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Uniformization of the Moduli Space of Pairs

joint with D. Hernández, E. Gómez and J.M. Muñoz.

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Aim.

Definition

Fix g, n and d. Let $\mathcal{U}_g^\infty(n,d)$ be the moduli functor whose set of rational points consists of 5-tuples (X,x,t,E,ϕ) where:

- X is a genus g projective, smooth and irreducible curve;
- $x \in X$ is a point;
- $t : \widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[z]]$;
- E is a rank n degree d v.b. on X;
- $\phi : \widehat{E}_x \simeq \widehat{\mathcal{O}}_{X,x}^{\oplus n}$.

$$\mathcal{U}_g(n,d) \dashleftarrow \mathcal{U}_g^\infty(n,d) \hookrightarrow \mathrm{Gr}(\mathbb{C}((z))^n) \circlearrowleft \text{Group}$$

- Scheme structure.
- Infinitesimal structure (a group acts on it $\rightsquigarrow$ *uniformization*).
- Map to the moduli space of pairs (X,E) (roughly mod out by changes of t and ϕ).

In order to illustrate the type of problems we deal with and to become familiar with the techniques, let me give an *overview* of these topics.

Contents

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Sato Grassmannian

Set $V = \mathbb{C}((z))^n$ and $V^+ = \mathbb{C}[[z]]^n$. [Sato, Segal-Wilson, Álvarez-Muñoz-P.]
The **Sato Grassmannian** of (V, V^+) is a **scheme** whose rational points are subspaces $W \subseteq V$ such that:

$$\mathrm{Gr}(V) := \left\{ \begin{array}{c} W \subseteq V \text{ s.t. } W \to V/V^+ \\ \text{has finite dimensional kernel and cokernel} \end{array} \right\}$$

Its connected components are labelled by the integers, $\mathrm{Gr}^\chi(V)$, where:

$$\chi := \dim_{\mathbb{C}} W \cap V^+ - \dim_{\mathbb{C}} V/(W + V^+) \in \mathbb{Z}$$

From now on, $\mathrm{Gr}(V)$ will denote the connected component labelled by 0. On $\mathrm{Gr}(V)$, one has the **determinant**, that generates $\mathrm{Pic}(\mathrm{Gr}(V))$:

$$\mathrm{Det} := \mathrm{Det}\left(\mathcal{W} \xrightarrow{\delta} V/V^+ \otimes \mathcal{O}_{\mathrm{Gr}(V)}\right)$$

and a global section $\det(\delta) \in H^0(\mathrm{Gr}(V), \mathrm{Det}^*)$, which allow us to introduce the τ-**function** of $W \in \mathrm{Gr}(V)$:

$$\tau_W(t_1, t_2, \dots) := \text{ normalization of } \det(\delta)((\sum_i \frac{t_i}{z^i})W)$$

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Central Extensions

The group $\mathrm{Gl}_{\mathbb{C}}(V)$ acts on $\mathrm{Gr}(V)$ and preserves the determinant bundle. Therefore, it makes sense to consider the group:

$$\widetilde{\mathrm{Gl}}_{\mathbb{C}}(V) := \left\{ \text{pairs } (\bar{g}, g) \text{ where } \begin{array}{ccc} \mathrm{Det} & \xrightarrow[\sim]{\bar{g}} & \mathrm{Det} \\ \downarrow & & \downarrow \\ \mathrm{Gr}(V) & \xrightarrow[\sim]{g} & \mathrm{Gr}(V) \end{array} \text{ and } g \in \mathrm{Gl}_{\mathbb{C}}(V) \right\}$$

Since $H^0(\mathrm{Gr}(V), \mathcal{O}^*) = \mathbb{C}^*$, one obtains a **central extension**

$$1 \to \mathbb{C}^* \to \widetilde{\mathrm{Gl}}_{\mathbb{C}}(V) \to \mathrm{Gl}_{\mathbb{C}}(V) \to 1$$

whose cocycle is given by:

$$c(g_1, g_2) = \det(\bar{g}_1 \circ (\overline{g_1 \circ g_2})^{-1} \circ \bar{g}_2) \qquad \text{for } (\bar{g}_i, g_i) \in \widetilde{\mathrm{Gl}}_{\mathbb{C}}(V)$$

These ideas, dating back to Tate, provide an alternative approach to the study of **reciprocity laws** [Anderson-Pablos, Muñoz-Pablos,etc].

