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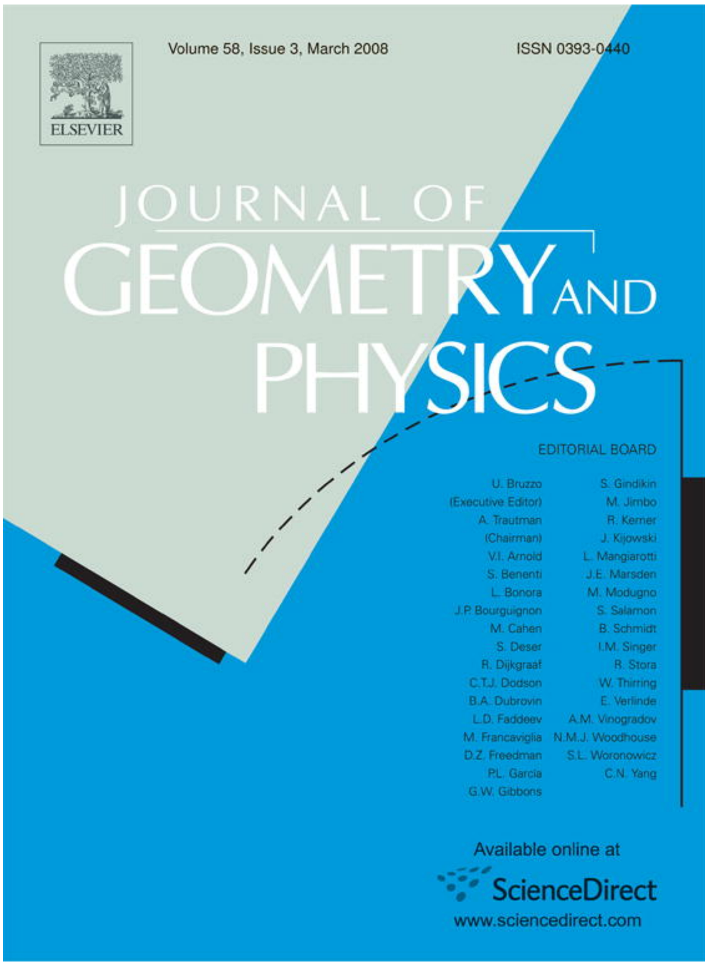

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Stability on the Sato Grassmannian. Applications to the moduli of vector bundles

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Abstract

The action of $\mathrm{Sl}(r, k[[z]])$ on the Sato Grassmannian is studied. Following ideas similar to those of GIT and to those used in the study of vector bundles, the (semi)stable points are introduced. It is shown that any point admits a Harder–Narasimhan filtration and that, if it is semistable, it has a Jordan–Hölder filtration. Finally, theses results are compared with the well-known theory of vector bundles on an algebraic curve.
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1. Introduction

This paper continues the study, started by Osipov [9], of the notion of stability of vector bundles in terms of the geometry of the Sato Grassmannian. We aim at establishing relationships between two well-known constructions of moduli spaces of vector bundles on an algebraic curve; namely, the one that uses GIT (e.g. [13]) and the one that considers pairs of bundles with a trivialization and uses Grassmannians (e.g. [7,1]). Very naively, the former should be the quotient of the latter by a certain group action. Under this perspective, one is naturally led to the study of the action of $\mathrm{Sl}(r, k[[z]])$ on the Sato Grassmannian. Our paper is a first step in this direction. The difficulties of this approach come into two flavors: the group is neither of finite type nor reductive; the Sato Grassmannian is not of finite type.

We begin with the study of $\mathrm{Sl}(r, k)$ acting on $\mathrm{Gr}(k((z))^{\oplus r})$. Then, in Section 3.1 GIT is applied to the action induced on certain natural finite type subschemes of the $\mathrm{Gr}(k((z))^{\oplus r})$ and it yields a numerical criterion for stability (Theorem 3.5) as well as the behaviour of stability under the natural morphisms (Proposition 3.6).

Section 3.2 begins with the definition of (semi)stable points of $\mathrm{Gr}(k((z))^{\oplus r})$ with respect to the action of $\mathrm{Sl}(r, k)$ which is motivated by Proposition 3.6. Then, some computations are needed to establish a numerical criterion of

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(semi)stability, see Theorem 3.11. These results allow us to offer a natural definition of (semi)stability with respect to the action of $\mathrm{Sl}(r, k[[z]])$ (Definition 3.12) as well as the corresponding numerical criterion in terms of the *slope*, which reminds us of the criterion for the case of vector bundles.

It should be observed that since the points of $\mathrm{Gr}(k((z))^{\oplus r})$ are k-subspaces of $k((z))^{\oplus r}$ and having a notion of (semi)stability, it makes sense to study filtrations of these subspaces. In particular, we prove in Section 3.3 that any point of $\mathrm{Gr}(k((z))^{\oplus r})$ admits a unique Harder–Narasimhan filtration (Theorem 3.27). Similarly, Section 3.4 deals with Jordan–Hölder filtrations of semistable objects (see Theorem 3.39).

As an application of our results we focus on the case of vector bundles on a punctured algebraic curve. It is well known that the Krichever map allows us to embed the moduli space of pairs of vector bundles with a formal trivialization $\{(E, \delta)\}$ into a Sato Grassmannian (see [7] and references therein). Thus, Section 4 shows explicitly how the Krichever map transforms (semi)stable objects, Harder–Narasimhan filtrations and Jordan–Hölder filtrations.

Upon finishing this introduction, some comments are in order. Osipov [9] has already studied how the (semi)stability of a vector bundle is stated in terms of Sato Grassmannians and has given a definition that is eventually proved to coincide with ours. Our aim was to relate Osipov's definition to GIT as well as possible.

The study of equations defining the set of (semi)stable points as well as the construction of quotients are two important problems that deserve further efforts. For the latter, a better and more explicit understanding of quotients under unipotent groups will be required. Moreover, it should be recalled that Sato Grassmannians have been extensively used in a variety of problems (e.g. string theory, multicomponent KP hierarchy, vertex algebras, ...). We believe that the interpretation of (semi)stability in the framework of those theories would be of great interest.

2. Infinite Grassmannian $\mathrm{Gr}(k((z))^{\oplus r})$

2.1. General theory

In this subsection we summarize some known results about the infinite Grassmannian. For more details on this subject readers are referred to [2,11].

