VECTOR FIELDS ON THE SATO GRASSMANNIAN

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ABSTRACT. An explicit basis of the space of global vector fields on the Sato Grassmannian is computed and the vanishing of the first cohomology group of the sheaf of derivations is shown.

1. INTRODUCTION

In this paper we compute explicitly the zero-th and first cohomology group of the sheaf of derivations on the Sato Grassmannian. It continues with the study of the properties of the Sato Grassmannian, $\mathrm{Gr}(\mathbb{C}((z)))$, started in [P].

We exhibit a basis of the vector space of global holomorphic vector fields on the Sato Grassmannian (Theorem 3.2) and show that the first cohomology group vanishes (Theorem 4.2). Our technique, although requires cumbersome notation, consists of performing computations with Čech cocycles with the help of an extension result of Hartogs type (Lemma 3.1).

Let us describe our results in a more detailed way. In the third section it is proved that any vector field on $\mathrm{Gr}(\mathbb{C}((z)))$ is induced by an endomorphism of $\mathbb{C}((z))$ and the expressions of these fields with respect to the coordinates on the charts of the covering are given. Our results will allow one to perform explicit computations with those vector fields associated to the action of subgroups of $\mathrm{Gl}(\mathbb{C}((z)))$. The study of such vector fields and group actions are specially relevant when studying the Sato Grassmannian from the point of view of dynamical systems and integrable systems ([DJKM, SS]). It deserves a special mention the relation with the characterization of Jacobian varieties ([M, Sh]) and with the infinitesimal structure of the moduli space of curves ([MP]).

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Furthermore, in this section it is also proved that vector fields are (essentially) induced by differential operators of $\mathbb{C}((z))$. Roughly speaking, this means that the differential operators generate infinitesimally the Grassmannian. This fact should have further consequences in the study of the moduli of opers in terms of infinite Grassmannian ([BF]).

In the last section it is proved that the first cohomology group vanishes. Naively, this would mean that the scheme $\operatorname{Gr}(V)$ could not admit non-trivial infinitesimal deformations.

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2. Background

Let us begin by recalling some basic facts on the scheme structure of the Sato Grassmannian of $\mathbb{C}((z))$ following [SS, P] (for an analytical approach to the infinite Grassmannian, see [SW]).

Let $V := \mathbb{C}((z))$ and $V^+ := \mathbb{C}[[z]]$. Let $\mathcal{S}$ be the set of strictly increasing sequences of integers $S = \{s_0, s_1, \dots\}$ such that $s_{i+1} = s_i + 1$ for all $i >> 0$. For $S \in \mathcal{S}$, let $V^S \subset V$ be the z-adic completion of $< z^{s_0}, z^{s_1}, \dots >$. Then, V will be endowed with the linear topology induced by $\{V^S\}_{S\in\mathcal{S}}$.

For a $\mathbb{C}$-scheme T, we define $\widehat{V}_T := V\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_T = \varprojlim (V\otimes_{\mathbb{C}}\mathcal{O}_T)/(V^S\otimes_{\mathbb{C}}\mathcal{O}_T)$ where $S \in \mathcal{S}$. Similarly, we introduce analogous notations for the completions of subspaces and quotients of V.

The infinite Grassmannian is the $\mathbb{C}$-scheme $\operatorname{Gr}(V)$ whose set of T-valued points is given by:

$$\operatorname{Gr}(V)(T) = \left\{ \begin{gathered} \text{submodules } L \subseteq \widehat{V}_T \text{ such that for every} \\ t \in T \text{ there exist a neighborhood } U \\ \text{and } S \in \mathcal{S} \text{ such that } \widehat{L}_U \oplus \widehat{V}^S_U \simeq \widehat{V}_U \end{gathered} \right\}$$

In particular, the set of $\mathbb{C}$-valued points is

$$\operatorname{Gr}(V)(\mathbb{C}) = \left\{ \begin{gathered} \text{subspaces } L \subseteq V, \text{ such that } L\cap V^+ \\ \text{and } V/L + V^+ \text{ are of finite dimension} \end{gathered} \right\}$$

