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On the finiteness of differential invariants

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Abstract

In this paper we show how Weil's theory of near points yields a new light on the classical approaches to the study of the differential invariants of a sheaf of tangent vector fields. We give conditions for the existence of invariant derivations for a sheaf of tangent vector fields, which allows to apply Lie's algorithm to obtain new differential invariants as quotients of Jacobian determinants of known ones. We give sufficient conditions for the asymptotic stability of the symbol of a sheaf of tangent vector fields and prove our main result, a finiteness theorem for the differential invariants of a sheaf of Lie algebras which simplifies and improves on the treatment given in J. Differential Geom. 10 (1975) 249–416.

Keywords: Jet; Formal derivation; Differential invariant; Invariant derivation; Finiteness theorem

1. Introduction

A jet of a smooth manifold M can be defined as an ideal $\mathfrak{p}$ of $C^\infty(M)$; namely, the jets of M are the kernels of the Weil A-points of M, where A is a Weil algebra. The advantage of this definition is that prolongations, tangent structures, and all related processes can be defined in terms of the ring of smooth functions of M, which makes them much more simple and natural. This presentation of the theory of jets was made in [18] for the classical higher order Grassmann bundles and in [1,10,14] for the general case.

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As tests to investigate how this point of view can be used in classical problems which have been treated usually from Ehresman's theory of jets, we applied it to some topics such as Lie pseudogroups [11,12], formal integrability [13] and the problem of the reduction of some systems of partial differential equations to first order systems with only one unknown function [3]. The present paper is another test of our theory, and the main result is a finiteness theorem for the differential invariants of a sheaf of Lie algebras (Theorem 19). Our result simplifies and improves on the corresponding theorem in [4].

In order to fix the notations we begin by summarizing, without proofs, some of the relevant material on [10].

A *Weil algebra* is a finite-dimensional local rational $\mathbb{R}$-algebra. For example,

$$\mathbb{R}_m^\ell = \mathbb{R}[x_1, \dots, x_m]/(x_1, \dots, x_m)^{\ell+1},$$

the ring of Taylor expansions in m variables, truncated at the order ℓ, is a Weil algebra.

Let A be a Weil algebra; an A-point of the smooth manifold M is a morphism of $\mathbb{R}$-algebras $p^A : C^\infty(M) \to A$ [19]. The set of all A-points of M is a manifold M^A. The tangent space to M^A at p^A can be identified canonically with the set of derivations from $C^\infty(M)$ into A, where A is a $C^\infty(M)$-module via the morphism p^A.

When p^A is an epimorphism, it is said to be *proper* or *regular*; the set of all the regular A-points of M is an open subset $\check{M}^A$ of M^A.

The kernel of each morphism $p^A \in \check{M}^A$ is an ideal $\mathfrak{p}^A$ of $C^\infty(M)$, called the *jet* of p^A; the set of all these kernels is a new manifold, $J^A M$. There is a canonical projection $\ker : \check{M}^A \to J^A M$ which makes of $\check{M}^A$ a principal fibre bundle over $J^A M$ whose structure group is $\mathrm{Aut}(A)$ (see [1]).

In this paper we restrict ourselves to Weil algebras of the type $A = \mathbb{R}_m^\ell$; we will write M_m^ℓ instead of $M^{\mathbb{R}_m^\ell}$, and $J_m^\ell M$ for the corresponding space of jets. In order to simplify the writing, we will suppose that M is fibred over an m-dimensional manifold X, and we will replace the whole jet space $J_m^\ell M$ by the open subset $J^\ell M$ of the "jets of sections" of the fibre bundle $\pi : M \to X$. The cases $\ell = 0$ and $\ell = -1$ will correspond to M and X, respectively. If $\ell \geqslant r$, $\pi_r^\ell : J^\ell M \to J^r M$ is the natural projection, and the ℓth source projection will be denoted by π^ℓ. If $\mathfrak{p}^\ell \in J^\ell M$, for each $r \leqslant \ell$ we will write $\mathfrak{p}^r$ instead of $\pi_r^\ell(\mathfrak{p}^\ell)$, when no confusion can arise.

In [10] we show that each jet $\mathfrak{p}^\ell \in J^\ell M$ can be understood as an algebra homomorphism from $C^\infty(M)$ onto $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$, where $x = \pi^\ell(\mathfrak{p}^\ell)$, whose restriction to $C^\infty(X)$ is the natural homomorphism onto $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$. There is a canonical bijection between such morphisms and $J^\ell M$. Thus, the Taylor imbedding $J^{\ell+r} M \to J^r(J^\ell M)$ allows us to consider each jet $\mathfrak{p}^{\ell+r} \in J^r(J^\ell M)$ as a morphism from $C^\infty(J^\ell M)$ onto $C^\infty(X)/\mathfrak{m}_x^{r+1}$; this fact will be used along the whole paper. We will use the same notation $\mathfrak{p}^\ell$ when we understand this jet as an ideal of $C^\infty(M)$, namely, the kernel of the algebra homomorphism $\mathfrak{p}^\ell : C^\infty(M) \to C^\infty(X)/\mathfrak{m}_x^{\ell+1}$.

A jet $\mathfrak{p}^\ell \in J^\ell M$ is *regular with respect to a subring* $\mathcal{A}$ of $C^\infty(M)$ if its restriction to $\mathcal{A}$ remains surjective; obviously, each jet is regular with respect to $C^\infty(X)$.

Let $\mathfrak{p}^\ell \in J^\ell M$; the kernel of the projection $T_{\mathfrak{p}^\ell} J^\ell M \to T_x X$ induced by the source map is called *vertical tangent space* and denoted by $V_{\mathfrak{p}^\ell} J^\ell M$. In [10] this space is canonically identified with the $C^\infty(M)$-module

$$\mathrm{Der}_{C^\infty(X)}\big(C^\infty(M), C^\infty(X)/\mathfrak{m}_x^{\ell+1}\big);$$

the space $Q_{\mathfrak{p}^\ell} J^\ell M$, kernel of the projection $V_{\mathfrak{p}^\ell} J^\ell M \to V_{\mathfrak{p}^{\ell-1}} J^{\ell-1} M$, is canonically isomorphic to the submodule of the derivations with values in $\mathfrak{m}_x^\ell / \mathfrak{m}_x^{\ell+1}$; from here in a straightforward way follows the isomorphism

$$Q_{\mathfrak{p}^\ell} J^\ell M \approx S^\ell T_x^*(X) \otimes_{\mathbb{R}} V_{\mathfrak{p}^0} M.$$

2. Formal derivations

Definition 1. By a formal derivation over a fibre bundle $\pi : M \to X$ we mean an $\mathbb{R}$-derivation from $C^\infty(X)$ into $C^\infty(J^\ell M)$, $\ell \geqslant -1$.

Each $\mathbb{R}$-derivation $\Phi : C^\infty(X) \to C^\infty(J^\ell M)$ can be understood as a smooth map $\tilde{\Phi} : J^\ell M \to TX$ whose value at $\mathfrak{p}^\ell \in J^\ell M$ is the vector $\tilde{\Phi}(\mathfrak{p}^\ell) \in T_{\pi(\mathfrak{p}^0)} X$ such that

$$\big(\tilde{\Phi}(\mathfrak{p}^\ell)\big) g = (\Phi g)(\mathfrak{p}^\ell)$$

for each function $g \in C^\infty(X)$.

Given a formal derivation $\Phi : C^\infty(X) \to C^\infty(J^\ell M)$, for each $s \geqslant -1$ we can define a mapping

$$\Phi^{(\ell+s+1)} : C^\infty(J^{\ell+s} M) \to C^\infty(J^{\ell+s+1} M)$$

by

$$(\Phi^{(\ell+s+1)} f)(\mathfrak{p}^{\ell+s+1}) = \tilde{\Phi}(\mathfrak{p}^\ell)\big(f(\mathfrak{p}^{\ell+s+1})\big)$$

for each $f \in C^\infty(J^{\ell+s} M)$, where $\mathfrak{p}^{\ell+s+1}$ is considered as an algebra homomorphism from $C^\infty(J^{\ell+s} M)$ onto $C^\infty(X)/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$ via the Taylor imbedding $J^{\ell+s+1} M \hookrightarrow J^1(J^{\ell+s} M)$ and $\tilde{\Phi}(\mathfrak{p}^\ell)$ as a derivation from $C^\infty(X)/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$ into $\mathbb{R}$. Each formal derivation $\Phi \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ defines, in this way, a derivation from $C^\infty(J^{\ell+s} M)$ into $C^\infty(J^{\ell+s+1} M)$.