Krichever Map

Let us fix (X, x, t). Then, for each pair (L, ϕ), where:

- L is a line bundle of degree $g-1$,
- $\phi : \widehat{L}_x \overset{\sim}{\to} \widehat{\mathcal{O}}_{X,x}$,

the cohomology exact sequence:

$$0 \to H^0(X, L) \to H^0(X-x, L) \to (\widehat{\mathcal{O}}_{X,x})_{(0)}/\widehat{\mathcal{O}}_{X,x} \to H^1(X, L) \to 0$$

shows that:

$$(L, \phi) \longmapsto (t \circ \phi)(H^0(X-x, L)) \in \operatorname{Gr}\left(\mathbb{C}((z))\right)$$

is well defined and it will be called the **Krichever Map**.

Originally defined in terms of Baker-Akhiezer functions and applied to the study of commutative rings of differential operators.

One could also vary (X, x, t).

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Example

Let $X = \mathbb{P}^1 = \operatorname{Proj}\mathbb{C}[x_0, x_1]$, $x = \infty = (0,1)$ be the point given by the equation $x_0 = 0$ and t the isomorphism $\widehat{\mathcal{O}}_{\mathbb{P}^1,\infty} \simeq \mathbb{C}[[z]]$ given by $\frac{x_0}{x_1} \mapsto z$. Then,

$$H^0(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{C}[\frac{x_1}{x_0}] \longmapsto \mathbb{C}[z^{-1}] \in \operatorname{Gr}^1(\mathbb{C}((z)))$$

Note that, $\mathbb{P}^1 \setminus \{\infty\} \simeq \operatorname{Spec}\mathbb{C}[z^{-1}]$ and that, further, $(\mathbb{P}^1, \infty, t)$ **can be reconstructed from its image by the Krichever map**, $\mathbb{C}[z^{-1}]$.

With the above notation, let us consider the point $(1,0) \in \mathbb{P}^1$, and the line bundle $\mathcal{O}_{\mathbb{P}^1}(2\cdot(1,0))$, whose Euler-Poincaré characteristic is 3, then:

$$H^0(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1}(2\cdot(1,0))) \simeq (\frac{x_0}{x_1})^2\mathbb{C}[\frac{x_1}{x_0}] \longmapsto z^2\mathbb{C}[z^{-1}] \in \operatorname{Gr}^3(\mathbb{C}((z)))$$

(the trivialization ϕ corresponds to multiplication by $(\frac{x_1}{x_0})^2$).
It is trivial that $\mathcal{O}_{\mathbb{P}^1}(2\cdot(1,0))$ and ϕ can be reconstructed from $z^2\mathbb{C}[z^{-1}]$ as $\mathbb{C}[z^{-1}]$-module.

Moduli of (L, ϕ)

This is the rank 1 case (we will restrict ourselves to degree $g-1$).
[Segal-Wilson, Shiota, Krichever, Mulase, Dubrovin, Mumford, Kyoto School, Álvarez-Muñoz-P., etc]:

$$\mathrm{Jac}(X) \xleftarrow[\text{mod out by } \mathbb{C}[[z]]^*]{\text{forget } \phi} \mathrm{Jac}^\infty(X) := \{(L,\phi) \text{ line bundle and triv.}\} \xhookrightarrow{K} \mathrm{Gr}(\mathbb{C}((z)))$$

Theorem (Álvarez-Muñoz-P.)

$\mathrm{Jac}^\infty(X)$ *is representable by a closed subscheme of* $\mathrm{Gr}(\mathbb{C}((z)))$.