We know that $\operatorname{Gr}(V)$ has an affine covering given by $\{\operatorname{Gr}^S\}_{S\in\mathcal{S}}$ where Gr^S represents the open subfunctor of $\operatorname{Gr}(V)$ given by

$$\operatorname{Gr}^S(T) := \{ \text{ sub-}\mathcal{O}_T\text{-modules } \mathcal{L} \subset \widehat{V}_T \text{ such that } \mathcal{L} \oplus \widehat{V}^S_T = \widehat{V}_T\}$$

The connected components of $\operatorname{Gr}(V)$ are labelled by the integers and are all isomorphic. *From now on,* $\operatorname{Gr}(V)$ *will denote the connected* *component containing the point* $z^{-1}\mathbb{C}[z^{-1}]$. It is easy to check that this component is covered by the open affine subschemes $\{\mathrm{Gr}^S\}_{S\in\mathcal{S}_0}$ where

$$\mathcal{S}_0 := \{S \in \mathcal{S} \,|\, \#(\mathbb{Z}_{<0} \cap S) = \#(\mathbb{Z}_{\geq 0} - S)\}$$

or, equivalently, those $S \in \mathcal{S}$ such that $\dim L_S \cap V^+ = \dim V/(L_S + V^+)$ where $L_S := <\{z^r | r \notin S\}>$.

Note that there is an isomorphism of functors

$$\begin{aligned} \mathrm{Hom}(L_S, V^S) &\overset{\sim}{\longrightarrow} \mathrm{Gr}^S \\ \phi &\longmapsto \mathrm{Graph}(\phi) := < l + \phi(l) \mid l \in L_S > \end{aligned}$$

Observe that there is an isomorphism of $\mathrm{Hom}(L_S, V^S)$ with an infinite dimensional affine space

$$\operatorname{Spec} \mathbb{C}[\{x_{ij} \mid i \notin S, j \in S\}] \simeq \mathrm{Hom}(L_S, V^S) \tag{2.1}$$

induced by the linear map that sends z^i to $\sum_{j\in S} x_{ij} z^j$.

Proposition 2.2. *Let $S_1, S_2 \in \mathcal{S}_0$ be two elements such that $S_1 \neq S_2$. Then, the restriction homomorphism associated to $\mathrm{Gr}^{S_1} \cap \mathrm{Gr}^{S_2} \hookrightarrow \mathrm{Gr}^{S_1}$ is the localization by an irreducible element.*

Proof. Consider a subspace $L \in \mathrm{Gr}^{S_1}$. Take $f^1 \in \mathrm{Hom}(L_{S_1}, V^{S^1})$ such that $L = \mathrm{Graph}(f^1)$. Writing f^1 as a map

$$L_{S_1 \cup S_2} \oplus L_{S_1}/L_{S_1\cup S_2} \to L_{S_2}/L_{S_1\cup S_2} \oplus V^{S_1\cap S_2}$$

where $L_{S_1}/L_{S_1\cap S_2}$ and $L_{S_2}/L_{S_1\cup S_2}$ are finite dimensional and $L_{S_1\cup S_2}$ and $V^{S_1\cap S_2}$ are infinite dimensional. One then has

$$f^1 = \begin{pmatrix} f^1_{11} & f^1_{12} \\ f^1_{21} & f^1_{22} \end{pmatrix}$$

where the block f^1_{12} has a finite number of columns and rows. Observe that the condition $L \in \mathrm{Gr}^{S_1} \cap \mathrm{Gr}^{S_2}$ is equivalent to say that

$$f^1_{12} \colon L_{S_1}/L_{S_1\cup S_2} \to L_{S_2}/L_{S_1\cup S_2}$$

is invertible. This means that $\mathrm{Gr}^{S_1} \cap \mathrm{Gr}^{S_2} = \operatorname{Spec} \mathbb{C}[\{x_{ij}\}]_{\det(f^1_{12})}$. Note that $\det(f^1_{12})$ is the determinant of the matrix $(x_{ij})_{\substack{i\in S_2 - S_1 \\ j\in S_1 - S_2}}$, which is irreducible in $\mathbb{C}[\{x_{ij} \mid i \notin S_1, j \in S_1\}]$. □

For our purposes we need an explicit relation between the coordinates of a given point with respect to two different charts of the covering. For $f^i \in \mathrm{Hom}(L_{S_i}, V^{S_i})$ $(i = 1, 2)$ the condition $\mathrm{Graph}(f^1) = \mathrm{Graph}(f^2)$ is equivalent to say that the subspaces generated by the columns of

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \\ f^1_{11} & f^1_{12} \\ f^1_{21} & f^1_{22} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \mathrm{Id} & 0 \\ f^2_{11} & f^2_{12} \\ 0 & \mathrm{Id} \\ f^2_{21} & f^2_{22} \end{pmatrix} \tag{2.3}$$

coincide (the columns of these matrices are the vectors of the corresponding subspaces and are written w.r.t. the decomposition $L_{S_1\cup S_2}\oplus L_{S_1}/L_{S_1\cup S_2}\oplus L_{S_2}/L_{S_1\cup S_2}\oplus V^{S_1\cap S_2}$).