For each jet $\mathfrak{p}^{\ell+s+1} \in J^{\ell+s+1} M$ we denote by $\Phi_{\mathfrak{p}^{\ell+s+1}}$ the vector of the tangent space $T_{\mathfrak{p}^{\ell+s}} J^{\ell+s} M$ defined as

$$\Phi_{\mathfrak{p}^{\ell+s+1}} = \tilde{\Phi}(\mathfrak{p}^\ell) \circ \mathfrak{p}^{\ell+s+1},$$

where the jet $\mathfrak{p}^{\ell+s+1}$ is considered as a morphism from $C^\infty(J^{\ell+s} M)$ onto $C^\infty(X)/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$ via the Taylor imbedding. It is obvious that for each $k \leqslant s$ the surjective submersion $\pi^{\ell+s}_{\ell+k} : J^{\ell+s} M \to J^{\ell+k} M$ maps $\Phi_{\mathfrak{p}^{\ell+s+1}}$ into $\Phi_{\mathfrak{p}^{\ell+k+1}}$, where $\mathfrak{p}^{\ell+k+1} = \pi^{\ell+s+1}_{\ell+k+1}(\mathfrak{p}^{\ell+s+1})$.

Remark 2. We will use the same notation Φ for the derivations $\Phi^{(\ell+s+1)}$; this way we simplify the writing. The context will show which one of those mappings we deal with.

Remark 3. For each $\mathfrak{p}^{\ell+1} \in J^{\ell+1} M$ the mapping

$$C^\infty(J^\ell M) \to C^\infty(X)/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}, \quad f \to f(\mathfrak{p}^{\ell+1}) - f(\mathfrak{p}^\ell),$$

is a derivation with values in $\mathfrak{m}_{\pi(\mathfrak{p}^0)}/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$.

Let U' be an open subset of X and $U=\pi^{-1}(U')$, with coordinates $y_1,\dots,y_m\in C^\infty(X)$, $y_1,\dots,y_m,y_{m+1},\dots,y_n\in C^\infty(M)$, respectively. The partial derivatives

$$\frac{\partial}{\partial y_i}: C^\infty(U')\to C^\infty(U'),\quad i=1,\dots,m,$$

span $\mathrm{Der}_{\mathbb{R}}(C^\infty(U'),C^\infty(J^{\ell+1}(U',U)))$; when considered as derivations from $C^\infty(J^\ell(U',U))$ into $C^\infty(J^{\ell+1}(U',U))$ they will be denoted by $\partial_1^{(\ell+1)},\dots,\partial_m^{(\ell+1)}$, as usual. Thus,

$$\big(\partial_i^{(\ell+1)}f\big)(\mathfrak{p}^{\ell+1})=\frac{\partial}{\partial y_i}\big(f(\mathfrak{p}^{\ell+1})\big),\quad i=1,\dots,m,$$

for each function $f\in C^\infty(J^\ell M)$. Hence the value of f at $\mathfrak{p}^{\ell+1}$ can be written as

$$f(\mathfrak{p}^{\ell+1})=f(\mathfrak{p}^\ell)+\sum_{i=1}^m\big(\partial_i^{(\ell+1)}f\big)(\mathfrak{p}^{\ell+1})\big(y_i(\mathfrak{p}^{\ell+1})-y_{i0}(\mathfrak{p}^\ell)\big),$$

where $y_{i0}(\mathfrak{p}^\ell)=y_i(\pi(\mathfrak{p}^0))\in\mathbb{R}$ $(i=1,\dots,m)$, that is to say, each y_{i0} is y_i itself, considered as belonging to $C^\infty(J^{\ell+1}M)$ via $(\pi^{\ell+1})^*$. In a similar way, we will use the notation f_0 for each function $f\in C^\infty(J^\ell M)$ when it is considered as belonging to $C^\infty(J^{\ell+1}M)$ via $(\pi_\ell^{\ell+1})^*$.

Proposition 4. *The set* $\mathrm{Der}_{\mathbb{R}}(C^\infty(X),C^\infty(J^{\ell+1}M))$ *is a locally free module over* $C^\infty(J^{\ell+1}M)$ *whose rank equals* $m=\dim X$. *If* $\Phi_1,\dots,\Phi_m$ *is a local basis, the prolongation of the ideal* (f) *of the ring* $C^\infty(J^\ell M)$ *to* $C^\infty(J^{\ell+1}M)$ *is locally generated by* $f_0,\Phi_1 f,\dots,\Phi_m f$, *where* $f_0=(\pi_\ell^{\ell+1})^*(f)$.

Proof. In the notations of the above remark, if $\Phi_1\dots\Phi_m$ is a local basis of the module $\mathrm{Der}_{\mathbb{R}}(C^\infty(X),C^\infty(J^{\ell+1}M))$, then

$$\frac{\partial}{\partial y_i}=\sum_{j=1}^m b_{ij}\Phi_j,\quad i=1,\dots,m,\ b_{ij}\in C^\infty(J^{\ell+1}M).$$

The prolongation of the ideal (f) of $C^\infty(J^\ell M)$ to $C^\infty(J^{\ell+1}M)$ is locally generated by $f_0,\partial_1^{(\ell+1)}f,\dots,\partial_m^{(\ell+1)}f$; it is clear that $f_0,\Phi_1 f,\dots,\Phi_m f$ is another generator system for this prolongation. $\square$

Definition 5. If $\Phi_1\dots\Phi_m$ is a local basis of $\mathrm{Der}_{\mathbb{R}}(C^\infty(X),C^\infty(J^{\ell+1}M))$ and $f\in C^\infty(J^\ell M)$, the functions $f_0,\Phi_1 f,\dots,\Phi_m f$ are called the *real components* of the prolongation of f to $C^\infty(J^{\ell+1}M)$ with respect to $\Phi_1,\dots,\Phi_m$.

Proposition 6. *A jet* $\mathfrak{p}^{\ell+1}\in J^{\ell+1}M$ *is regular over the subring* $\mathbb{R}[f_1,\dots,f_m]$ *of* $C^\infty(J^\ell M)$ *if and only if for any coordinate system* $y_1,\dots,y_m\in C^\infty(X)$ *on an open neighborhood of* $\pi(\mathfrak{p}^0)$ *the following condition holds*:

$$\det\big(\partial_i^{(\ell+1)}f_j\big)(\mathfrak{p}^{\ell+1})\neq 0.$$

Proof. When we consider the jet $\mathfrak{p}^{\ell+1}$ as a homomorphism from $C^\infty(J^\ell M)$ onto $C^\infty(X)/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$, the image of each function $f_1, \dots, f_m$ can be written in the form

$$f_j(\mathfrak{p}^{\ell+1}) = f_j(\mathfrak{p}^\ell) + \sum_{i=1}^m \bigl(\partial_i^{(\ell+1)} f_j\bigr)(\mathfrak{p}^{\ell+1})\bigl(y_i(\mathfrak{p}^{\ell+1}) - y_{i0}(\mathfrak{p}^\ell)\bigr), \quad j = 1, \dots, m.$$

Then $f_1(\mathfrak{p}^{\ell+1}) - f_1(\mathfrak{p}^\ell), \dots, f_m(\mathfrak{p}^{\ell+1}) - f_m(\mathfrak{p}^\ell)$ is a basis of $\mathfrak{m}_{\pi(\mathfrak{p}^0)}/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$ if and only if

$$\det\bigl(\partial_i^{(\ell+1)} f_j\bigr)(\mathfrak{p}^{\ell+1}) \neq 0,$$

which is our assertion. □

Proposition 7. *If $f_1, \dots, f_m$ are functions of $C^\infty(J^\ell M)$ such that the open subset $\mathcal{U} \subset J^{\ell+1}M$ of regular points over $\mathbb{R}[f_1, \dots, f_m]$ is not empty, then a local basis $\Phi_1, \dots, \Phi_m$ of* $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^{\ell+1}M))$*, with*

$$\Phi_i f_j = \delta_{ij},$$

can be chosen without any integration.

