q_h - Q_c)] x_{a,b,c,h} = T_h L_a' e_c x_{a,b,c,h} + x_b x_c x_h + x_c x_h x_a + x_a x_h x_b,
\]

and

\[
[2Q_c - \epsilon_c(Q_h - Q_c)] x_{a,b,c,h} = T_h L_b' e_c x_{a,b,c,h} + x_a x_b x_c + x_a x_h x_b + x_b x_h x_a,
\]

(29)

(30)

It follows, by substitution of \( \tau_c = 1/x_c(\kappa = h,c,a,b) \) into Eqs. (33), (34), and (35), that the optimal ratios of the time \( \tau_c/\tau_\kappa(\kappa = a,b) \) as well as \( \tau_\kappa/\tau_a \) are still given by Eq. (19), but that under the \( \Omega \) criterion the time ratio \( \tau_c/\tau_\kappa(\kappa = a,b) \) becomes

\[
\frac{\tau_c}{\tau_\kappa} = \sqrt{L_\kappa'(1 + \epsilon_c)/L_\kappa'(2 + \epsilon_c)} \quad (\kappa = a,b,h).
\]

(36)

Directly adding both sides of Eqs. (29), (30), and (31), we obtain

\[
12Q_c - \epsilon_c(Q_h - Q_c) \left[ \frac{x_{a,b,c,h}}{x_h} \right] = T_h L_a' e_c x_{a,b,c,h} + x_a x_b x_c + x_a x_h x_b + x_b x_h x_a + x_a x_h x_b,
\]

(37)

Substitution of Eqs. (6) and (7) into Eq. (37) leads to

\[
\Delta S = (2 + \epsilon_c)(L_c + L_c') + (1 + \epsilon_c)(L_a + L_b + L_h + L_h' x_a + L_h' x_b + L_h' x_b).
\]

(38)
It follows, after substituting Eq. (38) into Eq. (10), that the COP for the maximum $\Omega$ condition is

$$\varepsilon_\Omega^* = \frac{(2 + \varepsilon_C)L_x + (1 + \varepsilon_C)(L_a + L_b + L_h + L'_x x_a + L'_h x_a + L'_h x_h + L_c) + (1 + \varepsilon_C)(L'_a x_a + L'_h x_h + L'_h x_h)\varepsilon_C}{(2 + \varepsilon_C)L_x + (1 + \varepsilon_C)(L_a + L_b + L_h + L_c) + (1 + \varepsilon_C)(L'_a x_a + L'_h x_h + L'_h x_h)\varepsilon_C}. \quad (39)$$

Assuming that the irreversible entropy production for an isothermal or an adiabatic step is inversely proportional to the time for completing that process, i.e., $L_x = \Sigma_{\varepsilon_c}(\kappa = a, b, c, h)$, Eq. (39) becomes

$$\varepsilon_\Omega^* = \frac{3 + 2\varepsilon_C + 2\gamma}{4 + 3\varepsilon_C + 3\varepsilon_C\gamma}, \quad (40)$$

where

$$\gamma = \sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)\Sigma_{\varepsilon_C}/\Sigma_{\varepsilon_C}} + \sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)\Sigma_{\varepsilon_C}/\Sigma_{\varepsilon_C}},$$

which simplifies to $\gamma = \sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)\Sigma_{\varepsilon_C}/\Sigma_{\varepsilon_C}}$ in the ideal adiabatic refrigeration cycle, with use of Eqs. (33), (34), and (35). The value of $\gamma$ is a non-negative number, varying from 0 to $\infty$. Hence, the COP at the maximum $\Omega$ figure of merit, $\varepsilon_\Omega^*$, must be situated in the range

$$\frac{3 + 2\varepsilon_C}{4 + 3\varepsilon_C + 3\varepsilon_C\gamma} \leq \varepsilon_\Omega^* \leq \frac{3 + 2\varepsilon_C}{4 + 3\varepsilon_C + 3\varepsilon_C\gamma}. \quad (41)$$

The upper and lower bounds for the optimized COP at the maximum $\Omega$ figure of merit, $\varepsilon_\Omega^*$ and $\varepsilon_\Omega^+$, versus the Carnot COP $\varepsilon_C$, are plotted in Fig. 2(b).

As in the case of the $\chi$ figure of merit, the expression for the COP at the maximum $\Omega$ condition is similar to the corresponding one obtained in the model [19] with idealized adiabatic processes, and the internally nonadiabatic dissipation has no influence on the bounds of the COP. The here the optimal value of the COP, however, represents a broader context by including the nonadiabatic dissipation and the time required for completing any adiabat.