Since f^1_{12} is invertible

$$\left\langle \begin{matrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \\ f^1_{11} & f^1_{12} \\ f^1_{21} & f^1_{22} \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathrm{Id} & 0 \\ 0 & (f^1_{12})^{-1} \\ f^1_{11} & \mathrm{Id} \\ f^1_{21} & f^1_{22}(f^1_{12})^{-1} \end{matrix} \right\rangle = \left\langle \begin{matrix} \mathrm{Id} & 0 \\ -(f^1_{12})^{-1}f^1_{11} & (f^1_{12})^{-1} \\ 0 & \mathrm{Id} \\ f^1_{21}-f^1_{22}(f^1_{12})^{-1}f^1_{11} & f^1_{22}(f^1_{12})^{-1} \end{matrix} \right\rangle \tag{2.4}$$

(here angular brackets denote the linear span of the columns). Comparing (2.3) and (2.4) we obtain

$$\begin{aligned} f^2_{11} &= -(f^1_{12})^{-1}f^1_{11} \\ f^2_{12} &= (f^1_{12})^{-1} \\ f^2_{21} &= f^1_{21}-f^1_{22}(f^1_{12})^{-1}f^1_{11} \\ f^2_{22} &= f^1_{22}(f^1_{12})^{-1} \end{aligned} \tag{2.5}$$

3. Global Vector Fields

Lemma 3.1. *Let $S_1,S_2\in\mathcal{S}_0$ such that $S_1\neq S_2$. Let $\mathcal{M}$ be a locally free $\mathcal{O}_{\mathrm{Gr}(V)}$-module. Then, there exists an canonical isomorphism*

$$H^0(\mathrm{Gr}(V),\mathcal{M})\simeq H^0(\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2},\mathcal{M})$$

Proof. Let Z_i be the closed subscheme given by $\mathrm{Gr}(V)-\mathrm{Gr}^{S_i}$ (for $1\leq i\leq 2$).

Let S_i,S_j be in $\mathcal{S}_0$. Recalling that $\mathcal{O}_{\mathrm{Gr}^{S_i}}$ is a ring of polynomials and that the restriction homomorphism associated to $\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j}\hookrightarrow\mathrm{Gr}^{S_i}$ is the localization by an irreducible element, one shows that the restriction homomorphisms

$$\begin{aligned} H^0(\mathrm{Gr}^{S_i},\mathcal{M}) &\simeq H^0(\mathrm{Gr}^{S_i}-Z_1\cap Z_2,\mathcal{M}) \\ H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j},\mathcal{M}) &\simeq H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j}-Z_1\cap Z_2,\mathcal{M}) \end{aligned}$$

are isomorphisms.

Noting that the group $H^0(\mathrm{Gr}(V),\mathcal{M})$ (resp. $H^0(\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2},\mathcal{M})$) can be computed using Čech cohomology and that $\{\mathrm{Gr}^{S_i}\}_{S_i\in\mathcal{S}_0}$ (resp.

$\{\mathrm{Gr}^{S_i} - Z_1\cap Z_2\}_{S_i\in\mathcal{S}_0})$ covers $\mathrm{Gr}(V)$ (resp. $\mathrm{Gr}(V) - Z_1\cap Z_2 = \mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2}$) the result follows. □

Let $\mathcal{D}$ be the sheaf of derivations on $\mathrm{Gr}(V)$. From (2.1) one has that

$$H^0(\mathrm{Gr}^S,\mathcal{D}) = \mathrm{Der}_{\mathbb{C}}\left(\mathbb{C}[\{x_{ij}\}],\mathbb{C}[\{x_{ij}\}]\right) = \mathbb{C}[\{x_{ij}\}] << \{\frac{\partial}{\partial x_{ij}}\} >>$$

where $<< \{e_i\} >>$ denotes the completion of the vector space $< \{e_i\} >$ w.r.t. the topology given by the subspaces of finite codimension.