Proof. If the open subset $\mathcal{U}$ is not empty and we fix a point of $\mathcal{U}$, there exists a coordinate system $y_1, \dots, y_n \in C^\infty(M)$, in an open subset U of M, which represents U as an open subset $U' \times U''$, where U' is an open subset of X, with coordinates $y_1, \dots, y_m \in C^\infty(X)$, and $J^{\ell+1}(U', U) \cap \mathcal{U}$ is an open neighborhood of the point considered.

For each $\mathfrak{p}^{\ell+1} \in J^{\ell+1}(U', U)$ and $f \in C^\infty(J^\ell M)$ we have

$$f(\mathfrak{p}^{\ell+1}) - f(\mathfrak{p}^\ell) = \sum_{k=1}^m (\Phi_j f)(\mathfrak{p}^{\ell+1})\bigl(f_j(\mathfrak{p}^{\ell+1}) - f_j(\mathfrak{p}^\ell)\bigr), \quad j = 1, \dots, m,$$

where the $(\Phi_j f)(\mathfrak{p}^{\ell+1})$ are the coordinates of $f(\mathfrak{p}^{\ell+1}) - f(\mathfrak{p}^\ell) \in \mathfrak{m}_{\pi(\mathfrak{p}^0)}/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$ in the basis $\{f_j(\mathfrak{p}^{\ell+1}) - f_j(\mathfrak{p}^\ell)\}_{1 \leqslant j \leqslant m}$.

Since

$$(f \cdot h)(\mathfrak{p}^{\ell+1}) - (f \cdot h)(\mathfrak{p}^\ell) = \bigl(f(\mathfrak{p}^{\ell+1}) - f(\mathfrak{p}^\ell)\bigr)h(\mathfrak{p}^\ell) + f(\mathfrak{p}^\ell)\bigl(h(\mathfrak{p}^{\ell+1}) - h(\mathfrak{p}^\ell)\bigr)$$

for each $f, h \in C^\infty(J^\ell M)$, the mappings

$$C^\infty(U') \to C^\infty\bigl(J^{\ell+1}(U', U)\bigr), \quad g \to \Phi_j g, \quad j = 1, \dots, m,$$

are formal derivations. If we write

$$f_j(\mathfrak{p}^{\ell+1}) - f_j(\mathfrak{p}^\ell) = \sum_{i=1}^m \bigl(\partial_i^{(\ell+1)} f_j\bigr)(\mathfrak{p}^{\ell+1})\bigl(y_i(\mathfrak{p}^{\ell+1}) - y_{i0}(\mathfrak{p}^\ell)\bigr), \quad j = 1, \dots, m,$$

for the equations of change of basis on $\mathfrak{m}_{\pi(\mathfrak{p}^0)}/\mathfrak{m}^2_{\pi(\mathfrak{p}^0)}$, then we have

$$(\Phi_i f)(\mathfrak{p}^{\ell+1}) = \frac{\det \Lambda_i}{\det \Lambda}(\mathfrak{p}^{\ell+1}), \quad i = 1, \dots, m,$$

where $\Lambda = (\partial_i^{(\ell+1)} f_j)$ is the matrix whose ith row is $\partial_1^{(\ell+1)} f_i, \dots, \partial_m^{(\ell+1)} f_i$ and Λ_i is the matrix obtained by replacing the ith row of Λ by $\partial_1^{(\ell+1)} f, \dots, \partial_m^{(\ell+1)} f$. It is obvious that $\Phi_i f_i = 1$ and $\Phi_i f_j = 0$ when $i \neq j$. $\square$

The idea of Lie [7, Vol. I] in order to get a recurrence formula for the ℓ-jet prolongation of a vector field tangent to M is that such a prolongation must leave invariant the contact system; essentially the same idea can be found in [2], and in [17] a particular case is treated. In [15,16] the manifold M is supposed to be a product manifold $X \times Y$, and the proof of Lie is formalised after a previous study of the prolongations of one-parameter groups. Our method supposes a notable simplification of the process. The main idea is to consider the derivation from $C^\infty(M_m^\ell, \mathbb{R}_m^\ell)$ into $C^\infty(M_m^\ell, \mathbb{R}_m^\ell)$ given by the prolongation $D^{(\ell)}$ of the vector field D from M to M_m^ℓ (note that each tangent vector field $\mathcal{D}$ in a manifold V gives rise, by derivation of each coordinate, a derivation from $C^\infty(V^A)$ into $C^\infty(V^A)$, where A is an arbitrary Weil algebra); this derivation is completely determined by its restriction D to $C^\infty(M)$, when $C^\infty(M)$ is understood as a subalgebra of $C^\infty(M_m^\ell, \mathbb{R}_m^\ell)$. The derivation $D^{(\ell)}$ is projected into $D^{(\ell-1)}$ by definition.

In the notations of the proof of Proposition 7, let $\underline{U}_m^\ell$ be the open subset of points of U_m^ℓ regular over $\mathbb{R}[y_1, \dots, y_m]$, and $J^\ell(U', U)$ its image in $J^\ell M$. Each $f \in C^\infty(J^{\ell-1}(U', U))$ is an $\mathrm{Aut}(\mathbb{R}_m^{\ell-1})$-invariant function of $C^\infty(\underline{U}_m^{\ell-1})$; its prolongation of $f : (\underline{U}_m^{\ell-1})_m^1 \to \mathbb{R}_m^1$ to $\underline{U}_m^\ell$ is the specialization to $\underline{U}_m^\ell \subseteq (\underline{U}_m^{\ell-1})_m^1$; then, we can write

$$f = f_0 + \sum_{i=1}^m \bigl(\partial_i^{(\ell)} f\bigr)(y_i - y_{i0}), \tag{2.1}$$

understood as an identity between functions from $\underline{U}_m^\ell$ into $\mathbb{R}_m^1$, where f_0 and y_{i0} are f and y_i themselves, considered as functions of $C^\infty(J^\ell(U', U))$.

If $D^{(\ell-1)}$ is the prolongation to $M_m^{\ell-1}$ of a vector field D on M, then, by (2.1), $D^{(\ell-1)}$ associates to each $f \in C^\infty(J^{\ell-1}(U', U)) \subset C^\infty(\underline{U}_m^{\ell-1})$ the function ($\mathbb{R}_m^1$-valued)

$$D^{(\ell-1)} f = D^{(\ell-1)} f_0 + \sum_{i=1}^m \bigl(\partial_i^{(\ell)}(D^{(\ell-1)} f_0)\bigr)(y_i - y_{i0}), \tag{2.2}$$

where in the right-hand side $D^{(\ell-1)} f_0$ is $\mathbb{R}$-valued. On the other hand, if the second member of (2.1) is derived by $D^{(\ell)}$ we have

$$\begin{aligned} D^{(\ell-1)} f = D^{(\ell-1)} f_0 + \sum_{i=1}^m \bigl(& D^{(\ell)}\bigl(\partial_i^{(\ell)} f\bigr)(y_i - y_{i0}) \\ &+ \bigl(\partial_i^{(\ell)} f\bigr)(D^{(\ell-1)} y_i - D^{(\ell-1)} y_{i0})\bigr), \end{aligned} \tag{2.3}$$

where $D^{(\ell-1)} f_0 = (D^{(\ell-1)} f)_0$ and $D^{(\ell-1)} y_{i0} = (D^{(\ell-1)} y_i)_0$ by definition of f_0 and y_{i0}, and the function $D^{(\ell-1)} y_k \in C^\infty(\underline{U}_m^{\ell-1})$, understood as a function from $\underline{U}_m^\ell$ into $\mathbb{R}_m^1$, can be written as

$$D^{(\ell-1)} y_k = (D y_k)_0 + \sum_{i=1}^m \bigl(\partial_i^{(\ell)}(D y_k)\bigr)(y_i - y_{i0}). \tag{2.4}$$

Replacing each $D^{(\ell-1)}y_k$ in (2.3) by (2.4), it follows that

$$D^{(\ell-1)}f = D^{(\ell-1)}f_0 + \sum_{i=1}^{m}\left(D^{(\ell)}\big(\partial_i^{(\ell)}f\big) + \sum_{k=1}^{m}\partial_i^{(\ell)}(Dy_k)\partial_k^{(\ell)}f\right)(y_i - y_{i0}), \tag{2.5}$$

and by comparing the expressions (2.5) and (2.2) we get a recurrence formula

$$\partial_i^{(\ell)}\big(D^{(\ell-1)}f_0\big) = D^{(\ell)}\big(\partial_i^{(\ell)}f\big) + \sum_{k=1}^{m}\partial_i^{(\ell)}(Dy_k)\partial_k^{(\ell)}f,$$

which gives us the coordinates of the prolongation to jet spaces of a tangent vector field on M. Summarizing, we have

Proposition 8. *Let D be a tangent vector field on M, $y_1, \dots, y_m \in C^\infty(X)$ functionally independent functions on an open subset $U' \subset X$; on the open subset of the points of $J^\ell M$ regular over $\mathbb{R}[y_1, \dots, y_m]$, we have*

$$D^{(\ell)}\big(\partial_i^{(\ell)}f\big) = \partial_i^{(\ell)}\big(D^{(\ell-1)}f_0\big) - \sum_{k=1}^{m}\partial_i^{(\ell)}(Dy_k)\partial_k^{(\ell)}f \tag{2.6}$$

for each $f \in C^\infty(J^{\ell-1}M)$ (the function $D^{(\ell-1)}f_0$ in the right-hand side is $\mathbb{R}$-valued, as the other terms).