If the dissipations of the two adiabatic and two isothermal processes are respectively symmetric, i.e., $\Sigma_{\varepsilon_a} = \Sigma_{\varepsilon_h} = r\Sigma_{\varepsilon_b}$, then $\gamma = (2/\sqrt{r} + 1)(\sqrt{1 + \varepsilon_C}(2 + \varepsilon_C))$, and Eq. (39) becomes

$$\varepsilon_\Omega^*(r) = \frac{3 + 2\varepsilon_C + 4(2\varepsilon_C)\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)} + 6(2\varepsilon_C)\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)}\varepsilon_C}{4 + 3\varepsilon_C + 3\varepsilon_C\gamma}. \quad (42)$$

From Eq. (42), we find in such a case that the bounds of the COP for the maximum $\Omega$ figure of merit are achieved, $\frac{1}{2}\varepsilon_C \leq \varepsilon_\Omega^*(r) \leq \frac{3 + 2\varepsilon_C + 4(2\varepsilon_C)\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)} + 6(2\varepsilon_C)\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)}\varepsilon_C}{4 + 3\varepsilon_C + 3\varepsilon_C\gamma}$, when $r \rightarrow \infty$ and $r \rightarrow 0$, respectively. In the particular case when the dissipations of the four thermodynamic processes are symmetric, the COP can be obtained by the use of $r = 1$,

$$\varepsilon_\Omega(r = 1) = \frac{3 + 2\varepsilon_C + 6\varepsilon_C(1 + \varepsilon_C)(2 + \varepsilon_C)}{4 + 3\varepsilon_C + 9\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)}\varepsilon_C}. \quad (43)$$

Then the optimal time ratio of $\tau_a/\tau_c(\kappa = h, a, b)$ in Eq. (36) simplifies to $T_a/T_c = \sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)}(1 + \varepsilon_C)$ in this supersymmetric case, while the optimized times spent on the other three processes are equal ($\tau_b = \tau_h = \tau_h$). At the supersymmetric limit, the time ratios of $\tau_a/\tau_c$ with $\kappa = h, a, b$ and $\nu = h, c$ as functions of the Carnot COP $\varepsilon_C$, under the $\Omega$ and $\chi$ criteria, are plotted in Fig. 3 by using Eqs. (19), (20), and (36). Figure 3 shows that, whether under the $\chi$ or $\Omega$ criterion, the time taken for the cold isothermal process is larger than the one for the other three processes, on which the times spent are equal to each other. This result is in contrast to the fact that, for an irreversible heat engine [29], the hot isothermal process proceeds most slowly during a cycle, with equal times required for completing the cold isothermal and two adiabatic processes. This is not surprising, since the heat is transported into the system during the cold (hot) isothermal process for the refrigerator (heat engine), and the additional heat developed by the nonadiabatic dissipation is related to the high temperature $T_b$ (low temperature $T_a$) for the refrigerator (heat engine).

If an adiabatic process is ideal and thus isentropic ($L_a = L_h = 0$), for low-dissipation refrigerators with $L_h' = \Sigma_h$ and $L_h' = \Sigma_h$, the symmetric limit ($\Sigma_c = \Sigma_h$) gives rise to the following form of Eq. (39):

$$\varepsilon_\Omega^* = \frac{3 + 2\varepsilon_C + 2\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)} + 3(1 + \varepsilon_C)(2 + \varepsilon_C)\varepsilon_C}{4 + 3\varepsilon_C + 3\gamma}, \quad (44)$$

which can be simplified as

$$\varepsilon_\Omega^* = \frac{3 + 2\varepsilon_C + 2\sqrt{(1 + \varepsilon_C)(2 + \varepsilon_C)} + 3(1 + \varepsilon_C)(2 + \varepsilon_C)\varepsilon_C}{4 + 3\varepsilon_C + 3\gamma}.$$
Carnot-like refrigerators with the same optimization criterion but within the finite-time thermodynamics context and under the endoreversibility assumption [23]. At the symmetric limits (either with or without nonadiabatic dissipation) for the optimal COPs, $\epsilon_{\Omega}^{\pm}$ determined according to Eq. (43) and $\epsilon_{\Omega}^{\Sigma_{\chi}}$, given by Eq. (45), are also shown in Fig. 2(b). It is clear from Fig. 2(b) that the nonadiabatic dissipation leads to a very slight decrease in the COP.

