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Extensions of the Witt group and the moduli space of curves

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INTRODUCTION

We will construct a formal scheme on groups G and a family of natural line bundles on it $\{\Lambda_\beta\}$ satisfying:

$$\Lambda_\beta \simeq \Lambda_1^{(1-6\beta+6\beta^2)}$$

We will prove that these isomorphisms must be seen as a formal version of the Mumford isomorphisms.

This fact comes from the following property: G *uniformizes* (infinitesimally) the moduli space of curves.

REFERENCES

joint work with José M. Muñoz Porras

- "Equations of the moduli space of pointed curves in the infinite Grassmannian", J. Differ. Geom. **51** (1999), pp. 431–469

- "Automorphism group of $k((t))$: applications to the bosonic string", Comm. Math. Phys. **216** no. 3 (2001), pp. 609–634.

PLAN OF THE TALK

- Part I: Automorphism group of $k((z))$, G.
  1. Definition.
  2. Lie algebra.
- Part II: Infinite Grassmannians.
  1. Definition.
  2. Gl(V) and its extensions.
- Part III: Extensions of G.
  1. Construction.
  2. Formal Mumford formula.
- Part IV: G and the moduli of curves $\mathcal{M}_{g,1}^{\infty}$.
  1. Krichever morphism.
  2. Action of G on $\mathcal{M}_{g,1}^{\infty}$.
  3. Relation between Mumford formulae.

I.1 Automorphism group of $k((z))$

We consider the functor on the category of k-schemes given by:

$$A \longrightarrow G(A) := \mathsf{Aut}_{A-\mathsf{alg}}\, A((z))$$

(k algebraically closed of char. 0).

Observe that there is an inclusion:

$$\begin{aligned} G(A) &\hookrightarrow A((z)) \\ g &\mapsto g(z) \end{aligned}$$

Theorem. (Characterization of G)

$$G(A) = \left\{ \begin{array}{c} \sum a_i z^i \in A((z)) \text{ with } a_i \text{ nilpotent} \\ \text{for } i \leq 0 \text{ and } a_1 \text{ invertible} \end{array} \right\}$$

Theorem. *The functor G is representable by a formal k-scheme.*

It is a non-commutative group and of infinite dimension.

I.2 Lie Algebra of G

Recall that Lie(G) is defined by:

$$\text{Lie}(G) = T_{\text{Id}}G = G(k[\epsilon]/\epsilon^2) \times_{G(k)} \{\text{Id}\}$$

Therefore, it has a basis:

$$\text{Lie}(G) = <\{L_n \mid n \in \mathbf{Z}\}>$$

where L_n is the field associated to the automorphism:

$$z \longmapsto z(1 + \epsilon z^n)$$

One checks that:

$$[L_m, L_n] = (m-n)L_{m+n}$$

Theorem. *There is a canonical isomorphism of Lie algebras:*

$$\text{Lie}(G) \xrightarrow{\sim} k((z))\frac{\partial}{\partial z}$$

$$L_n \longmapsto z^{n+1}\frac{\partial}{\partial z}$$

which is compatible with their natural actions on $k((z))$.

In order to classify the extensions of G we will need the following:

Theorem.

$$\text{Ext}^1(G, k^*) \subseteq \text{Ext}^1(\text{Lie}(G), k) \simeq \text{Vir} \cdot k$$

where Vir $:= < \{\{d_n\}_n, c\} >$ *and* $[d_m, d_n] := (m-n)d_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c$ *and* c *is central.*

II.1. Infinite Grassmannians

Let $V = k((z)), V_+ = k[[z]]$.

The **infinite Grassmannian** of (V, V_+), $\mathrm{Gr}(V)$, is a k-scheme whose set of rational points is:

$$\mathrm{Gr}(V) = \left\{ \begin{array}{c} U \subset V \text{ such that the kernel and the cokernel} \\ \text{of } U \to V/V_+ \text{ are both of finite dimension} \end{array} \right\}$$

(It is a k-scheme of infinite type).

Recall the following very important result:

Theorem. (Sato) *The points of* $\mathrm{Gr}(V)$ *parametrize the set of solutions of the KP hierarchy.*

Its Picard group is isomorphic to $\mathbf{Z}$ and it is generated by the determinant bundle:

$$\mathbf{D} := \mathrm{Det}\big(\mathcal{U} \to V/V_+\big)$$

($\mathcal{U}$ being the submodule of $V \otimes \mathcal{O}_{\mathrm{Gr}(V)}$ corresponding to the universal object).