3. Invariant derivations of a sheaf of vector fields

Definition 9. Let D be a vector field on M and Φ a formal derivation belonging to $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$; the bracket $\{D, \Phi\}$ is the formal derivation from $C^\infty(X)$ into $C^\infty(J^\ell M)$ defined by

$$\big(\{D, \Phi\}g\big)(\mathfrak{p}^\ell) = D^{(\ell)}_{\mathfrak{p}^\ell}(\Phi g) - \Phi_{\mathfrak{p}^\ell}(Dg)$$

for each $\mathfrak{p}^\ell \in J^\ell M$, $g \in C^\infty(X)$, where $Dg \in C^\infty(M) \subseteq C^\infty(J^{\ell-1}M)$ with $\ell \geqslant 1$.

Over $C^\infty(X)$ we have the bracket defined as $\{D, \Phi\} = D^{(\ell)} \circ \Phi - \Phi \circ D$; we claim that an analogous equality holds over each algebra $C^\infty(J^r M)$, $r \geqslant 0$.

Proposition 10. *Let D be a tangent vector field on M and Φ a formal derivation of $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$; then the equality*

$$\{D, \Phi\} = D^{(\ell+s)} \circ \Phi - \Phi \circ D^{(\ell+s-1)}$$

holds over each ring $C^\infty(J^{\ell+s-1}M)$ $(s \geqslant 0)$.

Proof. In the above notations,

$$\left\{D, \frac{\partial}{\partial y_i}\right\} = -\sum_{k=1}^{m}\partial_i^{(\ell)}(Dy_k)\frac{\partial}{\partial y_k}$$

over $C^\infty(X)$. By Proposition 8,

$$\left\{D, \frac{\partial}{\partial y_i}\right\} = -\sum_{k=1}^{m} \partial_i^{(\ell)}(Dy_k)\partial_k^{(\ell)} = D^{(\ell)} \circ \partial_i^{(\ell)} - \partial_i^{(\ell)} \circ D^{(\ell-1)}$$

as a derivation from $C^\infty(J^{\ell-1}M)$ into $C^\infty(J^\ell M)$. Each formal derivation Φ can be written locally in the form

$$\Phi = \sum_{i=1}^{m} \varphi_i \frac{\partial}{\partial y_i} \quad \text{with } \varphi_i \in C^\infty(J^\ell M),$$

thus

$$\begin{aligned} D^{(\ell)} \circ \Phi - \Phi \circ D^{(\ell-1)} &= \sum_{i=1}^{m} (D^{(\ell)}\varphi_i)\partial_i^{(\ell)} + \sum_{i=1}^{m} \varphi_i \bigl(D^{(\ell)} \circ \partial_i^{(\ell)} - \partial_i^{(\ell)} \circ D^{(\ell-1)}\bigr) \\ &= \sum_{i=1}^{m} \left(D^{(\ell)}\varphi_i - \sum_{k=1}^{m} \varphi_k \bigl(\partial_k^{(\ell)}(Dy_i)\bigr) \right) \partial_i^{(\ell)} \\ &= \sum_{i=1}^{m} \bigl(D^{(\ell)}(\Phi y_i) - \Phi(Dy_i)\bigr)\partial_i^{(\ell)}, \end{aligned}$$

as a derivation from $C^\infty(J^{\ell-1}M)$ into $C^\infty(J^\ell M)$, which corresponds to the formal derivation

$$\{D, \Phi\} = \sum_{i=1}^{m} \bigl(D^{(\ell)}(\Phi y_i) - \Phi(Dy_i)\bigr) \frac{\partial}{\partial y_i}.$$

If for each $s \geqslant 0$, Φ is thought of as belonging to $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^{\ell+s}M))$, we can apply this argument again, which finishes the proof. $\square$

Definition 11. By an ℓth order differential invariant of a vector field D on M we will mean a first integral of its prolongation $D^{(\ell)}$ to $C^\infty(J^\ell M)$.

Let $\Phi \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ such that $\{D, \Phi\} = 0$ and f be a differential invariant of D on $J^{\ell-1}M$; then

$$0 = \{D, \Phi\} f = D^{(\ell)}(\Phi f),$$

that is, the function Φf is another differential invariant of D on $J^\ell M$.

Definition 12. Let $\mathcal{L}$ be a sheaf of vector fields on M; a formal derivation $\Phi \in \mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ is called an invariant derivation of $\mathcal{L}$ if for each differential invariant f of $\mathcal{L}$ the function Φf is also a differential invariant of $\mathcal{L}$.

If $f_1, \dots, f_m \in C^\infty(J^\ell M)$ are functionally independent, then $\det(\partial_i^{(\ell+1)} f_j) \in C^\infty(J^{\ell+1}M)$ is a polynomial of degree m, in the local coordinates $Y_{m+k,\alpha}$ $(k = 1, \dots,$

$n-m,\ 0<|\alpha|\leqslant \ell+1)$, whose coefficients are linear combinations of Jacobian determinants of $f_1,\dots,f_m$ with respect to the local coordinates in $J^\ell M$. When the function $\det(\partial_i^{(\ell+1)} f_j)$ does not vanish the open subset of points of $J^{\ell+1}M$ regular over $\mathbb{R}[f_1,\dots,f_m]$ is not empty. Under this condition, Lie's algorithm [6, p. 566], [8, p. 747], gives new differential invariants as quotients of Jacobian determinants of known differential invariants.

Proposition 13. *Let $\mathcal{L}$ be a sheaf of vector fields on M. If there is a point $\mathfrak{p}^{\ell+1}\in J^{\ell+1}M$ regular over the ring of differential invariants of $\mathcal{L}$ on $J^\ell M$, then there exists an open neighborhood of $\mathfrak{p}^{\ell+1}$ in $J^{\ell+1}M$ where it is possible to find, without any integration, a basis of* $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^{\ell+1}M))$ *formed by invariant derivations of $\mathcal{L}$.*

Proof. If $\mathfrak{p}^{\ell+1}$ is regular over the ring of differential invariants of $\mathcal{L}$ on $J^\ell M$, then there are some of them, $f_1,\dots,f_m\in C^\infty(J^\ell M)$, such that $\mathfrak{p}^{\ell+1}$ is regular over $\mathbb{R}[f_1,\dots,f_m]$. Proposition 7 associates to the functions $f_1,\dots,f_m$, without any integration, a basis $\Phi_1,\dots,\Phi_m$ of $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^{\ell+1}M))$ on the open subset of points of $J^{\ell+1}M$ regular over $\mathbb{R}[f_1,\dots,f_m]$. It remains to show that $\Phi_1 f,\dots,\Phi_m f$ are differential invariants of $\mathcal{L}$ on $C^\infty(J^{\ell+1}M)$, when f is a differential invariant of $\mathcal{L}$ on $J^\ell M$. Since f is a differential invariant of $\mathcal{L}$, for each local section D of $\mathcal{L}$ we have (by (2.6))

$$D^{(\ell+1)}\big(\partial_i^{(\ell+1)} f\big) = -\sum_{k=1}^m \big(\partial_i^{(\ell+1)}(Dy_k)\big)\partial_k^{(\ell+1)} f.$$

In the notations of the proof of Proposition 7, differentiating by columns the determinants we have

$$\begin{aligned}
D^{(\ell+1)}(\Phi_i f) &= D^{(\ell+1)}\frac{\det \Lambda_i}{\det \Lambda}\\
&= \frac{1}{\det \Lambda^2}\big[D^{(\ell+1)}(\det \Lambda_i)\det \Lambda - \det \Lambda_i\, D^{(\ell+1)}(\det \Lambda)\big]\\
&= \frac{1}{\det \Lambda^2}\left(-\sum_{k=1}^m \big(\partial_k^{(\ell+1)}(Dy_k)\big)\right)[\det \Lambda_i \det \Lambda - \det \Lambda_i \det \Lambda] = 0.
\end{aligned}$$

This completes the proof. □

Remark 14. The conditions of Proposition 13 are satisfied when $\mathcal{L}$ is a sheaf of vertical vector fields for $\pi: M\to X$, since all points of $J^\ell M$ are regular over $C^\infty(X)$ and the functions of this ring are first integrals of $\mathcal{L}$. This is the case for the sheaf of infinitesimal transformations of a Lie pseudogroup, vertical sheaf for the source projection $\alpha: M\times M\to M$.