Theorem 3.2. *Let $\mathcal{D}$ be the sheaf of derivations on $\mathrm{Gr}(V)$. Let $S\in\mathcal{S}_0$. Then the restriction map*

$$H^0(\mathrm{Gr}(V),\mathcal{D}) \hookrightarrow H^0(\mathrm{Gr}^S,\mathcal{D})$$

is injective and its image is the completion of the $\mathbb{C}$-vector space

$$\left\langle \left\{ \frac{\partial}{\partial x_{\alpha\beta}}, \sum_{\beta\in S} x_{\alpha_1\beta}\frac{\partial}{\partial x_{\alpha_2\beta}}, \sum_{\alpha\notin S} x_{\alpha\beta_1}\frac{\partial}{\partial x_{\alpha\beta_2}}, \sum_{\substack{\alpha\notin S\\ \beta\in S}} x_{\alpha_1\beta}x_{\alpha\beta_1}\frac{\partial}{\partial x_{\alpha\beta}} \right\}\right\rangle$$

where $\alpha_1,\alpha_2\notin S,\beta_1,\beta_2\in S$.

Proof. The previous lemma says that its suffices to compute $H^0(\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2},\mathcal{D})$ for any particular choice of S_1 and S_2. Therefore, let us take $S_1 = S$ and $S_2\in\mathcal{S}_0$ with $\#(S_1 - S_2) = \#(S_2 - S_1) = 1$. The proof is reduced to compute the kernel of the map

$$\begin{aligned} H^0(\mathrm{Gr}^{S_1},\mathcal{D})\times H^0(\mathrm{Gr}^{S_2},\mathcal{D}) &\overset{\delta}{\longrightarrow} H^0(\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2},\mathcal{D}) \\ (P,Q) &\longmapsto P|_{\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2}} - Q|_{\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2}} \end{aligned} \tag{3.3}$$

Recalling the expression (2.1) write

$$\begin{aligned} \mathrm{Gr}^{S_1} &= \mathrm{Spec}\,\mathbb{C}[\{x_{sr} \mid s\notin S_1, r\in S_1\}] \\ \mathrm{Gr}^{S_2} &= \mathrm{Spec}\,\mathbb{C}[\{y_{sr} \mid s\notin S_2, r\in S_2\}] \end{aligned}$$

Let introduce the following notation. Define integers i,j by the relations $S_1 - S_2 = \{i\}$ and $S_2 - S_1 = \{j\}$. Note that $\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2}\simeq \mathrm{Spec}\,\mathbb{C}[x_{rs}]_{x_{ji}}$. Let s denote any element in the complement of $S_1\cup S_2$ and r and any element in $S_1\cap S_2$.

Writing equations (2.5) for this case, we obtain the identities relating x's and y's

$$\begin{cases} y_{ir} = -x_{ji}^{-1}x_{jr} \\ y_{ij} = x_{ji}^{-1} \\ y_{sr} = x_{sr} - x_{si}x_{ji}^{-1}x_{jr} \\ y_{sj} = x_{si}x_{ji}^{-1} \end{cases} \qquad \begin{cases} x_{jr} = -y_{ir}y_{ij}^{-1} \\ x_{ji} = y_{ij}^{-1} \\ x_{sr} = y_{sr} - y_{sj}y_{ij}^{-1}y_{ir} \\ x_{si} = y_{sj}y_{ij}^{-1} \end{cases} \tag{3.4}$$

Taking differentials, and computing the corresponding vector fields, it turns out that the restriction homomorphism

$$H^0(\mathrm{Gr}^{S_2},\mathcal{D}) \longrightarrow H^0(\mathrm{Gr}^{S_1}\cap \mathrm{Gr}^{S_2},\mathcal{D})$$