1. The Krichever map is injective.
2. Its image is characterized by: $W \in \mathrm{Im}(K) \iff A \cdot W \subseteq W$ where $A = H^0(X - x, \mathcal{O}_X)$.
3. Let $\mathcal{W}$ be the universal submodule. $\mathrm{Jac}^\infty(X)$ is the locus where $A \otimes \mathcal{W} \to \mathcal{O}_{\mathrm{Gr}(\mathbb{C}((z)))}((z))/\mathcal{W}$ vanishes.
4. For any open affine subscheme $U = \operatorname{Spec} A$ of $\mathrm{Gr}(\mathbb{C}((z)))$, the quotient $\mathcal{O}_{\mathrm{Gr}(\mathbb{C}((z)))}((z))/\mathcal{W}|_U$ is an inverse limit of finite type A-modules.

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Relation with the KP hierarchy

From Sato Theory, we know that $\mathrm{Gr}(\mathbb{C}((z)))$ is the space of τ-functions of the KP hierarchy (for each W, its τ-function, $\tau_W(T)$, is a τ-functions of the KP hierarchy and every τ-function arises in this way).

Let us make explicit the KP flows. In this picture, there is a group $\mathbb{C}((z))^*$ acts by homotheties on $\mathrm{Gr}(\mathbb{C}((z)))$.
Furthermore, the group acts on $\mathrm{Jac}^\infty(X)$ and this action is transitive at the level of tangent spaces, more precisely:

$$\mathbb{C}((z))/A_W \xrightarrow{\sim} T_W \,\mathrm{Jac}^\infty(X)$$

for $W = K(L, \phi)$ and $A_W = \{f | f \cdot W \subseteq W\}$. And, further:

$$\mathbb{C}((z))/(A_W + \mathbb{C}[[z]]) \xrightarrow{\sim} T_L \,\mathrm{Jac}(X)$$

which **resembles the structure of the Jacobian as a double coset**.
The flows defined by $\{z^{-i} | i \geq 1\}$ are the KP-flows.
This is related to Mulase's characterization of Jacobian varieties as finite dimensional orbits.

Let us show how τ-functions are related to theta functions of Jacobians. Having in mind the diagram:

$$\begin{array}{ccccccccc} 0 \to & H^0(X,L) & \to & H^0(X-x,L) & \to & (\widehat{\mathcal{O}}_{X,x})_{(0)}/\widehat{\mathcal{O}}_{X,x} & \to & H^1(X,L) & \to 0 \\ & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\ 0 \to & W\cap\mathbb{C}[[z]] & \to & W & \to & \mathbb{C}((z))/\mathbb{C}[[z]] & \to & \mathbb{C}((z))/(W+\mathbb{C}[[z]]) & \to 0 \end{array}$$

$$K^* \operatorname{Det} \simeq \mathcal{O}_{\operatorname{Jac}(X)}(-\Theta)$$

Can we say something about sections?
Yes, given (X, x, z), fix a symplectic basis of $H_1(X, \mathbb{Z})$, and one defines a $g \times \infty$-matrix, A, and a quadratic form Q (depending on the expansions of differentials on $X - x$). Then ([Krichever]):

$$\tau_W(t) = \exp(Q(t))\,\theta(A(t) + \xi)$$

where:

- W is the image of (L, ϕ) by the Krichever map;
- and $\xi \in \mathbb{C}^g$ a preimage of L.

This exhibits a connection between geometric objects and hierarchies of partial differential equations.

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Moduli of Pointed Curves with Trivialization

Now, let X vary (and $E = \mathcal{O}_X$ trivial v.b.). [Arbarello-De Concini-Kac-Procesi, Kawamoto-Namikawa-Tsuchiya-Yamada, Ueno, Muñoz-P., etc].

Theorem (Muñoz-P.)

The moduli functor $\mathcal{M}^\infty$ parametrizing triples (X, x, t) is representable by a subscheme of the Sato Grassmannian.

The functor on groups:

$$G := \operatorname{Aut}_{\mathbb{C}-\text{alg}} \mathbb{C}((z))$$

acts on $\operatorname{Gr}(\mathbb{C}((z)))$ preserving the determinant and inducing an action:

$$G \times \mathcal{M}^\infty \longrightarrow \mathcal{M}^\infty$$

Let $\widetilde{G}$ be the extension of G deduced from it action on $\operatorname{Gr}(\mathbb{C}((z)))$. Then, Lie $\widetilde{G}$ is isomorphic to the **Virasoro algebra** (which generates the space of central extensions of Lie G).
Let us see some consequences for $\mathcal{M}^\infty$.