We begin with the definition of $\mathrm{Gr}(V, V_+)$ for an arbitrary pair (V, V_+), where V is a linear k-vector space and V_+ is a fixed k-subspace of V. We say that a subspace $A \subset V$ is commensurable with V_+ when $\dim_k (A+V_+)/(A\cap V_+) < \infty$ and we denote this by $A \sim V_+$. The pair (V, V_+) is assumed to satisfy

- $\bigcap_{A\sim V_+} A = (0)$
- $V = \varprojlim_{A\sim V_+} V/A$

The infinite Grassmannian $\mathrm{Gr}(V, V_+)$ (in short $\mathrm{Gr}(V)$ if we fix V_+) is the k-scheme ([2], Theorem 2.11) whose rational points are

$$\mathrm{Gr}(V) = \{k\text{-subspaces } W \subset V \text{ such that } \dim_k V/(V_+ + W) < \infty, \dim_k W \cap V_+ < \infty\}. \tag{2.1}$$

The index or characteristic of $L \in \mathrm{Gr}(V)$

$$\chi(L) = \dim_k(L \cap V_+) - \dim_k \left(\frac{V}{L + V_+}\right)$$

is locally constant as function of L. If $\mathrm{Gr}^\chi(V)$ denotes the set where the index takes the value $\chi \in \mathbb{Z}$, then

$$\mathrm{Gr}(V) = \coprod_{\chi\in\mathbb{Z}} \mathrm{Gr}^\chi(V)$$

is the decomposition in connected components ([2], Lemma 2.13).

In particular, if V is a finite dimensional vector space, the points of $\mathrm{Gr}^\chi(V)$ are those subspaces L where $\dim_k L = \chi + \dim_k(V/V_+)$.

We define the open k-subschemes of $\mathrm{Gr}(V)$ for each $A \sim V_+$

$$F_A := \{L \in \mathrm{Gr}(V) | L \oplus A = V\},$$

which define an open covering of $\mathrm{Gr}(V)$.

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Let us now consider the case $V := k((z))^{\oplus r}$, $V_+ := k[[z]]^{\oplus r}$, and the linear topology in V given by $\{z^m V_+|m \in \mathbb{Z}\}$ as a basis of neighbourhoods of (0). We define

$$e_i := \begin{cases} (z^{i/r}, 0, 0, \dots, 0), & i \equiv 0 \bmod r \\ (0, z^{(i-1)/r}, 0, \dots, 0), & i \equiv 1 \bmod r \\ \vdots, & \vdots \\ (0, 0, \dots, z^{(i-r+1)/r}), & i \equiv (r-1) \bmod r. \end{cases}$$

so that V_+ is the completion of the space $\langle \{e_i\}_{i\geq 0}\rangle$.

We define $\mathcal{S}_\chi$ ($\chi \in \mathbb{Z}$) as the set consisting of the strictly increasing sequences $S = \{s_0, s_1, \dots\} \subset \mathbb{Z}$ such we have an integer $i \gg 0$, for which $s_{n+1} = s_n + 1$, $\forall n \geq i$, and the index

$$i(S) := \#(\{s_0, s_1, \dots\} \setminus \mathbb{Z}_{\geq 0}) - \#(\mathbb{Z}_{\geq 0} \setminus \{s_0, s_1, \dots\})$$

is equal to χ.

For each $S \in \mathcal{S}_\chi$, we define the subspace A^S as the closure (w.r.t. the topology of V) of $\langle\{e_{s_i}, s_i \in S\}\rangle$. Given $S \in \mathcal{S}_\chi$, observe that $L \in F_{A^S}$ if and only if $L \oplus A^S = V$ and $\dim(A^S/A^S \cap V^+) - \dim(V^+/A^S \cap V^+) = \chi$. It follows that $\mathrm{Gr}^\chi(V) = \bigcup_{S\in\mathcal{S}_\chi} F_{A^S}$ and that

$$\chi(L) = \chi = i(S).$$

The Grassmannian $\mathrm{Gr}(V)$ carries a canonical line bundle on it; namely, the determinant bundle, Det_V, which is the determinant of the complex $\mathcal{L} \to V/V_+$, where $\mathcal{L}$ is the universal submodule. Furthermore, the determinant of the morphism of that complex gives rise to a canonical global section, Ω_+, of Det_V^* on the connected component $\mathrm{Gr}^0(V)$. Analogously, we can define other global sections of Det_V^* on $\mathrm{Gr}^\chi(V)$ for each A^S by replacing V_+ by A^S in the previous construction. The resulting global section is denoted by Ω_S.

Finally, recall that the Plücker morphism

$$\begin{aligned} \mathcal{P}_V : \mathrm{Gr}^\chi(V) &\longrightarrow \mathbb{P}\Omega^* \\ W &\longmapsto \{\Omega_S(W)\}_{S\in\mathcal{S}_\chi} \end{aligned}$$

(Ω being the k-subspace of $H^0(\mathrm{Gr}^\chi(V), \mathrm{Det}_V^*)$ generated by $\{\Omega_S\}_{S\in\mathcal{S}_\chi}$) is a closed immersion ([11], Theorem 2.3).

2.2. Description in terms of finite Grassmannians

The infinite Grassmannian is shown to be covered by an ascending chain of open subschemes U^m. Each of these open schemes is an inverse limit of open sets of finite Grassmannians.

Given $m, i \in \mathbb{N}$, let us consider the finite-dimensional spaces

$$V_{[-m,i)} := \frac{z^{-m}V_+}{z^i V_+} \simeq \langle\{e_k\}_{k=-rm,\dots,ri-1}\rangle.$$

Let us denote by $\mathrm{Gr}(V_{[-m,i)})$ the finite Grassmannian associated with the pair $(V_{[-m,i)}, V_{[0,i)})$ and by $\mathrm{Gr}^\chi(V_{[-m,i)})$ the connected component of index χ, whose rational points are the subspaces with dimension $\chi + rm$.

Proposition 2.1. *Fix integers m, χ. The map*

$$F_{m+1} \mapsto \Phi_m(F_{m+1}) := \frac{(F_{m+1} \cap V_{[-m,m+1)}) + V_{[m,m+1)}}{V_{[m,m+1)}} \subset \frac{V_{[-m,m+1)}}{V_{[m,m+1)}} \simeq V_{[-m,m)}$$

defines a surjective rational morphism $\mathrm{Gr}^\chi(V_{[-(m+1),m+1)}) \dashrightarrow \mathrm{Gr}^\chi(V_{[-m,m)})$ *whose domain is the open subscheme*

$$X^\chi_{m+1} := \left\{F_{m+1} s.t. F_{m+1} + V_{[-m,m+1)} = V_{[-(m+1),m+1)} and F_{m+1} \cap V_{[m,m+1)} = (0)\right\}.$$

Proof. The set X^χ_{m+1} consists of those F_{m+1} such that:

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- $F_{m+1} \oplus V_{[-m,m+1)} \to V_{[-(m+1),m+1)}$ is surjective
- $F_{m+1} \oplus V_{[m,m+1)} \to V_{[-(m+1),m+1)}$ is injective

and both conditions are open and nonempty.