is explicitly given by

$$\begin{aligned}
\frac{\partial}{\partial y_{ir}} &\mapsto -x_{ij}\frac{\partial}{\partial x_{jr}} - \sum_s x_{si}\frac{\partial}{\partial x_{sr}}\\
\frac{\partial}{\partial y_{ij}} &\mapsto -x_{ji}^2\frac{\partial}{\partial x_{ji}} - \sum_r x_{ji}x_{jr}\frac{\partial}{\partial x_{jr}} - \sum_{r,s} x_{si}x_{jr}\frac{\partial}{\partial x_{sr}} - \sum_s x_{si}x_{ji}\frac{\partial}{\partial x_{si}}\\
\frac{\partial}{\partial y_{sj}} &\mapsto x_{ji}\frac{\partial}{\partial x_{si}} + \sum_r x_{jr}\frac{\partial}{\partial x_{sr}}\\
\frac{\partial}{\partial y_{sr}} &\mapsto \frac{\partial}{\partial x_{sr}}
\end{aligned}$$

Now, let us compute the kernel of the map (3.3). A pair (P,Q) with $P=\sum_{\alpha,\beta}P_{\alpha\beta}(x)\frac{\partial}{\partial x_{\alpha\beta}}$ and $Q=\sum_{\alpha,\beta}Q_{\alpha\beta}(y)\frac{\partial}{\partial y_{\alpha\beta}}$ lies in $\operatorname{Ker}\delta$ if an only if

$$\begin{aligned}
0 &= P_{ji}(x) + x_{ji}^2 Q_{ij}(y)\\
0 &= P_{jr}(x) + x_{ji}Q_{ir}(y) + x_{ji}x_{jr}Q_{ij}(y)\\
0 &= P_{si}(x) + x_{si}x_{ji}Q_{ij}(y) - x_{ji}Q_{sj}(y)\\
0 &= P_{sr}(x) - Q_{sr}(y) + x_{si}Q_{ir}(y) + x_{si}x_{jr}Q_{ij}(y) - x_{jr}Q_{sj}(y)
\end{aligned} \tag{3.5}$$

Now one has to solve this system; that is, to find those fields P that extend to $\mathrm{Gr}(V)$.

The first equation and (3.4) imply that $y_{ij}^2P_{ji}(x)$ is a polynomial in y. Recalling the explicit expressions (3.4) one gets that $P_{\alpha\beta}(x)$ is a polynomial of degree less or equal than 2. Expressing y's in terms of x's, eliminating $Q_{\bullet}$ and solving the system, one gets the fields of the statement. □

From the theorem, it follows that if $H^0(\mathrm{Gr}(V),\mathcal{D})$ is endowed with the initial topology w.r.t. the inclusion in $H^0(\mathrm{Gr}^S,\mathcal{D})$, then it is complete.

For $a,b\in\mathbb{Z}$, let E_{ab} denote the endomorphism of V defined by $E_{ab}(z^\gamma):=z^b$ if $a=\gamma$ and 0 otherwise. Let $\mathrm{End}'(V)$ denote the $\mathbb{C}$-vector field generated by $\{E_{ab}|a,b\in\mathbb{Z}\}$.

Theorem 3.6. *The canonical morphism*

$$\mathrm{End}'(V) \longrightarrow H^0(\mathrm{Gr}(V),\mathcal{D})$$

has dense image.

Proof. For $a, b \in \mathbb{Z}$, the map $\mathrm{Id} + \epsilon E_{ab} : V \otimes \mathbb{C}[\epsilon]/\epsilon^2 \to V \otimes \mathbb{C}[\epsilon]/\epsilon^2$, where $\epsilon^2 = 0$, gives rise to an automorphism of $\mathrm{Gr}(V) \times \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ and, in particular, of the set of $\mathbb{C}[\epsilon]/\epsilon^2$-valued points of $\mathrm{Gr}(V)$; that is, to a vector field.

Consider $S \in \mathcal{S}_0$. For $L \in \mathrm{Gr}^S$ consider a basis of L of the type $\{z^r + \sum_s \lambda_{rs} z^s | r \notin S, s \in S\}$. Then, the coordinates $\{x_{ab}\}$ in the open subscheme Gr^S are given as follows

$$x_{ab}(L) := \lambda_{ab}$$

Then, the computation of the vector field consists essentially of writing down the coefficient of ϵ in the expression $x_{ab}\big((\mathrm{Id} + \epsilon E_{ab})(L)\big)$.