Formal Uniformization

Let (X, x, t) be given, then the orbit map at the level of tangent spaces is:

$$\text{Lie}\, G = \mathbb{C}((z))\frac{\partial}{\partial z} \twoheadrightarrow T_{(X,x,t)}\mathcal{M}^\infty \twoheadrightarrow T_X\mathcal{M} = H^1(X, \mathbb{T}_X)$$

where $\mathcal{M}$ denotes the moduli space of curves, the last map is induced by the forgetful map and the composition is the **Kodaira-Spencer map**.

Theorem (Formal Uniformization, Muñoz-P.)

The map between the formal completions:

$$\widehat{(G)}_{\text{Id}} \longrightarrow \widehat{(\mathcal{M}^\infty)}_{(X,x,t)}$$

is surjective.

The above surjectivity, an analogue of Lie's Theorem and Schlessinger formal smoothnees are the key points of its proof.

Mumford Formula

The action of G on $\mathrm{Gr}(\mathbb{C}((z))dz^{\otimes\beta})$ gives rise to a central extension $\widetilde{G}_\beta$ with cocycle (as extension of Lie algebras) c_β and they satisfies:

$$c_\beta = (6\beta^2 - 6\beta + 1)c_1$$

$$c_\beta = (6\beta^2 - 6\beta + 1)c_1$$

$$c_1(m,n) = \delta_{n+m,0}\left(\frac{m^3 - m}{6}\right) = 2 \cdot \mathrm{vir}(m,n)$$

where vir is the cocycle of the Virasoro Algebra.

This resembles the Mumford formula: let $\mathcal{M}_g^0$ be an open subscheme of the moduli of curves where there exists a universal curve:

$$\pi : \mathcal{X} \to \mathcal{M}_g^0$$

and let the Hodge bundles $\lambda_\beta := \mathrm{Det}\, R^\bullet \pi_* \omega_{\mathcal{X}}^{\otimes\beta}$ where $\omega_{\mathcal{X}}$ the relative dualizing sheaf. Then, Mumford's formula reads:

$$\lambda_\beta \simeq \lambda_1^{(6\beta^2 - 6\beta + 1)}$$

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Consider:

$$\begin{array}{ccc} \mathcal{M}_g^\infty & \xhookrightarrow{K_\beta} & \mathrm{Gr}(\mathbb{C}((z))dz^{\otimes\beta}) \\ {\scriptstyle p}\downarrow & & \\ \mathcal{M}_g^0 & & \end{array} \qquad K_\beta(X,x,t) := H^0(X-x,\omega_X^{\otimes\beta})$$

Theorem (Muñoz-P.)

$$K_\beta^* \,\mathrm{Det} \,\simeq\, p^*\lambda_\beta \qquad \text{on } \mathcal{M}_g^\infty$$

That is, the relation among cocycles **is** the infinitesimal version of the Mumford isomorphisms.

Notes:

- Mumford formula is highly relevant in String Theory (e.g. Polyakov measure, [Mumford, Belavin-Knizhnik, etc]).
- Explicit expressions and variation w.r.t. genus has been extensively studied ([Faltings, Beilinson-Manin, etc]).

The Group of Semilinear Automorphisms

From now on, we focus on `arXiv:1001.1719` [Hernández-Gómez-Muñoz-P.]

Definition

The group of semilinear automorphisms of $V = \mathbb{C}((z))^n$, $\mathrm{SGl}_{\mathbb{C}((z))}(V)$, consists of $\mathbb{C}$-linear automorphisms:

$$\gamma \colon V \xrightarrow{\sim} V$$

such that there exists $g \in G := \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}(\mathbb{C}((z)))$, satisfying:

$$\gamma(z \cdot v) \;=\; g(z) \cdot \gamma(v)$$

Proposition (Structure)

$$1 \to \mathrm{Gl}_{\mathbb{C}((z))}(V) \to \mathrm{SGl}_{\mathbb{C}((z))}(V) \to G = \mathrm{Aut}_{\mathbb{C}-alg}\, \mathbb{C}((z)) \to 1\,.$$

$$\mathfrak{sgl}_{\mathbb{C}((z))}(V) = \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V, V)$$

as Lie subalgebras of $\mathrm{End}_{\mathbb{C}}\, \mathbb{C}((z))^n$.