The irreversible entropy production for the instantaneous adiabatic process in which $t_{a} = 1/x_{a} \rightarrow 0$ and $t_{b} = 1/x_{b} \rightarrow 0$ may be assumed to be constant and independent of time, while low dissipation is adopted to describe the irreversible isothermal process, because of the fact pointed out in Sec. III A. That is, $L_{a} = \Sigma_{a}$ and $L_{b} = \Sigma_{b}$, while $L_{c} = \Sigma_{c}$, and $L_{h} = \Sigma_{h}$, with $\Sigma_{k} (k = a, b, c, h)$ being constants independent of time. Substituting these relations into Eq. (39), we find that the value of $\epsilon_{C}/2$ as the lower bound of $\epsilon_{\Omega}^{\pm}$ is achieved when $\Sigma_{a} \rightarrow \infty$ or $\Sigma_{b} \rightarrow \infty$. Thus the values of $\epsilon_{\Omega}^{\Sigma_{\chi}}$ are situated in the range $\epsilon_{C}/2 \leq \epsilon_{\Omega}^{\pm} \leq \epsilon_{\Omega}^{\ast}$, where the upper bound $\epsilon_{\Omega}^{\ast}$, defined by Eq. (41), can be achieved for the two isentropic adiabatic processes ($\Sigma_{a} = \Sigma_{b} = 0$).

Note that even the lower bound of COP under the $\Omega$ criterion is finite and considerably larger than the upper bound obtained under the $\chi$ optimization criterion, as shown in Fig. 4. It is therefore indicated that, in comparison with the $\chi$ criterion, the objective function $\Omega$ can be adopted as one guide to design more efficient refrigerators.

C. Comparison between our prediction and experimental data

It would be instructive to compare our theoretical predictions with the observed COPs of some real refrigerators. Our theoretical prediction versus the data for real refrigerators [37] at different values of temperature are plotted in Fig. 4, which shows that the theoretical results agree well with the experimental refrigerator data, whether at maximum $\chi$ or maximum $\Omega$ figure of merit. In the case when the low-dissipation assumption is valid for the isothermal as well as the adiabatic processes, applying the $\Omega$ criterion to optimization of the refrigerator cycle, we find that there are relatively small differences even between the lower and upper bounds ($\epsilon_{\Omega}^{\pm}$ and $\epsilon_{\Omega}^{\ast}$) of the COP for the refrigerator cycle. If the entropy production in the adiabatic process is assumed to be constant and independent of time, while the low-dissipation assumption is valid for the two isothermal processes, the bounds of $\epsilon_{\Omega}^{\pm}$ are given by $\epsilon_{\Omega}^{\Sigma_{\chi}}$, as expected. Hence, our result suggests that internally nonadiabatic dissipation indeed affects the performance of heat devices and thus cannot be considered negligible in comparisons with the experimental data.
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APPENDIX: DERIVATION OF EQUATION (22)

Using Eq. (8), we can rewrite Eq. (21) as

\[
\frac{(Q_h - Q_c)(L_a + L_b + L_h + L_c)}{Q_h/T_h - Q_c/T_c} = 2Q_h T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h) \tag{A1}
\]

or

\[
\frac{T_h T_c (Q_h - Q_c)(L_a + L_b + L_h + L_c)}{Q_h T_c - Q_c T_h} = 2Q_h T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h). \tag{A2}
\]

Equation (A2), together with \(\varepsilon^*_x = \frac{T_h}{T_c} \varepsilon_c\) and \(\varepsilon_C = \frac{T_h}{T_c} \varepsilon_c\), can be used to derive Eq. (A3):

\[
\frac{1}{\varepsilon^*_x} - \frac{1}{\varepsilon_C} = \frac{T_h (L_a + L_b + L_c) (Q_h - Q_c)}{2T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h)}
\]

\[
= \frac{1}{2T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h)}
\]

\[
\frac{1}{2T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h)}
\]

\[
= \frac{1}{T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h)}
\]

After some simple reshuffling, Eq. (A3) can be expressed in the form

\[
\frac{1}{\varepsilon^*_x} - \frac{1}{\varepsilon_C} = \frac{1}{(T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h))}
\]

\[
= \frac{(2T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h))}{(T_h L'_c x_c - T_c L'_c x_c Q_c + Q_c T_h (L'_a x_a + L'_b x_b + L'_h x_h))}
\]

\[
= \frac{1}{\varepsilon^*_x} - \frac{1}{\varepsilon_C} + \frac{1}{\varepsilon_C} \frac{L'_c x_c}{(L'_a x_a + L'_b x_b + L'_h x_h)}.
\tag{A4}
\]