One gets the following cases

- if $a, b \notin S$, then the field is $-\sum_{\beta \in S} x_{b\beta} \frac{\partial}{\partial x_{a\beta}}$;
- if $a \notin S, b \in S$, it is $\frac{\partial}{\partial x_{ab}}$;
- if $a \in S, b \notin S$, it is $-\sum_{\alpha \notin S,, \beta \in S} x_{\alpha a} x_{b\beta} \frac{\partial}{\partial x_{\alpha\beta}}$;
- if $a, b \in S$, it is $\sum_{\alpha \notin S} x_{\alpha a} \frac{\partial}{\partial x_{\alpha b}}$.

The conclusion now follows from the Theorem 3.2. □

Remark 3.7. One can study the the extension of the above map to a suitable completion of $\mathrm{End}'(V)$, in that case the identity map generates the kernel. We refer the reader to [SW] for the relation with the restricted linear group and to [P] for the relation with the automorphisms group of $\mathrm{Gr}(V)$.

Recall that there exists the Plücker embedding $\mathrm{Gr}(V) \hookrightarrow \mathbb{P}\Omega(\mathcal{S})^*$ that identifies the Grassmannian with a closed subscheme of that projective space ([P]). The previous theorems allows us to compute global sections of the tangent bundles in both cases. In particular, we obtain the following

Corollary 3.8. *A vector field on* $\mathbb{P}\Omega(\mathcal{S})^*$ *induces a vector field on* $\mathrm{Gr}(V)$ *if and only if it is induced by an endomorphism of* V.

Remark 3.9 (Finite dimensional case). Our arguments also work for "standard" Grassmannians of a quasicoherent sheaf of modules; in particular, Grassmannians of finite dimensional vector spaces, projective spaces and relative Grassmannians. When V is a finite dimensional vector space our claims coincide with the known results; for instance, the kernel of the map of Theorem 3.6 is generated by the identity and $\dim H^0(\mathrm{Gr}(V), \mathcal{D}) = (\dim V)^2 - 1$ (see also [K]).

Theorem 3.10. *The canonical map*

$$\mathrm{Diff}(\mathbb{C}((z))) \hookrightarrow H^0(\mathrm{Gr}(V), \mathcal{D})$$

is injective and dense.

Proof. Observe that $\mathrm{Diff}(\mathbb{C}((z)))$ is generated by the multiplication by z, which shifts the degree by 1, and by $\frac{\partial}{\partial z}$, which shifts the degree by 1. Note that every differential operator can be expressed as a sum of homogeneous differential operators.

Following [SS], we introduce the operators Λ, which is the multiplication by z^{-1}, and $K = z\circ\frac{\partial}{\partial z}\circ z$. Since $\Lambda K - K\Lambda = \mathrm{Id}$, a homogeneous differential operator of degree n (where $n \in \mathbb{Z}$) is of the type $p(K\Lambda)\Lambda^{-n}$ where p is a polynomial.

Note that the identity (as endomorphisms of V)

$$p(K\Lambda)\Lambda^{-n} = \sum_{l\in\mathbb{Z}} p(l+n)E_{l,l+n}$$

implies the injectivity of the map of the statement. From the definition of the topology of $H^0(\mathrm{Gr}(V),\mathcal{D})$, it follows that the closure of the image of $\mathrm{Diff}(\mathbb{C}((z)))$ contains the vector fields induced by the operators E_{ab}. One concludes from Theorem 3.6. □

Remark 3.11. The methods of this section can be adapted for other computations. For instance, in the case of a relative Grassmannian $\mathrm{Gr}(V)\to S$ one can show that $H^0(\mathrm{Gr}(V),\mathcal{O}) = H^0(S,\mathcal{O}_S)$.

4. Vanishing of the first cohomology group

Lemma 4.1. *Let $S_1, S_2, S_3 \in \mathcal{S}_0$ be pairwise distinct. Let $\mathcal{M}$ be a locally free $\mathcal{O}_{\mathrm{Gr}(V)}$-module. Then, there exists an canonical isomorphism*

$$H^1(\mathrm{Gr}(V),\mathcal{M}) \simeq H^1(\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2}\cup\mathrm{Gr}^{S_3},\mathcal{M})$$