Central Extensions

$\mathrm{SGl}_{\mathbb{C}((z))}(V)$ acts on $V_{n,\beta} = (\mathbb{C}((z))(dz)^{\otimes\beta})^n$ by:

$$\mu_{n,\beta}\big(\gamma(z\cdot v)\big) = g'(z)^{\beta}\cdot\gamma(v)\,.$$

One shows that $\mu_{n,\beta}$ induces an action of $\mathrm{SGl}_{\mathbb{C}((z))}(V)$ on $\mathrm{Gr}(V)$, preserving the determinant and, therefore, it gives rise to a central extension:

$$1 \to \mathbb{C}^* \to \widetilde{\mathrm{SGl}}^{\beta}_{\mathbb{C}((z))}(V) \to \mathrm{SGl}_{\mathbb{C}((z))}(V) \to 1$$

Let $c_{n,\beta}$ be the associated cocycle (between their Lie alg.)

Note that each central extension:

$$1 \to \mathbb{C}^* \to \widetilde{G}_{\beta} \to G \to 1$$

can be pulled back to $\mathrm{SGl}_{\mathbb{C}((z))}(V)$, by the surjection $\mathrm{SGl}_{\mathbb{C}((z))}(V) \xrightarrow{p_n} G$, yielding a central extension:

$$1 \to \mathbb{C}^* \to \mathrm{SGl}_{\mathbb{C}((z))}(V)\times_G \widetilde{G}_{\beta} \to \mathrm{SGl}_{\mathbb{C}((z))}(V) \to 1\,.$$

Let us denote by $\mathrm{vir}_{n,\beta}$ its cocycle.

It is easy to check $\text{vir}_{n,\beta} = n \cdot \text{vir}$.

Theorem

$$c_{n,\beta} = \beta c_{n,1} + (1-\beta)c_{n,0} + 6n\beta(\beta-1)\,\text{vir}$$

For the proof one computes explicitly the value of the cocycles on a basis of $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$ using the formula:

$$c(g_1, g_2) \;=\; \det(\bar{g}_1 \circ (\overline{g_1 \circ g_2})^{-1} \circ \bar{g}_2)$$

Writing the formula between the associated bitorsors:

$$\mathbb{L}_{n,\beta} \;\simeq\; \mathbb{L}_{n,1}^{\otimes\beta} \otimes \mathbb{L}_{n,0}^{\otimes(1-\beta)} \otimes p^* \Lambda_1^{\otimes 6n\beta(\beta-1)} \,,$$

where p is the natural projection map $\text{SGl}_{\mathbb{C}((z))}(V) \to G$.

Note: in the rank 1 case, the space of central extensions is known to be three dimensional and generated by $c_{n,1}, c_{n,0}, \text{vir}$ ([Arbarello-De Concini-Kac-Procesi].

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Open problems

$\mathrm{SGl}_{\mathbb{C}((z))}(V)$ acts on the Sato Grassmannian preserving the determinant and, thus, there are representations of it on the Fock space:

$$\widetilde{\mathrm{SGl}}_{\mathbb{C}((z))}(V) \longrightarrow \mathrm{Gl}\left((H^0(\mathrm{Gr}(V), \mathrm{Det}^*)\right)$$

and, similarly, infinite dimensional Lie algebra representations.
$\widetilde{\mathrm{SGl}}_{\mathbb{C}((z))}(V)$ is plenty of interesting subalgebras (Virasoro algebras, etc) and very closely related to the so-called $\mathcal{W}$-algebras in mathematical physics (and Atiyah algebras of Beilinson).

1. Since $\mathrm{Gr}(V)$ parametrizes τ-functions for the KP, what τ-functions solve the p.d.e. coming from those Lie algebras? (Recall that Kontsevich studied certain KP τ-functions in the context of the String Equation).
2. Can we interpret the Lie algebra representations of these algebras and take advantage from the fact that our moduli spaces are embedded into the Sato Grassmannian?