Second, let us compute the index of $\Phi_m(F_{m+1})$

$$\chi\left(\Phi_m(F_{m+1})\right) = \dim(F_{m+1} \cap V_{[-m,m+1)}) - rm =$$

since $F_{m+1} \cap V_{[m,m+1)} = (0)$ and, thus

$$\begin{aligned} &= \dim F_{m+1} + \dim V_{[-m,m+1)} - \dim(F_{m+1} + V_{[-m,m+1)}) - rm \\ &= \dim F_{m+1} - r(m+1) = \chi(F_{m+1}) \end{aligned}$$

because $F_{m+1} + V_{[-m,m+1)} = V_{[-(m+1),m+1)}$.

Finally, in order to see that the morphism is surjective it suffices to notice that

$$F_m = \Phi_m(F_m \oplus \langle e_{-rm-r}, \dots, e_{-rm+1} \rangle)$$

and that $F_m \oplus \langle e_{-rm-r}, \dots, e_{-rm+1} \rangle$ is in X^{χ}_{m+1} for every $F_m \in \mathrm{Gr}^{\chi}(V_{[-m,m)})$. □

Remark 2.2. The choice of $\{e_i\}_{i\in\mathbb{Z}}$ yields a basis on $V_{[-m,m)}$ for all m. The Plücker coordinates of a point $F_m \in \mathrm{Gr}^{\chi}(V_{[-m,m)})$ can be computed as follows (recall that $l := \dim_k(F_m) = \chi + rm$). Fixing a basis in F_m, for a sequence of indices $-rm \leq i_1 < \cdots < i_l \leq rm - 1$ and the complementary sequence $-rm \leq j_1 < \cdots < j_{2rm-l} \leq rm - 1$, the Plücker coordinate $p_{i_1,\dots,i_l}(F_m)$ is equal to the determinant of the matrix of the natural map

$$F_m \longrightarrow \frac{V_{[-m,m)}}{\langle e_{j_1}, \dots, e_{j_{2rm-l}} \rangle}$$

with respect to the basis chosen in F_m and the basis $\{e_i\}$ in $V_{[-m,m)}$. This is usually understood as a minor of the matrix consisting of the coordinates w.r.t. $\{e_i\}$ of a basis of F_m. A straightforward computation shows that the Plücker coordinates of F_{m+1} and $\Phi_m(F_{m+1})$ are related by

$$p_{i_1,\dots,i_l}(\Phi_m(F_{m+1})) = p_{-rm-r,\dots,-rm-1,i_1,\dots,i_l}(F_{m+1}).$$

Remark 2.3. Note that the definition of the morphism Φ_m only depends on the filtration $\{z^m V_+\}_{m\in\mathbb{Z}}$ and not on the basis $\{e_i\}_{i\in\mathbb{Z}}$ of V.

Definition 2.4 (*Property* (m)). A point $F \in \mathrm{Gr}^{\chi}(V)$ is said to satisfy the property (m) if $F \cap z^m V_+ = (0)$ and $F + z^{-m} V_+ = V$. Observe that if F satisfies the property (m_0) then it satisfies the property (m) for every $m \geq m_0$.

Lemma 2.5. *It holds that*

$$\left\{F \in \mathrm{Gr}^{\chi}(V) \text{ such that } F \text{ satisfies property } (m)\right\} = \bigcup_{S\in\mathcal{S}^m_{\chi}} F_{A^S} \tag{2.2}$$

where $\mathcal{S}^m_{\chi}$ *consists of those sequences* $S = \{s_0 < s_1 < \cdots\} \in \mathcal{S}_{\chi}$ *such that* $\{rm, rm+1, \dots\} \subset S \subset \{-rm, -rm+1, -rm+2, \dots\}$.

Proof. Note that the subspace F satisfies the property (m) if and only if $F + \langle\{e_i\}_{i\geq -rm}\rangle = V$ and $F \cap \langle\{e_i\}_{i\geq rm}\rangle = (0)$. These two conditions are satisfied if and only if there exists $S \in \mathcal{S}$ such that $F \oplus A^S = V$ and $\langle\{e_i\}_{i\geq rm}\rangle \subset A^S \subset \langle\{e_i\}_{i\geq -rm}\rangle$. Recalling that the open subschemes F_A cover $\mathrm{Gr}^{\chi}(V)$, the claim follows. □

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We have the following diagram:

$$
\begin{array}{ccccccccc}
\cdots \to & X_3^\chi \cap \Phi_2^{-1}(X_2^\chi \cap \Phi_1^{-1}(X_1^\chi)) & \xrightarrow{\Phi_2} & X_2^\chi \cap \Phi_1^{-1}(X_1^\chi) & \xrightarrow{\Phi_1} & X_1^\chi & \xrightarrow{\Phi_0} & \mathrm{Gr}^\chi(V_{[0,0)}) \\
& \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & X_3^\chi \cap \Phi_2^{-1}(X_2^\chi) & \xrightarrow{\Phi_2} & X_2^\chi & \xrightarrow{\Phi_1} & \mathrm{Gr}^\chi(V_{[-1,1)}) & & \\
& \downarrow & & \downarrow & & & & \\
\cdots \longrightarrow & X_3^\chi & \xrightarrow{\Phi_2} & \mathrm{Gr}^\chi(V_{[-2,2)}) & & & & \\
& \downarrow & & & & & & \\
\cdots \longrightarrow & \mathrm{Gr}^\chi(V_{[-3,3)}) & & & & & &
\end{array} \tag{2.3}
$$

where each row is considered as an inverse system. We index the rows by $0, \dots, m, m+1, \dots$ and the columns by $0, 1, \dots, i, i+1, \dots$ from right to left. Each square of the diagram is cartesian, so if $U_{m,i}$ denotes the term of the diagram lying in the m-th row and i-th column; it holds that

$$U_{m,i} = \begin{cases} \mathrm{Gr}^\chi(V_{[-m,m)}) & \text{for } i = m \\ \Phi_{i-1}^{-1}(U_{m,i-1}) \cap X_i^\chi = U_{m,i-1} \times_{\mathrm{Gr}^\chi(V_{[-i,i)})} X_i^\chi & \text{for } i > m \end{cases}$$