4. Theorem on finiteness

Definition 15. The symbol at $\mathfrak{p}^\ell\in J^\ell M$ of a sheaf of vector fields $\mathcal{L}$ on M is the vector space $G^\ell_{\mathfrak{p}^\ell} = L_{\mathfrak{p}^\ell}\cap Q_{\mathfrak{p}^\ell} J^\ell M$, where $L_{\mathfrak{p}^\ell}$ is the value of $\mathcal{L}$ at $\mathfrak{p}^\ell$.

The *prolongation* of a subspace E^ℓ of $S^\ell T_x^*(X) \otimes V_p M$ is

$$E^{\ell(1)} = \left[T_x^*(X) \otimes E^\ell\right] \cap \left[S^{\ell+1} T_x^*(X) \otimes V_p M\right],$$

which can be understood as the space of polynomials belonging to $S^{\ell+1} T_x^*(X) \otimes V_p M$ whose first derivatives lie in E^ℓ; the *s-prolongation of E^ℓ* is defined to be the prolongation of $E^{\ell(s-1)}$. Following Kuranishi [5], if a subspace E^ℓ of $S^\ell T_x^*(X) \otimes V_p M$ is given for each $\ell \geqslant 0$ and $E^{\ell+1} \subseteq E^{\ell(1)}$ for each ℓ, then there is an integer $k \geqslant 0$ such that $E^{\ell+1} = E^{\ell(1)}$ for $\ell > k$ (in other words, the family $\{E^\ell\}$ is asymptotically stable). The following result gives sufficient conditions for the asymptotic stability of the symbol of a sheaf of vector fields.

Remark 16. If $G^\ell_{\mathfrak{p}^\ell}$ is the symbol at $\mathfrak{p}^\ell$ of a sheaf $\mathcal{L}$ of vector fields on M, we will denote its s-prolongation by $G^{\ell(s)}_{\mathfrak{p}^\ell}$; for each jet $\mathfrak{p}^{\ell+s}$ in the fibre of $\mathfrak{p}^\ell$ this is a subspace of $Q_{\mathfrak{p}^{\ell+s}} J^{\ell+s} M \approx S^{\ell+s} T_x^*(X) \otimes V_p M$ which does not depend on $\mathfrak{p}^{\ell+s}$, but only on $\mathfrak{p}^\ell$.

Proposition 17. *Let $\mathcal{L}$ be a sheaf of vector fields on M, which defines on $J^\ell M$ an involutive distribution regular with constant rank in a neighborhood of $\mathfrak{p}^\ell \in J^\ell M$, and I the ideal of a germ of maximal integral submanifold of $\mathcal{L}$ through $\mathfrak{p}^\ell$. If $\mathfrak{p}^{\ell+s}$ is an integral jet of the s-prolongation I^s of I in the fibre of $\mathfrak{p}^\ell$, then we have*

$$G^{\ell+s}_{\mathfrak{p}^{\ell+s}} \subseteq G^{\ell(s)}_{\mathfrak{p}^\ell}.$$

Proof. According to the definitions, the space $G^{\ell(s)}_{\mathfrak{p}^\ell}$ is the s-symbol at $\mathfrak{p}^\ell$ of the system of partial differential equations I, that is, $G^{\ell(s)}_{\mathfrak{p}^\ell}$ is the subspace of $Q_{\mathfrak{p}^{\ell+s}} J^{\ell+s} M$ whose elements, considered as derivations from $C^\infty(J^\ell M)$ into $C^\infty(X)/\mathfrak{m}_x^{s+1}$, kill the ideal I. If $\mathfrak{p}^{\ell+s}$ is an integral jet of the ideal I^s, then each function $f \in I$ vanishes at $\mathfrak{p}^{\ell+s}$, when $\mathfrak{p}^{\ell+s}$ is thought of as a jet of $J^s(J^\ell M)$, that it to say, an $\mathbb{R}$-algebra homomorphism from $C^\infty(J^\ell M)$ onto $C^\infty(X)/\mathfrak{m}_x^{\ell+1}$. Since $D^{(\ell)} I \subseteq I$ for all $D \in \mathcal{L}(M)$, we have $(D^{(\ell)} f)(\mathfrak{p}^{\ell+s}) = 0$ for all $f \in I$.

Each vector $D^{(\ell+s)}_{\mathfrak{p}^{\ell+s}} \in G^{\ell+s}_{\mathfrak{p}^{\ell+s}}$ is the value at $\mathfrak{p}^{\ell+s}$ of a vector field $D \in \mathcal{L}(M)$, namely, it is a residue class of the derivation

$$\mathfrak{p}^{\ell+s} \circ D^{(\ell)} : C^\infty(J^\ell M) \to C^\infty(X)/\mathfrak{m}_x^{s+1}, \quad f \to (D^{(\ell)} f)(\mathfrak{p}^{\ell+s}),$$

with respect to the submodule of derivations which annihilate the ideal $\mathfrak{p}^{\ell+s}$, the kernel of the homomorphism $\mathfrak{p}^{\ell+s}$, where $D^{(\ell)}$ is the prolongation of D to $J^\ell M$ and $\mathfrak{p}^{\ell+s} \in J^s(J^\ell M)$. Each derivation from $C^\infty(J^\ell M)$ into $C^\infty(X)/\mathfrak{m}_x^{s+1}$ whose residue class is $D^{(\ell+s)}_{\mathfrak{p}^{\ell+s}}$ differs from $\mathfrak{p}^{\ell+s} \circ D^{(\ell)}$ in a derivation which annihilates the ideal $\mathfrak{p}^{\ell+s} \in J^s(J^\ell M)$, hence it kills the ideal I since $\mathfrak{p}^{\ell+s}$ is an integral point of I^s; and we conclude that $D^{(\ell+r)}_{\mathfrak{p}^{\ell+s}} \in G^{\ell(s)}_{\mathfrak{p}^\ell}$. □

Remark 18. Proposition 17 holds, in particular, when $\mathcal{L}$ is a sheaf of Lie algebras on M, on the open subset of $J^\ell M$, where $\mathcal{L}$ has maximal rank.

The differential invariants of a sheaf of Lie algebras form a sheaf of regular rings on the open subset of $J^\ell M$, where $\mathcal{L}$ has maximal rank. If there exists an integer $\ell_0 \geqslant 0$ such that this sheaf has nonconstant sections for $\ell = \ell_0$, then the same holds for $\ell \geqslant \ell_0$, because $C^\infty(J^{\ell_0}M) \subseteq C^\infty(J^\ell M)$ and the first integrals of $\mathcal{L}$ in $J^{\ell_0}M$ are first integrals of $\mathcal{L}$ in $J^\ell M$, too. In general, the dimensions of the regular rings of differential invariants do not get stationary; our next result provides local basis of these rings from a local basis of differential invariants of a certain order and their invariant derivatives.

The finiteness theorem for a sheaf of vector fields on M depends only on the fact that $\mathcal{L}$ verifies the conditions of Frobenius' theorem and the existence of a basis of formal derivations which are invariant derivations for $\mathcal{L}$, because the stability of the symbols of higher orders is granted by Proposition 17. For simplicity we state it for a sheaf of Lie algebras.