Proof. From Proposition 1.4.1 of Chapter III of [EGA], the group $H^1(\mathrm{Gr}(V),\mathcal{M})$ can be computed from the covering $\{\mathrm{Gr}^S\}$ using Čech cohomology. Since $\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2}\cup\mathrm{Gr}^{S_3} = \mathrm{Gr}(V) - Z_1\cap Z_2\cap Z_3$, it is enough to check that the restriction homomorphisms

$$\begin{aligned} H^0(\mathrm{Gr}^{S_i},\mathcal{M}) &\simeq H^0(\mathrm{Gr}^{S_i} - Z_1\cap Z_2\cap Z_3,\mathcal{M}) \\ H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j},\mathcal{M}) &\simeq H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j} - Z_1\cap Z_2\cap Z_3,\mathcal{M}) \\ H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j}\cap\mathrm{Gr}^{S_k},\mathcal{M}) &\simeq H^0(\mathrm{Gr}^{S_i}\cap\mathrm{Gr}^{S_j}\cap\mathrm{Gr}^{S_k} - Z_1\cap Z_2\cap Z_3,\mathcal{M}) \end{aligned}$$

are isomorphisms for any $S_i, S_j, S_k \in \mathcal{S}_0$. This fact follows from arguments similar to those of the proof of Lemma 3.1. □

Theorem 4.2. *The following holds*

$$H^1(\mathrm{Gr}(V),\mathcal{D}) = 0$$

Proof. It is enough to show that $H^1(\mathrm{Gr}^{S_1}\cup\mathrm{Gr}^{S_2}\cup\mathrm{Gr}^{S_3},\mathcal{D})=0$ for a particular choice of $S_1,S_2,S_3\in\mathcal{S}_0$ (pairwise distinct). Take S_1,S_2,S_3 such that $\#(S_\alpha-S_\beta)=1$ for $\alpha\neq\beta$. Introduce i,j,k by the relations $\{i,k\}=S_1-(S_2\cap S_3)$, $\{j,k\}=S_2-(S_1\cap S_3)$ and $\{i,j\}=S_3-(S_1\cap S_2)$.

Let

$$\begin{aligned}\mathrm{Gr}^{S_1} &= \operatorname{Spec}\mathbb{C}[\{x_{sr}\mid s\notin S_1, r\in S_1\}]\\ \mathrm{Gr}^{S_2} &= \operatorname{Spec}\mathbb{C}[\{y_{sr}\mid s\notin S_2, r\in S_2\}]\\ \mathrm{Gr}^{S_3} &= \operatorname{Spec}\mathbb{C}[\{z_{sr}\mid s\notin S_3, r\in S_3\}]\end{aligned}$$

where x's and z's are related as follows (similarly to (3.4))

$$\begin{aligned}z_{ir} &= -x_{ki}^{-1}x_{kr}\\ z_{ik} &= x_{ki}^{-1}\\ z_{sr} &= x_{sr}-x_{si}x_{ki}^{-1}x_{kr}\\ z_{sk} &= x_{si}x_{ki}^{-1}\end{aligned}\tag{4.3}$$

It follows that

$$\begin{aligned}H_{12} &:= H^0(\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2},\mathcal{D}) = \mathbb{C}[\{x_{sr}\}]_{x_{ji}} << \{\frac{\partial}{\partial x_{sr}}\} >>\\ H_{13} &:= H^0(\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_3},\mathcal{D}) = \mathbb{C}[\{x_{sr}\}]_{x_{ki}} << \{\frac{\partial}{\partial x_{sr}}\} >>\\ H_{23} &:= H^0(\mathrm{Gr}^{S_2}\cap\mathrm{Gr}^{S_3},\mathcal{D}) = \mathbb{C}[\{y_{sr}\}]_{y_{kj}} << \{\frac{\partial}{\partial y_{sr}}\} >>\\ H_{123} &:= H^0(\mathrm{Gr}^{S_1}\cap\mathrm{Gr}^{S_2}\cap\mathrm{Gr}^{S_3},\mathcal{D}) = \mathbb{C}[\{x_{sr}\}]_{x_{ji}x_{ki}} << \{\frac{\partial}{\partial x_{sr}}\} >>\end{aligned}$$

Then, to prove the statement one has to check that the kernel of

$$\begin{aligned}H_{12}\times H_{13}\times H_{23} &\xrightarrow{\mathcal{Z}} H_{123}\\ (A,B,C) &\longmapsto A-B+C\end{aligned}$$

is contained into the image of

$$\begin{aligned}H^0(\mathrm{Gr}^{S_1},\mathcal{D})\times H^0(\mathrm{Gr}^{S_2},\mathcal{D})\times H^0(\mathrm{Gr}^{S_3},\mathcal{D}) &\xrightarrow{\mathcal{B}} H_{12}\times H_{13}\times H_{23}\\ (P,Q,R) &\mapsto (P-Q,P-R,Q-R)\end{aligned}$$

Let us consider $(A,B,C)\in\operatorname{Ker}\mathcal{Z}$. One has to show that its class in the quotient $\operatorname{Ker}\mathcal{Z}/\operatorname{Im}\mathcal{B}$ vanishes.