Theorem 2.6. *Let U^m be the inverse limit $\varprojlim_{i \geq m} U_{m,i}$ for the maps Φ_m.*

There is an isomorphism

$$U^m \simeq \{F \in \mathrm{Gr}^\chi(V) \mid F \text{ satisfies theproperty } (m)\}$$

and, in particular, the open sets U^m define a covering of the infinite Grassmannian

$$\mathrm{Gr}^\chi(V) = \bigcup_{m>0} U^m = \bigcup_{m>0} \varprojlim_{i \geq m} U_{m,i}.$$

Proof. Let us give a map from the open set of $\mathrm{Gr}^\chi(V)$ of the points with the property (m_0) to U^{m_0}. Let $F \in \mathrm{Gr}^\chi(V)$ be such a point. We define

$$F_{[-m,i)} := \frac{(F \cap z^{-m} V_+) + z^i V_+}{z^i V_+} \subset V_{[-m,i)} \quad i \geq m \geq m_0.$$

Then, for every $m \geq m_0$, the following statements are satisfied:

- $F_{[-m,m)} \in \mathrm{Gr}^\chi(V_{[-m,m)})$
- $F_{[-(m+1),m+1)} \in X_{m+1}^\chi$
- $\Phi_m(F_{[-(m+1),m+1)}) = F_{[-m,m)}$.

The last property implies that $\Phi_i(F_{[-(i+1),i+1)}) = F_{[-i,i)}$ for every $i \geq m$. For $i > m$, $F_{[-i,i)}$ is a the point of $U_{m,i}$. Note that $\{F_{[-i,i)}\}_{i \geq m} \in U^m$ for every $m \geq m_0$.

We now construct the inverse map. Let $\{F_{m_0,i}\}_{i \geq m_0} \in \varprojlim_{i \geq m_0} U_{m_0,i}$. For $m \geq m_0$, we consider

$$F_m := \varprojlim_{i \geq m} (F_{m_0,i} \cap V_{[-m,i)}) \subset z^{-m} V_+$$

where the morphisms of this inverse system are induced by the inclusion $F_{m_0,i+1} \cap V_{[-m_0,i+1)} \hookrightarrow F_{m_0,i}$ and by the projection $V_{[-m_0,i+1)} \twoheadrightarrow V_{[-m_0,i)}$. This procedure gives us a family of subspaces $F_m \subseteq z^{-m} V_+$,where $m \geq m_0$. Moreover, since $F_m \subseteq F_{m+1}$, it makes sense to consider the subspace $F := \bigcup_{m \geq m_0} F_m$ of V. It is straightforward to check that $F \in \mathrm{Gr}^\chi(V)$ and that it satisfies the property (m_0).

By the very constructions, the two maps given above are the inverse of each other.

Finally, from the identity (2.1) we have that given $F \in \mathrm{Gr}^\chi(V)$ there exist $m_1, m_2 \in \mathbb{N}$ (depending on F) such that $F \cap z^{m_1} V_+ = 0$ and $F + z^{-m_2} V_+ = V$. Taking $m_0 = \max\{m_1, m_2\}$, one concludes that F fulfils the property (m_0). Therefore, the open sets U^m cover $\mathrm{Gr}^\chi(V)$. □

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The very construction yields the following:

Corollary 2.7. *Let $F \in \mathrm{Gr}^{\chi}(V)$ be a point satisfying the property (m_0) and let F_m and $\{F_{[-i,i)}\}_{i\geq m_0}$ be the subspaces corresponding to F as in the proof of Theorem 2.6.*

For every $i \geq m \geq m_0$, it holds that

$$\chi(F) = \chi(F_m) = \chi(F_{[-i,i)}) = \chi(F_{[-m,i)}).$$

3. Stability for the action of $\mathrm{Sl}(r, k[[z]])$

3.1. Stability on $\mathrm{Gr}(V_{[-m,m)})$ for the action of $\mathrm{Sl}(r,k)$

Mumford's criterion [6] is applied to the finite Grassmannians $\mathrm{Gr}(V_{[-m,m)})$ and a numerical criterion of stability is obtained (Theorem 3.5). In Chapter 4.4 of [6] or in [8] one finds a detailed study of the stability notion on finite Grassmannians for the action of the group of automorphisms of the vector space. We adapt those computations to our case.

Since the group $\mathrm{Sl}(r,k)$ is a subgroup of $\mathrm{Sl}(r,k((z)))$, it acts naturally on $k((z))^{\oplus r}$. The subspaces $z^{-m}V_+$, $z^m V_+$ are invariant by this action, so $\mathrm{Sl}(r,k)$ acts naturally on the spaces $V_{[-m,m)}$ and on its Grassmannians $\mathrm{Gr}(V_{[-m,m)})$. Moreover, this action lifts to an action on the determinant bundle, eventually giving rise to a natural linearization. Using the above-mentioned basis of $V_{[-m,m)}$, the group $\mathrm{Sl}(r,k)$ can be seen as a subgroup of $\mathrm{Sl}(2mr,k) \subset \mathrm{Gl}(V_{[-m,m)})$ by the following immersion:

$$\begin{aligned} \mathrm{Sl}(r,k) &\hookrightarrow \mathrm{Sl}(2rm,k) \\ A &\to \begin{pmatrix} A & 0 & & \\ 0 & A & & \\ & & \ddots & \\ 0 & & & A \end{pmatrix}. \end{aligned}$$

Let $F_m \in \mathrm{Gr}^{\chi}(V_{[-m,m)})$ be a point of the finite Grassmannian, with characteristic χ and dimension $q := rm + \chi$. Let $(x^j_{-rm}, \ldots, x^j_{rm-1})$ be the coordinates of a basis of F_m w.r.t. the basis $\{e_i\}$ of $V_{[-m,m)}$. We denote by $p_{i_1 \ldots i_q}(F_m)$ the minor of order $q \times q$ of the $(2rm \times q)$-matrix (x^j_i), formed by the columns of index $-rm \leq i_1 < \cdots i_q \leq rm-1$. These minors define the Plücker coordinates of F_m.

Let $\lambda(t)$ be a 1-parameter subgroup of $\mathrm{Sl}(r,k)$.

Definition 3.1 (*[8], Page 104*). We shall denote by $\mu(F_m,\lambda)$ the unique integer μ such that the limit $\lim_{t\to 0} t^{\mu}\lambda(t)F_m$ exists and is different from zero.