Representability of $\mathcal{U}_g^\infty(n,d)$

Recall that for g, n and d, the set of rational points of the moduli functor $\mathcal{U}_g^\infty(n,d)$ consists of 5-tuples (X,x,t,E,ϕ) where:

- X is a genus g projective, smooth and irreducible curve;
- $x \in X$ is a point;
- $t : \widehat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[z]]$;
- E is a rank n degree d v.b. on X;
- $\phi : \widehat{E}_x \simeq \widehat{\mathcal{O}}_{X,x}^{\oplus n}$.

Theorem

$\mathcal{U}_g^\infty(n,d)$ is representable by a subscheme of the Sato Grassmannian Gr(V).

The data (x,t,ϕ) *rigidify* the pair (X,E) so that it has no automorphisms.
Stability is no longer required, but the moduli space fails to be of finite type.

Proof of Representability

1. **Krichever map** is now:

$$(X, x, t, E, \phi) \mapsto H^0(X - x, E)$$

2. **Injectivity**: for $W = H^0(X - x, E)$, observe that:

$$(X, x, t) \text{ is recovered from } A_W := \{f \in \mathbb{C}((z)) | f \cdot W \subseteq W\}$$

   Then, W is an A_W-module that allows us to recover E on $X := \operatorname{Spec} A_W \cup \{x\}$.

3. **Characterization**:

$$W \text{ lies in the image of Krichever} \iff \begin{cases} A_W \in \operatorname{Gr}\left(\mathbb{C}((z))\right) \\ A_W \text{ is a regular ring} \end{cases}$$

4. **Condition**: the set of pairs $(A, W) \in \operatorname{Gr}\left(\mathbb{C}((z))\right) \times \operatorname{Gr}\left(V\right)$ such that $A \cdot A = A$, A regular and $A \cdot W = W$ is a subscheme. Then, study the projection $(A, W) \mapsto W$.

Tangent Space

Let $W \in \mathrm{Gr}\left(V\right)$, then the tangent space is:

$$T_W \,\mathrm{Gr}\left(V\right) \;=\; \mathrm{Hom}_{\mathbb{C}}(W, V/W)$$

Let $\mathcal{M}_g^\infty$ be the moduli space of triples (X, x, t). Pick $\mathcal{X} = (X, x, z)$. Then, the forgetful map induces:

$$T_{\mathcal{X}}\mathcal{M}_g^\infty \;\longrightarrow\; T_X\mathcal{M}_g \;=\; H^1(X, \mathbb{T}_X)$$

and, further, one shows that:

$$T_{\mathcal{X}}\mathcal{M}_g^\infty \;=\; \varprojlim H^1(X, \mathbb{T}_X(mx)) \;\simeq\; \mathrm{Der}_{\mathbb{C}}(A, \mathbb{C}((z))/A)$$

where $A = H^0(X - x, \mathcal{O}_X)$. For $\mathcal{X} = (X, x, z)$ given, let $\mathcal{U}_{\mathcal{X}}^\infty(n, d)$ be the moduli space of pairs (E, ϕ) on X and let $\mathcal{U}_{\mathcal{X}}(n, d)$ be the moduli space of v.b. E on X. Recall that $T_E\mathcal{U}_{\mathcal{X}}(n, d) \simeq H^1(X, \mathcal{E}nd(E))$. Furthermore:

$$T_{(E,\phi)}\mathcal{U}_{\mathcal{X}}^\infty(n, d) \;=\; \varprojlim H^1(X, E^* \otimes E(-mx)) \;\simeq\; \mathrm{Hom}_{A-\mathrm{mod}}(W, V/W)$$

where $W = H^0(X - x, E)$.