Theorem 19 (Theorem on finiteness). *Let $\mathcal{L}$ be a sheaf of Lie algebras on M such that there exists an open neighborhood of $\mathfrak{p}^\ell \in J^\ell M$, where $\mathcal{L}$ is regular with constant rank $\dim J^\ell M - s_\ell > 0$, and let $\{\mathfrak{p}^{\ell+s}\}_{s\geqslant 0}$ be a family of jets over $\mathfrak{p}^\ell$, where the symbol of $\mathcal{L}$ at $\mathfrak{p}^{\ell+s}$ is $G^{\ell+s}_{\mathfrak{p}^{\ell+s}} = G^{\ell(s)}_{\mathfrak{p}^\ell}$. If the module $\operatorname{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ has a basis of invariant derivations of $\mathcal{L}$ on an open neighborhood of $\mathfrak{p}^\ell$, then $\mathcal{L}$ is regular with constant rank in an open neighborhood of each $\mathfrak{p}^{\ell+s}$ and the ring of differential invariants of $\mathcal{L}$ in this neighborhood is generated by the real components, with respect to a basis of invariant derivations of $\mathcal{L}$, of the prolongation to $C^\infty(J^{\ell+s}M)$ of a basis of differential invariants of $\mathcal{L}$ on $J^\ell M$.*

Proof. Let $J_1^{(\ell)}, \dots, J_{s_\ell}^{(\ell)}$ be a local basis of the ring of differential invariants of $\mathcal{L}$ in an open neighborhood of $\mathfrak{p}^\ell \in J^\ell M$ and $\Phi_1, \dots, \Phi_m$ a local basis of the module $\operatorname{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$, where Φ_i are invariant derivations for $\mathcal{L}$. We first show that

$$J_1^{(\ell)}, \dots, J_{s_\ell}^{(\ell)}, \Phi_1 J_1^{(\ell)}, \dots, \Phi_1 J_{s_\ell}^{(\ell)}, \dots, \Phi_m J_1^{(\ell)}, \dots, \Phi_m J_{s_\ell}^{(\ell)}$$

generate the ring of differential invariants of $\mathcal{L}$ in an open neighborhood of $\mathfrak{p}^{\ell+1} \in J^{\ell+1}M$.

Between the functions $J_1^{(\ell)}, \dots, J_{s_\ell}^{(\ell)}, \Phi_1 J_1^{(\ell)}, \dots, \Phi_1 J_{s_\ell}^{(\ell)}, \dots, \Phi_m J_1^{(\ell)}, \dots, \Phi_m J_{s_\ell}^{(\ell)}$ there are $\dim J^{\ell+1}M - \dim L_{\mathfrak{p}^{\ell+1}}$ functionally independent in an open neighborhood $\mathcal{U}$ of $\mathfrak{p}^{\ell+1}$, since

$$\operatorname{rank} \frac{\partial(\Phi_1 J_1^{(\ell)}, \dots, \Phi_m J_{s_\ell}^{(\ell)})}{\partial(Y_{m+1,\alpha_{m+1}}, \dots, Y_{n,\alpha_n})}(\mathfrak{p}^{\ell+1}) = \dim Q_{\mathfrak{p}^{\ell+1}} J^{\ell+1}M - \dim G^{\ell(1)}_{\mathfrak{p}^{\ell+1}}$$

for $|\alpha_i| = \ell + 1$ (because $G^{\ell(1)}_{\mathfrak{p}^{\ell+1}}$ is the symbol of $(J_1^{(\ell)}, \dots, J_{s_\ell}^{(\ell)})$ at $\mathfrak{p}^{\ell+1}$), and

$$\dim J^{\ell+1}M - \dim L_{\mathfrak{p}^{\ell+1}} = \dim J^\ell M + \dim Q_{\mathfrak{p}^{\ell+1}} J^{\ell+1}M - \dim L_{\mathfrak{p}^\ell} - \dim G^{\ell(1)}_{\mathfrak{p}^{\ell+1}}$$

by hypothesis.

Choosing $\mathcal{U}$ smaller if necessary, we can suppose that $\dim L_{\mathfrak{q}^{\ell+1}} \geqslant \dim L_{\mathfrak{p}^{\ell+1}}$ for each $\mathfrak{q}^{\ell+1} \in \mathcal{U}$. The canonical projection maps the vector subspace of $T_{\mathfrak{p}^{\ell+1}} J^{\ell+1}M$ annihilated by

$$d_{\mathfrak{p}^{\ell+1}} J_1^{(\ell)}, \dots, d_{\mathfrak{p}^{\ell+1}} J_{s_\ell}^{(\ell)}, d_{\mathfrak{p}^{\ell+1}} \Phi_1 J_1^{(\ell)}, \dots, d_{\mathfrak{p}^{\ell+1}} \Phi_m J_{s_\ell}^{(\ell)}$$

onto $L_{\mathfrak{p}^\ell}$ with kernel $G^{\ell(1)}_{\mathfrak{p}^\ell}$; since the sequence

$$0 \to G^{\ell(1)}_{\mathfrak{p}^\ell} \to L_{\mathfrak{p}^{\ell+1}} \to L_{\mathfrak{p}^\ell} \to 0$$

is exact, this subspace equals $L_{\mathfrak{p}^{\ell+1}}$. For each $\mathfrak{q}^{\ell+1}$ the subspace of $T_{\mathfrak{q}^{\ell+1}} J^{\ell+1}M$ annihilated by the 1-forms $d_{\mathfrak{q}^{\ell+1}} J_1^{(\ell)}, \dots, d_{\mathfrak{q}^{\ell+1}} J_{s_\ell}^{(\ell)}, d_{\mathfrak{q}^{\ell+1}} \Phi_1 J_1^{(\ell)}, \dots, d_{\mathfrak{q}^{\ell+1}} \Phi_m J_{s_\ell}^{(\ell)}$ contains $L_{\mathfrak{q}^{\ell+1}}$ and its dimension equals $\dim L_{\mathfrak{p}^{\ell+1}}$ hence

$$\dim L_{\mathfrak{q}^{\ell+1}} = \dim L_{\mathfrak{p}^{\ell+1}},$$

that is to say, $\mathcal{L}$ is regular with constant rank in $\mathcal{U}$ and the functions

$$J_1^{(\ell)}, \dots, J_{s_\ell}^{(\ell)}, \Phi_1 J_1^{(\ell)}, \dots, \Phi_1 J_{s_\ell}^{(\ell)}, \dots, \Phi_m J_1^{(\ell)}, \dots, \Phi_m J_{s_\ell}^{(\ell)}$$

generate the ring of differential invariants of $\mathcal{L}$ in $\mathcal{U}$. The proof is completed by recurrence. □

We write $\mathfrak{I}^\ell(M)$ for the open subset of invertible jets of $J^\ell(M, M \times M)$, that is to say, the open subset of regular jets over the rings $\alpha^* C^\infty(M)$ and $\beta^* C^\infty(M)$, where α and β are the source and target projection, respectively (see [11]). Each Lie pseudogroup of transformations of M is a solution of a nonlinear Lie equation on M, and its infinitesimal transformations are vector fields on M which generate local one-parameter groups of transformations belonging to a Lie pseudogroup. These infinitesimal transformations form a solution sheaf of a linear Lie equation; when this linear Lie equation is completely integrable the Lie pseudogroup is regular. From Theorem 19 it follows the finiteness of a basis of differential invariants of a regular Lie pseudogroup, because the remark after Proposition 13 guarantees the existence of a local basis of invariant derivations for the sheaf of its infinitesimal transformations.

Corollary 20 (Lie). *Let $\mathcal{L}$ be the sheaf of infinitesimal transformations of a Lie pseudogroup $\mathcal{P}$ of transformations of a manifold M, integral sheaf of a linear Lie equation $H^\ell \subset J^\ell TM$, whose prolongations $H^{\ell+s}$ are subfibre bundles of $J^{\ell+s}TM$ for each $s > 0$. For each $\varphi \in \mathcal{P}$ and $p \in \operatorname{Dom}\varphi$ there exists an integer $\ell_0 = \ell_0(j_p^\ell \varphi) \geqslant \ell$ such that, for each $s \geqslant 0$, the real components of the prolongation to $C^\infty(\mathfrak{I}^{\ell_0+s}(M))$ of a basis of the ring of differential invariants of $\mathcal{L}$ in an open neighborhood of $j_p^{\ell_0}\varphi \in \mathfrak{I}^{\ell_0}(M))$ form a local basis of the ring of differential invariants of $\mathcal{L}$ in $\mathfrak{I}^{\ell_0+s}(M)$.*

Remark 21. The study of differential invariants, on the manifolds $J^\ell M$, of a regular Lie pseudogroup is a particular case of the theory for a sheaf of Lie algebras developed here.