Theorem 3.2 (*Theorem 4.9 and Proposition 4.11 of [8]*). *F_m is (semi)stable for the action of $\mathrm{Sl}(r,k)$ if and only if*

$$\mu(gF_m, \lambda^*)(\geq) > 0, \quad \forall g \in \mathrm{Sl}(r,k) \tag{3.1}$$

and for every 1-parameter group, λ^*, of the form

$$\lambda^*(t) := \begin{pmatrix} t^{n_1} & 0 & & \\ 0 & t^{n_2} & & \\ & & \ddots & \\ 0 & & & t^{n_r} \end{pmatrix} \tag{3.2}$$

where $n_i \in \mathbb{Z}$ satisfy $n_1 + n_2 + \cdots + n_r = 0$, $n_1 \geq n_2 \geq \cdots \geq n_r$, and some n_i is different from zero. We say $n = (n_1, \ldots, n_r)$ is admissible if the n_i's satisfy these relations.

Let $\sigma := (i_1 < i_2 < \cdots < i_q)$ be the multiindex with $-rm \leq i_1, i_2, \ldots, i_q \leq rm-1$. And, for each $j \in \{1,\ldots,r\}$, let σ_j be the number of indices i_l $(1 \leq l \leq q)$ with $i_l = (j-1) \pmod r$, i.e. the number of indices i_l that correspond to some $(0,\ldots,z^s,0,\ldots,0)$ where z^s is in the j-th entry.

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Lemma 3.3. *Let $g \in \mathrm{Sl}(r, k)$ be arbitrary, then*

$$\mu(gF_m, \lambda^*) = \max\left\{-\sum_{i=1}^{r} n_i\sigma_i : p_\sigma(gF_m) \neq 0\right\} \tag{3.3}$$

where $p_\sigma(gF_m)$ is the Plücker coordinate associated with the columns with indices $\sigma := (i_1 < i_2 < \cdots < i_q)$.

Proof. This results from the action by the 1-parameter groups λ^*, (3.2), and from the definition given before of $\mu(gF_m, \lambda^*)$. □

Let us compute the maximum of the previous Lemma. For every $1 \le l \le r$, we consider

$$V^l := k((z)) \oplus \overset{l}{\cdots} \oplus k((z)) \oplus (0) \oplus \overset{r-l}{\cdots} \oplus (0) \subset k((z))^{\oplus r}$$
$$V_+^l := k[[z]] \oplus \overset{l}{\cdots} \oplus k[[z]] \oplus (0) \oplus \overset{r-l}{\cdots} \oplus (0) \subset k[[z]]^{\oplus r}$$

and we define the spaces $V^l_{[-m,m)}$, with $1 \le l \le r$, as

$$V^l_{[-m,m)} := \frac{z^{-m}V_+^l}{z^m V_+^l} \subseteq V_{[-m,m)}.$$

A basis of $V^l_{[-m,m)}$ is given by the equivalence classes of the vectors e_{rk+i}, with $-m \le k \le m-1$ and $i = 0, 1, \ldots, l-1$.

Lemma 3.4. *For every $g \in \mathrm{Sl}(r, k)$ and every 1-parameter group λ^* of the form (3.2), it holds that*

$$\mu(gF_m, \lambda^*) = -qn_r + \sum_{i=1}^{r-1} \dim_k(gF_m \cap V^i_{[-m,m)})(n_{i+1} - n_i).$$

Proof. For convenience we shall write the matrices with respect to $\{e_{-rm}, \ldots, e_{rm-1}\}$ with its rows and columns reordered as if we had chosen

$$e_{-rm}, e_{r(-m+1)}, \ldots, e_{r(m-1)},$$
$$e_{-rm+1}, e_{r(-m+1)+1}, \ldots, e_{r(m-1)+1}, \ldots,$$
$$e_{-rm+(r-1)}, e_{r(-m+1)+(r-1)}, \ldots, e_{r(m-1)+(r-1)}$$

as a basis. With this convention, $\lambda^* \in \mathrm{Sl}(2rm, k)$ is expressed by the diagonal matrix

$$\begin{pmatrix} t^{n_1} & 0 & & \cdots & & 0 \\ & \ddots & & & & \\ 0 & & t^{n_1} & & & \\ & & & \ddots & & \\ & & & & t^{n_r} & \\ & & & & & \ddots \\ 0 & & & & & t^{n_r} \end{pmatrix}.$$

Let $s_i := \dim_k(gF_m \cap V^i_{[-m,m)})$ for each $i = \{1, \ldots, r\}$. Take a basis $(x^1, \ldots x^q)$ of gF_m whose first s_i vectors lie in $gF_m \cap V^i_{[-m,m)}$ and consider the coordinates (x_i^j) of these vectors with respect to the basis of $V_{[-m,m)}$ ordered as above. The $q \times q$-minors of this matrix define the Plücker coordinates of gF_m. Due to the choice of the basis, there exists a multi-index $\sigma = (i_1 < \cdots < i_q)$ with $p_\sigma(gF_m) \neq 0$ and with s_1 indices in the family $\{-rk\}_{k=-m,\ldots,m-1}$, s_2 indices in the family $\{-rk, -rk+1\}_{k=-m,\ldots,m-1}$, etc. Hence, for this multi-index the values σ_i satisfy: $\sigma_1 + \cdots + \sigma_j = s_j$ $(1 \le j \le r)$. Moreover, for any other multi-index $\bar{\sigma}$ such that $\bar{\sigma}_1 + \cdots + \bar{\sigma}_j < s_j$ for some $1 \le j \le r$, the $(q \times q)$-minor

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of (x_i^j) associated with this multi-index would be a determinant of the form

$$p_{\bar\sigma}(gF_m) = \det \begin{pmatrix} * & 0 & \\ & s_j\times(q-\sum_{i=1}^{j}\bar\sigma_i) & \\ * & & * \end{pmatrix}$$

with a $(s_j \times (q - \sum_{i=1}^{j}\bar\sigma_i))$ block of 0's, where there are linearly dependent rows when $\sum_{i=1}^{j}\bar\sigma_i < s_j$. Thus $p_{\bar\sigma}(gF_m) = 0$ for such a multi-index $\bar\sigma$.