Note that the fibre of $\mathbf{D}$ at a rational point U is:

$$\wedge^{\max}(U \cap V_+) \otimes \wedge^{\max}(V/U + V_+)^*$$

II.2 General linear group

$\mathsf{Gl}(V)$ will denote the functor from the category of k-schemes to the category of groups whose rational points are:

$$\left\{ \begin{array}{c} k\text{-linear automorphisms of } k((z)),\ \phi, \text{ such that} \\ \text{the following condition holds:} \\ \text{if } \dim(A + k[[z]])/A \cap k[[z]] < \infty, \\ \text{then } \dim(\phi(A) + k[[z]])/\phi(A) \cap k[[z]] < \infty \end{array} \right\}$$

Theorem. *The group functor* $\mathsf{Gl}(V)$ *acts on* $\mathsf{Gr}(V)$*; that is, for* $\phi \in \mathsf{Gl}(V)$ *one has:*

$$\begin{aligned} \mathsf{Gr}(V) &\to \mathsf{Gr}(V) \\ U &\mapsto \phi(U) \end{aligned}$$

Theorem. *The group* G *is canonically a subgroup of* $\mathsf{Gl}(V)$*.*

II.3 Central Extension of $\mathsf{Gl}(V)$

The following facts:

- $H^0(\mathsf{Gr}(V), \mathcal{O}) = k$
- the action preserves the determinant bundle:

$$\phi^* \mathbf{D} \simeq \mathbf{D}$$

allows us to consider a natural central extension of $\mathsf{Gl}(V)$:

$$0 \to k^* \to \widetilde{\mathsf{Gl}}(V) \to \mathsf{Gl}(V) \to 0$$

where $\widetilde{\mathsf{Gl}}(V)$ is given by the following expression:

$$\widetilde{\mathsf{Gl}}(V) := \left\{ \begin{array}{ccc} \mathbf{D} & \xrightarrow[\sim]{\bar{g}} & \mathbf{D} \\ \downarrow & & \downarrow \\ \mathsf{Gr}(V) & \xrightarrow[\sim]{g} & \mathsf{Gr}(V) \end{array} \quad \begin{array}{c} \text{commutative} \\ \text{and } g \in \mathsf{Gl}(V) \end{array} \right\}$$

Considering $k[\epsilon]/\epsilon^2$-valued points we may compute the cocycle of the associated extension of Lie algebras.

For $1 + \epsilon D_i \in \mathsf{Gl}(V)$ $(i = 1, 2)$ with decompositions:

$$D_i = \begin{pmatrix} D_i^{--} & D_i^{+-} \\ D_i^{-+} & D_i^{++} \end{pmatrix}$$

w.r.t. the splitting $V = V_- \oplus V_+ = z^{-1}k[z^{-1}] \oplus k[[z]]$, one gets:

$$c(D_1, D_2) = \mathsf{Tr}(D_1^{+-} D_2^{-+} - D_2^{+-} D_1^{-+})$$

III.1 Central Extensions of G

Since $G \subset \mathsf{Gl}(V)$, the central extension $\widetilde{\mathsf{Gl}}(V)$ gives rise to:

$$0 \to k^* \to \tilde{G} \to G \to 0$$

and therefore:

$$0 \to k \to \mathsf{Lie}(\tilde{G}) \to \mathsf{Lie}(G) \to 0$$

whose associated cocycle is computed from the previous formula:

$$c(L_m, L_n) = \delta_{m,-n}\left(\frac{m^3 - m}{6}\right)$$

This result shows that $\tilde{G}$ behaves like a "Virasoro group".

III.2 Mumford formula on G

For any $\beta \in \mathbf{Z}$, one consider the action of G on $V_\beta := k((z))dz^{\otimes\beta}$:

$$\begin{aligned} G \times k((z))dz^{\otimes\beta} &\to k((z))dz^{\otimes\beta} \\ (g, f(z)dz^{\otimes\beta}) &\mapsto f(g(z))(g'(z))^\beta dz^{\otimes\beta} \end{aligned}$$

Repeating the previous constructions for $\mathrm{Gr}(V_\beta)$ and $\mathrm{Gl}(V_\beta)$ one gets a central extension:

$$0 \to k^* \to \widetilde{G}_\beta \to G \to 0$$

Let Λ_β be the invertible sheaf on G defined by $\widetilde{G}_\beta$, then it holds:

Theorem. (formal Mumford formula)

$$\Lambda_\beta \simeq \Lambda_1^{\otimes(1-6\beta+6\beta^2)} \qquad \forall\beta \in \mathbf{Z}$$

Proof.