5. Finite-dimensional Lie algebras

In the following result we give conditions for the existence of a basis of invariant derivations of a Lie algebra.

Proposition 22. *Let L be a finite Lie algebra of vector fields on M and $J^\ell M$ a jet manifold, where L (prolongated to this jet space) has at some point maximal rank $r = \dim L$. If the ring of differential invariants of L contains nonconstant functions, then in the open subset of $J^\ell M$, where L has rank r, $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ has local basis of invariant derivations of L.*

Proof. Let $\mathcal{U}_\ell$ be the open subset of $J^\ell M$, where L has maximal rank r. By hypothesis we have $r < \dim J^\ell M$. For each $\mathfrak{p}^\ell \in \mathcal{U}_\ell$, let us choose an open neighborhood U of $p = \mathfrak{p}^0$ in M and a coordinate system $y_1, \ldots, y_n \in C^\infty(M)$ which represents it as an open subset $U' \times U''$, where U' is an open subset of X, $y_1, \ldots, y_m \in C^\infty(X)$, and $J^\ell(U', U)$ contains an open neighborhood of $\mathfrak{p}^\ell$ in $\mathcal{U}_\ell$.

A sufficient condition for Φ to be an invariant derivation of L is that $\{D, \Phi\} = 0$ for every $D \in L$; if we write this equation in local coordinates we obtain

$$D^{(\ell)}(\Phi y_i) - \Phi(D y_i) = 0, \quad i = 1, \ldots, m.$$

To solve this system of partial differential equations we introduce $\mathbb{R}^m$ as a space of parameters, with coordinates $\lambda_1, \ldots, \lambda_m$. On the manifold $J^\ell(U', U) \times \mathbb{R}^m$ we take the formal derivations $\Phi = \sum_{i=1}^m \lambda_i (\partial/\partial y_i)$, and the vector space $\mathcal{L}'$, generated by the vector fields

$$D^{(\ell)} + \sum_{i=1}^m \Phi(D y_i) \frac{\partial}{\partial \lambda_i} \quad \text{with } D \in L.$$

The Lie bracket of two vector fields of $\mathcal{L}'$ is

$$\begin{aligned}
&\left[D^{(\ell)} + \sum_{i=1}^m \Phi(D y_i) \frac{\partial}{\partial \lambda_i}, D'^{(\ell)} + \sum_{j=1}^m \Phi(D' y_j) \frac{\partial}{\partial \lambda_j} \right] \\
&\quad = [D^{(\ell)}, D'^{(\ell)}] + \sum_{j=1}^m D^{(\ell)}\big(\Phi(D' y_j)\big) \frac{\partial}{\partial \lambda_j} \\
&\qquad - \sum_{j=1}^m D'^{(\ell)}\big(\Phi(D y_j)\big) \frac{\partial}{\partial \lambda_j} \\
&\qquad + \sum_{i,j=1}^m \left(\Phi(D y_i) \frac{\partial \Phi(D' y_j)}{\partial \lambda_i} - \Phi(D' y_i) \frac{\partial \Phi(D' y_j)}{\partial \lambda_i} \right) \frac{\partial}{\partial \lambda_j}.
\end{aligned}$$

From (2.6) it follows that

$$\begin{aligned}
D^{(\ell)}\big(\Phi(D' y_j)\big) &= D^{(\ell)}\left(\sum_{k=1}^m \lambda_k \partial_k^{(\ell)}(D' y_j) \right) = \sum_{k=1}^m \lambda_k D^{(\ell)}\big(\partial_k^{(\ell)}(D' y_j)\big) \\
&= \sum_{k=1}^m \lambda_k \left(\partial_k^{(\ell)}\big(D(D' y_j)\big) - \sum_{i=1}^m \partial_k^{(\ell)}(D y_i) \partial_i^{(\ell)}(D' y_j) \right) \\
&= \sum_{k=1}^m \lambda_k \partial_k^{(\ell)}\big(D(D' y_j)\big) - \sum_{i=1}^m \Phi(D y_i) \frac{\partial \Phi(D' y_j)}{\partial \lambda_i},
\end{aligned}$$

and therefore

$$\left[D^{(\ell)} + \sum_{i=1}^{m} \Phi(Dy_i)\frac{\partial}{\partial\lambda_i}, D'^{(\ell)} + \sum_{j=1}^{m} \Phi(D'y_j)\frac{\partial}{\partial\lambda_j}\right]$$
$$= [D^{(\ell)}, D'^{(\ell)}] + \sum_{j=1}^{m} \Phi\big([D, D']y_j\big)\frac{\partial}{\partial\lambda_j},$$

that is to say, $\mathcal{L}'$ is involutive on $J^\ell(U', U) \times \mathbb{R}^m$.

If $D_1, \ldots, D_r$ is a basis of L, each vector field $D \in L$ is a linear span, with scalar coefficients, $D = \sum_{j=1}^{r} \alpha_j D_j$, then

$$D^{(\ell)} + \sum_{i=1}^{m} \Phi(Dy_i)\frac{\partial}{\partial\lambda_i} = \sum_{j=1}^{r} \alpha_j \left(D_j^{(\ell)} + \sum_{i=1}^{m} \Phi(D_j y_i)\frac{\partial}{\partial\lambda_i}\right).$$

Consequently, $\mathcal{L}'$ is regular with rank r.

The ring of first integrals of $\mathcal{L}'$ is locally generated by the differential invariants of L in $J^\ell M$ and m functions $J'_1, \ldots, J'_m$ whose Jacobian determinant with respect to $\lambda_1, \ldots, \lambda_m$ does not vanish. If we solve the system $J'_i = 1$, $J'_k = 0$, $k \neq i$, we obtain $\lambda_1 = \varphi_{i1}, \ldots, \lambda_m = \varphi_{im}$, where $\varphi_{i1}, \ldots, \varphi_{im}$ are smooth functions in an open subset of $J^\ell M$; in this open subset, the m formal derivations $\Phi_1, \ldots, \Phi_m$ defined by

$$\Phi_i = \sum_{j=1}^{m} \varphi_{ij}\frac{\partial}{\partial y_j}, \quad i = 1, \ldots, m,$$

are a local basis of $\mathrm{Der}_R(C^\infty(X), C^\infty(J^\ell M))$ and each Φ_i satisfies $\{D, \Phi_i\} = 0$. □

Remark 23. Let I be the ideal of a closed submanifold $\mathcal{R}^\ell$ of $J^\ell M$ invariant by L. If the open subset of $J^\ell M$, where L has rank $r = \dim L$ is not empty, around each point of it we can choose some differential invariants $F_1, \ldots, F_k$ of L on $J^\ell M$, as local generators of the system of partial differential equations I. The prolongation of the ideal I to $C^\infty(J^{\ell+s}M)$ is not, in general, a regular ideal. However, according to Proposition 22 there is in $\mathrm{Der}_{\mathbb{R}}(C^\infty(X), C^\infty(J^\ell M))$ a local basis of invariant derivations $\Phi_1, \ldots, \Phi_m$ of L, and by Proposition 4 the sth prolongation of the ideal I is locally generated by $F_1, \ldots, F_k$ and their derivatives up to order s with respect to $\Phi_1, \ldots, \Phi_m$, that is, by differential invariants of L in $C^\infty(J^{\ell+s}M)$.

Remark 24. In the proof of Proposition 22, $\mathcal{L}'$ is involutive since L is involutive. However, the equality $\operatorname{rank}\mathcal{L}' = \operatorname{rank} L$ follows from the fundamental fact that L is a finite-dimensional Lie algebra. Indeed, if $D = \sum_{j=1}^{r} \alpha_j D_j$, where the coefficients α_j are non-constant functions, then

$$D^{(\ell)} + \sum_{i=1}^{m} \Phi(Dy_i)\frac{\partial}{\partial\lambda_i}$$
$$= \sum_{j=1}^{r} \alpha_j \left(D_j^{(\ell)} + \sum_{i=1}^{m} \Phi(D_j y_i)\frac{\partial}{\partial\lambda_i}\right) + \sum_{i=1}^{m}\left(\sum_{j=1}^{r} \Phi\alpha_j (D_j y_i)\right)\frac{\partial}{\partial\lambda_i}$$

and the rank of $\mathcal{L}'$ can be greater than the rank of L on $J^\ell M$, contrary to Ovsiannikov's assertion in his Theorem of [16, p. 326]. With this method some invariant derivations of L can be obtained, but in general it does not provide a local basis of formal derivations.