Let us now consider $\bar\sigma$ any multiindex with $p_{\bar\sigma}(gF_m) \neq 0$. It holds that $\bar\sigma_1 + \cdots + \bar\sigma_j \geq s_j$ for every $j \in \{1, \ldots, r\}$ (and the equality holds for the previously mentioned multi-index σ), and hence

$$\begin{aligned} -\sum_{j=1}^{r} n_j\bar\sigma_j = & - n_r(\bar\sigma_1+\cdots+\bar\sigma_r) + (n_r - n_{r-1})(\bar\sigma_1+\cdots+\bar\sigma_{r-1}) \\ & + (n_{r-1}-n_{r-2})(\bar\sigma_1+\cdots+\bar\sigma_{r-2}) + \cdots + (n_2-n_1)\bar\sigma_1 \\ & \leq -qn_r + (n_r-n_{r-1})s_{r-1} + (n_{r-2}-n_{r-1})s_{r-2} + \cdots + (n_2-n_1)s_1 = -\sum_{j=1}^{r} n_j\sigma_j. \end{aligned}$$

We conclude from (3.3) that

$$\begin{aligned} \mu(gF_m,\lambda^*) &= -\sum_{i=1}^{r}[\dim_k(gF_m\cap V^i_{[-m,m)}) - \dim_k(gF_m\cap V^{(i-1)}_{[-m,m)})]n_i \\ &= -qn_r + \sum_{i=1}^{r-1}\dim_k(gF_m\cap V^i_{[-m,m)})(n_{i+1}-n_i). \quad \square \end{aligned} \tag{3.4}$$

Let us study the situation when $\mu(gF_m,\lambda^*)$ is positive (resp. nonnegative). The right-hand side of equality (3.4) is a linear function of $n := (n_1,\ldots,n_r)$. For admissible n we have $n_1 > 0$ and $n_r < 0$. Let $x_i := \frac{n_{i+1}-n_i}{n_r} \geq 0$ for $1 \leq i \leq r-1$. To give an admissible n is equivalent to giving $n_r < 0$ and $x_1,\ldots,x_{r-1} \in \frac{1}{n_r}\mathbb{Z}$, with

$$x_1 \geq 0, \ldots, x_{r-1} \geq 0,\ x_1 + 2x_2 + \cdots + (r-1)x_{r-1} = r. \tag{3.5}$$

With these new data,

$$\mu(gF_m,\lambda^*) = -n_r\left[q - \sum_{i=1}^{r-1}\dim_k(gF_m\cap V^i_{[-m,m)})x_i\right].$$

The right-hand side is positive (resp. nonnegative) if and only if

$$q - \sum_{i=1}^{r-1}\dim_k(gF_m\cap V^i_{[-m,m)})x_i\,(\geq) > 0.$$

for any $x_1,\cdots,x_{r-1} \in \mathbb{Q}$ satisfying (3.5), i.e. we have a problem of linear programming. This expression is positive (resp. non-negative) if and only if it is so in the vertices of the set bounded by the inequalities (3.5). These points, for each $i = 1,\ldots,r-1$, have coordinates $x_i = \frac{r}{i}$ and $x_j = 0, \forall j \neq i$. Therefore we obtain:

$$\begin{aligned} \mu(gF_m,\lambda^*)\,(\geq) > 0 &\Leftrightarrow \dim_k(gF_m\cap V^i_{[-m,m)})\,(\leq) < \frac{qi}{r} \Leftrightarrow \\ &\Leftrightarrow \dim_k(F_m\cap g^{-1}V^i_{[-m,m)})\,(\leq) < \frac{qi}{r} \end{aligned}$$

for every $1 \leq i \leq r-1$.

Theorem 3.5. *Let $F_m \in \mathrm{Gr}^\chi(V_{[-m,m)})$. Then, the following conditions are equivalent:*

(1) *F_m is a (semi)stable point of $\mathrm{Gr}^\chi(V_{[-m,m)})$.*

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(2) $\frac{1}{i}\dim_k(F_m \cap gV^i_{[-m,m)})\ (\leq)<\ \frac{1}{r}\dim_k(F_m)$, *for every* $g \in \mathrm{Sl}(r,k)$ *and for every* $1 \leq i \leq r-1$.
(3) $\frac{1}{i}\chi(F_m \cap gV^i_{[-m,m)})\ (\leq)<\ \frac{1}{r}\chi(F_m)$, *for every* $g \in \mathrm{Sl}(r,k)$ *and every* $1 \leq i \leq r-1$.

Proof. For the equivalence of (1) and (2), we use the previously proven relation in the case of the element g^{-1}. To see the relation with the characteristics, we simply recall the relation between the characteristic and the dimension in the case of Grassmannians of finite dimensional vector spaces. □

The behaviour of the stability under the rational map Φ_m follows from Proposition 2.1 and Theorem 3.5 and is given by

Proposition 3.6. *If* $F_m \in \mathrm{Gr}^\chi(V_{[-m,m)})$ *is a (semi)stable point for the action of* $\mathrm{Sl}(r,k)$*, then any* $F_{m+1} \in \Phi_m^{-1}(F_m)$ *is (semi)stable for the same action.*

3.2. *Stability on* $\mathrm{Gr}(k((z))^{\oplus r})$

We introduce a notion of (semi)stability of points of $\mathrm{Gr}(V)$ and give a numerical criterion (Theorem 3.11) based on the case of finite Grassmannians (Theorems 3.5 and 2.6). This subsection ends with an intrinsic definition of (semi)stability for the action of $\mathrm{Sl}(r,k[[z]])$ (Definition 3.12) compatible with GIT that is analogous to the one given in [9] and equivariant by certain automorphisms of $\mathrm{Gr}(V)$.

Motivated by Theorem 2.6 and Proposition 3.6 it is natural to define

Definition 3.7. A point $F \in \mathrm{Gr}(V)$ is (semi)stable for the action of $\mathrm{Sl}(r,k)$ if there exists $m \in \mathbb{N}$ and $i \geq m$ such that $F_{[-i,i)} \in U_{m,i} \subset \mathrm{Gr}(V_{[-i,i)})$ is (semi)stable. Here $\{F_{[-i,i)}\}_{i\geq m}$ and F are related as described in the proof of Theorem 2.6. We denote the set of the stable and semistable points of $\mathrm{Gr}(V)$ by $(\mathrm{Gr}(V))^s$ and $(\mathrm{Gr}(V))^{ss}$, respectively.

Definition 3.8. Let H^l be a l-dimensional $k((z))$-subspace of V. For a k-subspace $F \subseteq H^l$ such that $\dim_k(F \cap (H^l \cap V_+)) < \infty$ we define $\chi(F)$ as

$$\chi(F,H^l) := \begin{cases} \dim_k(F\cap V_+ \cap H^l) - \dim_k\left(\dfrac{H^l}{F+(V_+\cap H^l)}\right), & \text{if } \dim_k\left(\dfrac{H^l}{F+(V_+\cap H^l)}\right) < \infty \\ -\infty, & \text{otherwise.}\end{cases}$$

Whenever H^l is clear from the context, it will be omitted and we shall simply write $\chi(F)$.