- $\mathrm{Ext}^1(G, k^*) \subseteq \mathrm{Ext}^1(\mathrm{Lie}(G), k)$
- the cocyle of Lie algebras of $\widetilde{G}_\beta$ is:

$$c_\beta(L_m, L_n) = \delta_{m,-n}\Big(\frac{m^3 - m}{6}\Big)(1 - 6\beta + 6\beta^2)$$

IV.1 Moduli of curves

Consider the scheme $\mathcal{M}_{g,1}^{\infty}$ whose rational points are triples (C, p, z) where:

- C is a complete and integral curve of arithmetic genus g.
- p is a smooth point.
- ϕ is an isomorphism $\widehat{\mathcal{O}}_p \simeq k[[z]]$.

The Krichever morphism is defined by:

$$\begin{aligned} K\colon \mathcal{M}_{g,1}^{\infty} &\longrightarrow \mathrm{Gr}^{1-g} V \\ (C, p, \phi) &\mapsto H^0(C-p, \mathcal{O}_C) \subset k((z)) \end{aligned}$$

Theorem.

- *The Krichever morphism is a closed immersion.*
- *A point $U \in \mathrm{Gr}(V)$ lies on $\mathrm{Im}(K)$ if it is a subalgebra of $k((z))$; that is, if and only if:*

$$\begin{cases} U \cdot U \subseteq U \\ k \subset U \end{cases}$$

IV.2 G acts on $\mathcal{M}_{g,1}^{\infty}$

Since the points of $\mathcal{M}_{g,1}^{\infty}$ are those subspaces of $k((z))$ which are subalgebras, it follows that there is an action:

$$G \times \mathcal{M}_{g,1}^{\infty} \longrightarrow \mathcal{M}_{g,1}^{\infty}$$

Then, for a rational point $X := (C, p, \phi) \in \mathcal{M}_{g,1}^{\infty}$, one gets:

$$p_X \colon G \times \{(C, p, \phi)\} \longrightarrow \mathcal{M}_{g,1}^{\infty}$$

Let $\widehat{\ }$ the completion and let $\widehat{p}_X$ be the morphism:

$$\widehat{p}_X \colon \widehat{G} \times \{X\} \to (\mathcal{M}_{g,1}^{\infty})\widehat{\ }_X \to (\mathcal{M}_g)\widehat{\ }_C$$

where $\mathcal{M}_g$ is the moduli space of (complete and integral) curves of genus g.

Theorem. *The morphism $\widehat{p}_X$ is surjective for all $X = (C, p, z)$.*

IV.3 Mumford formula

Then consider the line bundle on $\mathcal{M}_g$:

$$\lambda_\beta := \mathrm{Det}(R^\bullet \pi_* \omega_{\mathcal{C}}^{\otimes\beta})$$

where $\omega_{\mathcal{C}}$ is the sheaf of relative differentials of $\pi : \mathcal{C} \to \mathcal{M}_g$ ($\mathcal{C}$ being the universal curve).

Theorem. (Mumford)

$$\lambda_\beta \simeq \lambda_1^{\otimes(1-6\beta+6\beta^2)} \qquad \forall \beta \in \mathbf{Z}$$

Theorem.

$$\Lambda_\beta|_{\widehat{G}} \simeq \widehat{p}_X^* \lambda_\beta$$

Lemma.

$$K_\beta^* \mathbf{D} \simeq p^* \lambda_\beta$$

where K_β is the following morphism:

$$\begin{aligned} K_\beta \colon \mathcal{M}_{g,1}^\infty &\hookrightarrow \mathrm{Gr}(k((z))dz^{\otimes\beta}) \\ (C, p, \phi) &\mapsto H^0(C - p, \omega_C^{\otimes\beta}) \subset k((z))dz^{\otimes\beta} \end{aligned}$$