Theorem

Let $\mathcal{E} = (X, x, t, E, \phi) \in \mathcal{U}_g^\infty(n, d)$*, then:*

$$T_{\mathcal{E}}\mathcal{U}_g^\infty(n, d) \simeq \mathcal{D}^1_{A/\mathbb{C}}(W, V/W)$$

where $A = H^0(X - x, \mathcal{O})$, $W = H^0(X - x, E)$ *and* $\mathcal{D}^1$ *denotes the Lie algebra of first order differential operators with scalar symbol.*

Proof.

$$T_W\mathcal{U}_g^\infty(n, d) = \{\overline{W} \in T_W \operatorname{Gr}(V) \text{ s.t. } A_{\overline{W}} \in T_A \operatorname{Gr}(\mathbb{C}((z)))\}$$

Search for $f \in \operatorname{Hom}_{\mathbb{C}}(W, V/W)$ s.t. $\exists g \in \operatorname{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W)$ with:

$$\overline{W} = \{w + \epsilon f(w),\ w \in W\} \text{ is a module over } < a + \epsilon g(a),\ a \in A > \}$$

Or, what amounts to the same, $f(aw) = af(w) + g(a)w$.
Thus, f belongs to $\mathcal{D}^1_{A/\mathbb{C}}(W, V/W)$.
The converse is straightforward. □

Theorem (Formal Uniformization)

The group $\mathrm{SGl}_{\mathbb{C}((z))}(V)$ *acts on* $\mathcal{U}_g^\infty(n,d)$*. Furthermore, this action is locally transitive, i.e. for* $\mathcal{E} = (X, x, t, E, \phi) \in \mathcal{U}_g^\infty(n,d)$*, then:*

$$\Big(\mathrm{SGl}_{\mathbb{C}((z))}(V)\Big)^{\widehat{}}_{\mathrm{Id}} \twoheadrightarrow \Big(\mathcal{U}_g^\infty(n,d)\Big)^{\widehat{}}_{\mathcal{E}}$$

Let us put everything together.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(X-x, \mathcal{E}nd_X E) & \to & \mathfrak{gl}_{\mathbb{C}((z))}(V) & \to & T_{(E,\phi)}\mathcal{U}_X^\infty(n,d) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(X-x, \mathcal{D}^1_{X/\mathbb{C}}(E,E)) & \to & \mathfrak{sgl}_{\mathbb{C}((z))}(V) & \to & T_{\mathcal{E}}\mathcal{U}_g^\infty(n,d) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(X-x, \mathbb{T}_X) & \to & \mathfrak{g} & \to & T_{(X,x,z)}\mathcal{M}_g^\infty & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Which is identified (via the Krichever map) with the diagram...

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X-x, \mathcal{E}nd_X E) & \longrightarrow & \mathfrak{gl}_{\mathbb{C}((z))}(V) & \longrightarrow & T_{(E,\phi)}\mathcal{U}_X^{\infty}(n,d) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X-x, \mathcal{D}^1_{X/\mathbb{C}}(E,E)) & \longrightarrow & \mathfrak{sgl}_{\mathbb{C}((z))}(V) & \longrightarrow & T_{\mathcal{E}}\mathcal{U}_g^{\infty}(n,d) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X-x, \mathbb{T}_X) & \longrightarrow & \mathfrak{g} & \longrightarrow & T_{(X,x,z)}\mathcal{M}_g^{\infty} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{End}_{A_W} W & \longrightarrow & \operatorname{End}_{\mathbb{C}((z))} V & \longrightarrow & \operatorname{Hom}_{A_W}(W, V/W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{D}^1_{A_W/\mathbb{C}}(W) & \longrightarrow & \mathcal{D}^1_{\mathbb{C}((z))/\mathbb{C}}(V) & \longrightarrow & \mathcal{D}^1_{A_W/\mathbb{C}}(W, V/W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Der}_{\mathbb{C}}(A_W) & \longrightarrow & \operatorname{Der}_{\mathbb{C}}\left(\mathbb{C}((z))\right) & \longrightarrow & \operatorname{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Note that the rank 1 case is closely related to [Arbarello-De Concini-Kac-Procesi].

A relation in the Picard group of the moduli of pairs

Question: Similarly to the case of the Mumford formula in the case of the moduli of curves, we now wonder if the formula of cocycles on $\mathrm{SGl}_{\mathbb{C}((z))}(V)$ is an infinitesimal version of a relation in the Picard group of the moduli of pairs (X, E).

$\mathcal{M}_g$ be the moduli space of genus g smooth projective curves.
$\mathcal{M}_g^0$ be the open subscheme of curves without automorphisms ($g > 2$).
$p_{\mathcal{X}} : \mathcal{X} \to \mathcal{M}_g^0$ be the universal curve.
$\mathcal{U}_{\mathcal{X}}(n, d) \to \mathcal{M}_g^0$ be the moduli of s.s. v.b. over $\mathcal{X}$ of rk n deg d. [Simpson, etc.].