Remark 25. Let L be an r-dimensional Lie algebra, and $\ell \geqslant 0$ the first integer such that L has maximal rank $r < \dim J^\ell M$ on $J^\ell M$. The symbol of L is $G^{\ell+k+1} = G^{\ell+1(k)} = 0$ at each jet $\mathfrak{p}^{\ell+k+1} \in J^{\ell+k+1}M$ ($k \geqslant 0$) over a jet $\mathfrak{p}^\ell \in J^\ell M$, where $\dim L_{\mathfrak{p}^\ell} = r$. From Proposition 22 and Theorem 19 it follows that each differential invariant of L is a function of differential invariants of order $\ell + 1$ and those which are obtained by invariant derivations. A basis of differential invariants of order ℓ and their invariant derivations give all differential invariants of a basis of order $\ell + 1$, except those belonging to a regular ring of dimension $\dim G^{\ell(1)}$.

6. Examples

To illustrate our general theory we include two examples.

6.1. *A finite-dimensional Lie algebra*

Let us consider the canonical projection $\mathbb{R}^2_{(x,u)} \to \mathbb{R}_{(x)}$ and use the notation u_k for the coordinate in $J^\ell\mathbb{R}^2$ which corresponds to the derivative $d^k u/dx^k$. Let L be the three-dimensional Lie algebra generated by the vector fields

$$D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial u}, \quad D_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Using formula (2.6) we have that the prolongation of L to $J^1\mathbb{R}^2$ is generated by

$$D_1^{(1)} = \frac{\partial}{\partial x}, \quad D_2^{(1)} = \frac{\partial}{\partial u}, \quad D_3^{(1)} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},$$

hence its rank equals 2 on the open subset U_1, where $xu \neq 0$; for $\ell \geqslant 2$, the prolongation to $J^\ell\mathbb{R}^2$ is generated by the vector fields

$$D_1^{(\ell)} = \frac{\partial}{\partial x}, \quad D_2^{(\ell)} = \frac{\partial}{\partial u}, \quad D_3^{(\ell)} = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} - \sum_{k=1}^{[\ell/2]} u_{2k}\frac{\partial}{\partial u_{2k}},$$

hence its rank is 3 on the open subset U_ℓ given by the condition $xu\dots u_\ell \neq 0$. Consequently, $G^\ell_{\mathfrak{p}^\ell} = 0$ for each $\mathfrak{p}^\ell \in U_\ell$ if $\ell \neq 2$, whereas $G^2_{\mathfrak{p}^2}$ is the vector space spanned by $(\partial/\partial u_2)_{\mathfrak{p}^2}$ for each $\mathfrak{p}^2 \in U_2$. Thus, the symbol G^ℓ is stable for $\ell \geqslant 3$.

From the form of the generators of the prolongation of L to $J^\ell\mathbb{R}^2$ it follows that u_1 generates the rings of differential invariants of orders 1 and 2, and u_1, u_3 that of order 3. If $\ell \geqslant 2$, Proposition 13 guarantees the existence of a basis of the module of formal derivations $\mathrm{Der}_{\mathbb{R}}(C^\infty(\mathbb{R}), C^\infty(J^\ell\mathbb{R}^2))$ formed by invariant derivations for L, hence the differential invariants of order 3 are a basis of the ring of differential invariants of L. Using the formulas from the proof of Proposition 13, we have that, for $\ell \geqslant 2$, $\partial^{(\ell)}/u_2$ is an invariant derivation of L, where $\partial^{(\ell)}$ denotes the total derivative with respect to x.

Remark 26. Observe that $G^{1(1)} \neq G^2$; according to the dimension count which appears in the proof of Theorem 19, L does not have any differential invariant of second order.

6.2. The classical example

The following example, due to Lie [9], was treated by Tresse, Kumpera, and other authors (see [4] and references therein).

Let us consider the trivial bundle $\mathbb{R}^3_{(x,y,z)} \to \mathbb{R}^2_{(x,y)}$ defined by the natural projection. In order to simplify the writing we will use the classical notations for the coordinates in the corresponding jet spaces, that is, $\{x, y, z, p, q\}$ in $J^1\mathbb{R}^3$ and those and r, s, t in $J^2\mathbb{R}^3$. When we deal with jets of higher order, $Z_{(h,k)}$ will be the coordinate corresponding to $\partial^{h+k}z/(\partial x^h \partial y^k)$. Finally, $\partial_x^{(\ell)}$ and $\partial_y^{(\ell)}$ will denote the total derivatives with respect to x and y.

Consider the sheaf of Lie algebras $\mathcal{L}$ locally generated in $\mathbb{R}^3$ by the vector fields of the form $D = f(x)(\partial/\partial x) - zf'(x)(\partial/\partial z)$, where f is any smooth function of x. According to formula (2.6), we have

$$D^{(1)} = f\frac{\partial}{\partial x} - f'\left(z\frac{\partial}{\partial z} + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}\right) - f''\left(z\frac{\partial}{\partial p}\right),$$

$$\begin{aligned} D^{(2)} = f\frac{\partial}{\partial x} &- f'\left(z\frac{\partial}{\partial z} + 2p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + 3r\frac{\partial}{\partial r} + 2s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t}\right) \\ &- f''\left(z\frac{\partial}{\partial p} + 3p\frac{\partial}{\partial r} + q\frac{\partial}{\partial s}\right) - f'''\left(z\frac{\partial}{\partial r}\right). \end{aligned}$$

The ℓth prolongation of D has the form

$$D^{(\ell)} = f\frac{\partial}{\partial x} - \sum_{k=1}^{\ell+1} f^{(k)} D_k^\ell,$$

where $D_1^\ell, \dots, D_{k+1}^\ell$ do not depend on f and the first term of D_k^ℓ is $z(\partial/\partial Z_{(k-1,0)})$; consequently, in the open subset U_ℓ of $J^\ell\mathbb{R}^3$, where $z \neq 0$, $\mathcal{L}$ is regular with rank $\ell + 2$. Furthermore, if $\ell \geqslant 1$ the expression of the field $D^{(\ell)}$ shows that, for each $\mathfrak{p}^\ell \in U_\ell$, the symbol $G^\ell_{\mathfrak{p}^\ell}$ is spanned by the value at $\mathfrak{p}^\ell$ of $\partial/\partial Z_{(\ell,0)}$, which guarantees the stability of G^ℓ for $\ell \geqslant 1$.

An easy calculus shows that $\mu = y$, $\nu = q/z$ generate the ring of differential invariants of order 1; there are points on U_2 regular over $\mathbb{R}[\mu, \nu]$, because the determinant of the matrix

$$\varLambda = \begin{pmatrix} \partial_x^{(2)}\mu & \partial_x^{(2)}\nu \\ \partial_y^{(2)}\mu & \partial_y^{(2)}\nu \end{pmatrix}$$

is different from zero at each point of an open subset of U_2. Thus, according to Proposition 13 there exists a basis of the module $\mathrm{Der}_{\mathbb{R}}(C^\infty(\mathbb{R}^2), C^\infty(J^2\mathbb{R}^3))$ formed by invariant derivations of $\mathcal{L}$ and, according to Theorem 19, the differential invariants of order 2 are a basis of differential invariants of $\mathcal{L}$. But in this case we can omit this argument and obtain, by a simple calculus, a basis of invariant derivations of first order. In fact, a sufficient condition for a formal derivation $\varPhi : C^\infty(\mathbb{R}^2) \to C^\infty(J^2\mathbb{R}^3)$ to be an invariant derivation for

$\mathcal{L}$ is that $\{D, \Phi\} = 0$ for each section D of $\mathcal{L}$; if we write $D = f(\partial/\partial x) - f'z(\partial/\partial z)$ and $\Phi = \lambda_1(\partial/\partial x) + \lambda_2(\partial/\partial y)$, we have

$$\{D, \Phi\} = (D^{(1)}\lambda_1 - \lambda_1 f')\frac{\partial}{\partial x} + D^{(1)}\lambda_2\frac{\partial}{\partial y},$$

and one can see easily that

$$\Phi_1 = \frac{\partial}{\partial y}, \qquad \Phi_2 = \frac{1}{z}\frac{\partial}{\partial x}$$

is the basis we were looking for. Thus, Theorem 19 gives that the differential invariants of first order of $\mathcal{L}$ are a basis.

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