Let us write down the condition $(A,B,C)\in\operatorname{Ker}\mathcal{Z}$. For $A=\sum_{\alpha,\beta}A_{\alpha\beta}\frac{\partial}{\partial x_{\alpha\beta}}\in H_{12}$ and similarly for $B\in H_{13}$ and $C\in H_{23}$, the condition reads

(4.4)

$$\begin{aligned} 0 &= A_{ji}(x) - B_{ji}(x) - x_{ji}^2 C_{ij}(y) \\ 0 &= A_{jr}(x) - B_{jr}(x) - x_{ji} C_{ir}(y) - x_{ji} x_{jr} C_{ij}(y) \\ 0 &= A_{si}(x) - B_{si}(x) - x_{si} x_{ji} C_{ij}(y) + x_{ji} C_{sj}(y) \\ 0 &= A_{sr}(x) - B_{sr}(x) + C_{sr}(y) - x_{si} C_{ir}(y) - x_{si} x_{jr} C_{ij}(y) + x_{jr} C_{sj}(y) \end{aligned}$$

The identity $\mathcal{B}(0,Q,0) = (-Q,0,Q)$ means that there is an element $(\bar{A},\bar{B},\bar{C})$ in the class of (A,B,C) such that the coefficients of $\frac{\partial}{\partial y_{\alpha\beta}}$ in $\bar{C}$, $\bar{C}_{\alpha\beta} = \sum_l \bar{C}^l_{\alpha\beta}(y) y^l_{kj}$, are polynomials in y^{-1}_{kj} with coefficients in $\mathbb{C}[\{y_{sr} | s \neq k, r \neq j\}]$ and with $\bar{C}^0_{\alpha\beta}(y) = 0$; or, what amounts to the same, we may assume that $C^l_{\alpha\beta}(y) = 0$ for $l \geq 0$.

Observe that $\mathcal{B}(P,0,0) = (P,P,0)$ and, thus, we can assume that $A = \sum_{\alpha,\beta,l} A^l_{\alpha\beta}(x) x^l_{ji} \frac{\partial}{\partial x_{\alpha\beta}} \in H_{12}$ with $A^l_{\alpha\beta}(x) \in \mathbb{C}[\{x_{sr} | s \neq j, r \neq i\}]$ satisfies $A^l_{\alpha\beta}(x) = 0$ for all $l \geq 0$.

Bearing in mind the first equation of (4.4), the expressions of A_{ji} and C_{ij}, the identity $y^{-1}_{kj} = x^{-1}_{ki} x_{ji}$ and the fact that B is a polynomial in x_{ji} it follows that $A_{ji} = 0$. Similar arguments show that $A_{\alpha\beta} = 0$ for all α, β; and, therefore, $A = 0$.

Now, the conditions (4.4) mean that B, C glue together and give rise to a vector field in $(\mathrm{Gr}^{S_1} \cap \mathrm{Gr}^{S_3}) \cup (\mathrm{Gr}^{S_2} \cap \mathrm{Gr}^{S_3})$ and, using similar arguments as in Lemma 3.1, a vector field $R \in H^0(\mathrm{Gr}(V)^{S_3}, \mathcal{D})$. That is, there exists R such that $\mathcal{B}(0,0,R) = (0,B,C)$. And the conclusion follows. □

Remark 4.5 (Picard group). Similar arguments can be used to get another proof of the known fact $H^1(\mathrm{Gr}(V), \mathcal{O}^*) \simeq \mathbb{Z}$. In the relative case $\mathrm{Gr}(V) \to S$, one gets $H^1(\mathrm{Gr}(V), \mathcal{O}^*) \simeq \mathbb{Z} \times H^1(S, \mathcal{O}^*)$.

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