An universal vector bundle on $\mathcal{U}_{\mathcal{X}}(n, d)$ may not exist [Mestrano-Ramanan].

E be a relatively s.s. v.b. on $\mathcal{X} \times_{\mathcal{M}_g^0} S$ (for a scheme S).
E yields a map $S \to \mathcal{U}_{\mathcal{X}}(n,d)$. Let p be the composition

$$S \to \mathcal{U}_{\mathcal{X}}(n,d) \to \mathcal{M}_g^0$$

p be the composition $S \to \mathcal{U}_{\mathcal{X}}(n,d) \to \mathcal{M}_g^0$.
$\pi_{\mathcal{X}}, \pi$ be the projections of $\mathcal{X} \times_{\mathcal{M}_g^0} S$ onto the first and second factors.
ω is the dualizing sheaf of $\mathcal{X} \to \mathcal{M}_g^0$.
$Ł_\beta$ is the line bundle on S defined as follows:

$$Ł_\beta := \operatorname{Det} R^\bullet \pi_*(E \otimes \pi_{\mathcal{X}}^* \omega^{\otimes \beta}) \qquad \text{for } \beta \in \mathbb{Z}$$

Theorem

There is an isomorphism of line bundles over S:

$$Ł_\beta \xrightarrow{\sim} Ł_1^{\otimes \beta} \otimes Ł_0^{\otimes(1-\beta)} \otimes p^* \lambda_1^{\otimes 6n\beta(\beta-1)},$$

where λ_1 is the Hodge bundle on $\mathcal{M}_g^0$; that is, $\operatorname{Det} R^\bullet (p_{\mathcal{X}})_* \omega$.

Comparing Formulas

We continue with the above notations.
Assume that a lift of $S \to \mathcal{U}_{\mathcal{X}}(n,d)$ to $\mathcal{U}_g^\infty(n,d)_{ss}$ is given; in other words, that one has a 5-tuple of the type $(\mathcal{X} \times_{\mathcal{M}_g^0} S, x, z, E, \phi)$ or, equivalently:

$$\begin{array}{ccc} & & \mathcal{U}_g^\infty(n,d) \\ & \nearrow & \downarrow \\ S & \longrightarrow & \mathcal{U}_{\mathcal{X}}(n,d) \end{array}$$

Assume that a $\mathbb{C}$-valued point of S is given and let $U \in \mathcal{U}_g^\infty(n,d)$ be its image by the Krichever map. Using our previous results, we have:

$$\begin{array}{ccccc} \mathrm{SGl}_{\mathbb{C}((z))}(V) \simeq \mathrm{SGl}_{\mathbb{C}((z))}(V) \times \{U\} & \xrightarrow{\mu_U} & \mathcal{U}_g^\infty(n,d) & & \\ \downarrow{\scriptstyle \bar{\mu}_U} & \nearrow & \downarrow & \searrow & \\ S & \xrightarrow[\bar{p}]{} & \mathcal{U}_{\mathcal{X}}(n,d) & \longrightarrow & \mathcal{M}_g^0 \end{array}$$

(assuming that the orbit of U falls inside S, i.e. μ_U factors through S and we obtain $\bar{\mu}_U$, see the dashed arrow).

Theorem

The pullback of the formula:

$$Ł_{\beta} \stackrel{\sim}{\to} Ł_1^{\otimes\beta} \otimes Ł_0^{\otimes(1-\beta)} \otimes p^*\lambda_1^{\otimes 6n\beta(\beta-1)}$$

by:

$$\bar{\mu}_U : \ \mathrm{SGl}_{\mathbb{C}((z))}(V) \longrightarrow S$$

is precisely the formula:

$$\mathbb{L}_{n,\beta} \simeq \mathbb{L}_{n,1}^{\otimes\beta} \otimes \mathbb{L}_{n,0}^{\otimes(1-\beta)} \otimes p^*\Lambda_1^{\otimes 6n\beta(\beta-1)}$$

Thanks for your atention!

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