Note that if a k-subspace $F \subset V$ satisfies the condition that $\chi(F,H^l)$ is finite, then $F \in \mathrm{Gr}(H^l, V_+ \cap H^l)$ and $\chi(F,H^l)$ coincides with the characteristic of F as a point of this Grassmannian.

Lemma 3.9. *Let* $F \in \mathrm{Gr}^\chi(V)$ *be a point satisfying the property* (m_0) *and let* F_m *and* $\{F_{[-i,i)}\}_{i\geq m_0}$ *be the subspaces corresponding to* F *as in the proof of Theorem* 2.6.

For every $g \in \mathrm{Sl}(r,k)$, $i \geq m \geq m_0$ *and* $1 \leq l \leq r-1$ *it holds that*

$$\chi(F_m \cap gz^{-m}V^l_+) = \chi(F_{[-m,i)} \cap gV^l_{[-i,i)}).$$

Proof. Since $gV_+ = V_+$ for $g \in \mathrm{Sl}(r,k)$, one has that $gz^{-m}V_+ = z^{-m}V_+$. Suppose that $\chi(F \cap gV^l)$ is finite or, what amounts to the same, that $F \cap gV^l \in \mathrm{Gr}(gV^l)$. Then, the statement follows from Corollary 2.7 since the m-th, and $[m,i)$-th subspaces associated to $F \cap gV^l$ are given by

$$\left(F\cap gV^l\right)_m = F\cap gV^l \cap z^{-m}V^l_+ = F_m \cap gz^{-m}V^l_+$$
$$\left(F\cap gV^l\right)_{[-m,i)} = \frac{(F\cap gV^l\cap z^{-m}V^l_+)+z^iV^l_+}{z^iV^l_+} = F_{[-m,i)} \cap gV^l_{[-i,i)}. \quad \square$$

Lemma 3.10. *Let* F *be as in Lemma* 3.9*. For every* $g \in \mathrm{Sl}(r,k)$, $m \geq m_0$*, it holds that*

(1) $0 \leq \chi(F_m \cap gz^{-m}V^l_+) - \chi(F_{m+1} \cap gz^{-m-1}V^l_+) \leq l$.

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(2) $\chi(F \cap gV^l) \leq \chi(F_m \cap gz^{-m}V_+^l)$.
(3) $\chi(F \cap gV^l) = \lim_{m\to\infty} \chi(F_m \cap gz^{-m}V_+^l)$.

Proof. (1) Observe that $\chi(F_m \cap gz^{-m}V_+^l)$ and $\chi(F_{m+1} \cap gz^{-m-1}V_+^l)$ are, respectively, the Euler–Poincaré characteristics of the first two columns of the following exact sequence of complexes (written vertically)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F \cap gz^{-m}V_+^l) \oplus gV_+^l & \longrightarrow & (F \cap gz^{-m-1}V_+^l) \oplus gV_+^l & \longrightarrow & \frac{(F\cap gz^{-m-1}V_+^l)}{(F\cap gz^{-m}V_+^l)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & gz^{-m}V_+^l & \longrightarrow & gz^{-m-1}V_+^l & \longrightarrow & \frac{gz^{-m-1}V_+^l}{gz^{-m}V_+^l} & \longrightarrow & 0 \end{array}$$

Now the claim follows from the additivity property of the Euler–Poincaré characteristic.

(2) The claim follows from similar arguments as above for the following sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (F \cap gz^{-m}V_+^l) \oplus gV_+^l & \longrightarrow & (F \cap gV^l) \oplus gV_+^l & \longrightarrow & \frac{(F\cap gV^l)}{(F\cap gz^{-m}V_+^l)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & gz^{-m}V_+^l & \longrightarrow & gV^l & \longrightarrow & \frac{gV^l}{gz^{-m}V_+^l} & \longrightarrow & 0 \end{array}$$

(3) The two preceding items imply that $\chi(F \cap gV^l) \leq \lim_{m\to\infty} \chi(F_m \cap gz^{-m}V_+^l)$. If $\lim_{m\to\infty} \chi(F_m \cap gz^{-m}V_+^l) = -\infty$, the statement holds. If the limit is finite, then we have m' such that

$$\chi(F_m \cap gz^{-m}V_+^l) = \chi(F_{m+1} \cap gz^{-m-1}V_+^l) \quad \forall m \geq m'.$$

Having in mind the sequence of cokernels of the diagram of item 2, the condition is equivalent to saying that

$$\frac{\frac{gV^l}{gz^{-m}V_+^l}}{\frac{(F\cap gV^l)}{(F\cap gz^{-m}V_+^l)}} = \frac{gV^l}{F \cap gV^l + gz^{-m}V_+^l}$$

is finite dimensional and does not depend on m. However, this can happen if and only if it is equal to zero and the conclusion follows. □

Let us now give the numerical criterion for an arbitrary point of $\mathrm{Gr}(V)$ with $V = k((z))^{\oplus r}$.

Theorem 3.11. *A point $F \in \mathrm{Gr}^\chi(V)$ is (semi)stable for the action of $\mathrm{Sl}(r,k)$ if and only if for every $g \in \mathrm{Sl}(r,k)$ and for every $1 \leq l \leq r-1$,*

$$\frac{1}{l}\chi(F \cap gV^l)\,(\leq) < \frac{1}{r}\chi(F).$$

Proof. Definition 3.7 and Lemma 3.9 imply that a point $F \in \mathrm{Gr}^\chi(k((z))^{\oplus r})$ is (semi)stable if and only if there are $m \in \mathbb{N}$ and $i \geq m$ such that $F_{[-m,i)} \in \mathrm{Gr}(V_{[-i,i)})$ is (semi)stable. Applying Theorem 3.5 and Lemma 3.9, one has that it is equivalent to the inequality

$$\frac{1}{l}\chi(F_i \cap g(z^{-i}V_+^l))\,(\leq) < \frac{1}{r}\chi(F), \quad \forall g \in \mathrm{Sl}(r,k)$$

for all $1 \leq l \leq r-1$ and for all $i \geq m$. The claim now follows from Lemma 3.10 part 2. □

Note that $\mathrm{Sl}(r, k[[z]])$ acts on V, leaving the subspace V_+ invariant. Actually, from [11] we know that this group acts on $\mathrm{Gr}(V)$ and that this action lifts to an action on Det_V^*. This fact was proved in [9] (Lemma 3) in terms of the analytical space structure of the infinite Grassmannian ([15,12]). This group is indeed a subgroup of the *restricted linear group* of [15,12] or of the *bicontinuous linear group* of [11].

Finally, we are ready to introduce the notion of (semi)stability for the group $\mathrm{Sl}(r, k[[z]])$.

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Definition 3.12. A point $F \in \mathrm{Gr}^\chi(V)$ is called (semi)stable for the action of $\mathrm{Sl}(r, k[[z]])$ if $T(F)$ is (semistable) for the action of $\mathrm{Sl}(r, k)$ for all T in the subgroup $\{T \in \mathrm{Sl}(r, k[[z]])$ s.t. $T|_{z=0} = \mathrm{Id}\}$.

Let us make a remark on the motivation underlying this definition. The group $\mathrm{Sl}(r, k[[z]])$ is a group that acts on $k((z))^{\oplus r}$, preserving the filtration $\{z^m \cdot k[[z]]^{\oplus r}\}_{m\in\mathbb{Z}}$. Hence, it induces actions on each finite Grassmannian $\mathrm{Gr}(V_{[-m,m)})$ that are compatible with the morphisms considered in Section 2.2 and such that the pullbacks of the determinant bundles are again the determinant bundles. Since we are concerned with actions and (in the future) with quotients, it is natural to impose that (semi)stability should be a notion on the orbit.

Definition 3.13. The rank of $E \in \mathrm{Gr}^\chi(V)$, denoted by $r(E)$, is the dimension of V over $k((z))$ and the slope of E, $\mu(E)$, is $\mu(E) := \frac{\chi(E)}{r(E)}$.

Bearing in mind that $\mathrm{Sl}(r, k[[z]])$ is generated by the subgroups $\mathrm{Sl}(r, k)$ and $\{T \in \mathrm{Sl}(r, k[[z]])$ s.t. $T|_{z=0} = \mathrm{Id}\}$ and that it acts transitively on the set of $k((z))$-subspaces of V, the numerical criterion of stability (Theorem 3.11) is generalized in the following form:

Theorem 3.14. *Let $F \in \mathrm{Gr}^\chi(V)$. Then F is (semi)stable for the action of* $\mathrm{Sl}(r, k[[z]])$ *if every non-trivial $k((z))$-subspace $H^l \subset V^r$ fulfills $\mu(F \cap H^l)\,(\le) < \mu(F)$.*

Remark 3.15. An alternative approach would consist of the study of the action of the subgroup $\{T \in \mathrm{Sl}(r, k[[z]])$ s.t. $T|_{z=0} = \mathrm{Id}\}$ instead of ours which is based on the automorphisms. However, that would require an explicit *geometrical invariant theory* of unipotent groups since that subgroup is an inverse limit of unipotent groups.

3.3. Harder–Narasimhan filtration

We prove the existence of a unique Harder–Narasimhan filtration for each point of the infinite Grassmannian (Theorem 3.27). Our view is motivated by the works [13,3,10], which were carried out for the case of vector bundles on algebraic curves. Our reference for basic facts on Category theory is [5].

Definition 3.16. We define $\mathcal{G}$ to be the category whose objects are pairs (E, E_0^+), where E_0^+ is a free $k[[z]]$-module of finite rank and

$$E \in \mathrm{Gr}(E_0^+ \otimes_{k[[z]]} k((z)), E_0^+).$$

The set of homomorphisms between two objects, $\mathrm{Hom}_{\mathcal{G}}((E, E_0^+), (F, F_0^+))$, is the set formed by the $k[[z]]$-linear morphisms, $T : E_0^+ \to F_0^+$ with $(T \otimes 1)(E) \subset F$ (here $T \otimes 1$ is the linear map from $E_0^+ \otimes_{k[[z]]} k((z))$ to $F_0^+ \otimes_{k[[z]]} k((z))$ induced by T).

For an object (E, E_0^+), we define $E_0 := E_0^+ \otimes_{k[[z]]} k((z))$. For the sake of simplicity, we shall refer to this object as $E \in \mathrm{Gr}(E_0)$ and the set of homomorphisms $\mathrm{Hom}_{\mathcal{G}}((E, E_0^+), (F, F_0^+))$ will be written as $\mathrm{Hom}_{\mathcal{G}}(E, F)$. Similarly, the morphism $T \otimes 1 : E_0 \to F_0$ induced by a morphism $T \in \mathrm{Hom}_{\mathcal{G}}(E, F)$ will be also denoted by T and will be called the *underlying linear map*.

We give an abelian group structure for each set $\mathrm{Hom}_{\mathcal{G}}(E, F)$. Let $T, S \in \mathrm{Hom}_{\mathcal{G}}(E, F)$. We define the addition of T with S as the induced by the sum of the corresponding underlying maps. It is immediate to see that $\mathcal{G}$ is an additive category w.r.t. this addition law.

Since an object of $\mathcal{G}$ is a point of an infinite Grassmannian, we may consider its rank and slope as those given in Definition 3.13.

Similarly, we point out that Definition 3.12 yields a notion of (semi)stability for objects of $\mathcal{G}$. Indeed, let E_0^+ be as above. Let us fix a $k[[z]]$-linear isomorphism $T : E_0^+ \xrightarrow{\sim} k[[z]]^{\oplus r}$. Then, there is an action of $\mathrm{Sl}(r, k[[z]])$ on $\mathrm{Gr}(E_0, E_0^+)$ (by conjugation by T). Then, the notion of (semi)stability may be transported from $\mathrm{Gr}(k((z))^{\oplus r}, k[[z]]^{\oplus r})$ to $\mathrm{Gr}(E_0, E_0^+)$ via the isomorphism of schemes

$$\mathrm{Gr}(E_0, E_0^+) \simeq \mathrm{Gr}(k((z))^{\oplus r}, k[[z]]^{\oplus r})$$

induced by T. This notion of (semi)stability does not depend on T because of Definition 3.12.

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Lemma 3.17. *Let E, F be objects of $\mathcal{G}$ and let $T \in \operatorname{Hom}_{\mathcal{G}}(E, F)$. Then*

(1) $\operatorname{Ker} T \cap E \in \operatorname{Gr}(\operatorname{Ker} T, \operatorname{Ker} T \cap E_0^+)$ *together with the natural morphism* $i : \operatorname{Ker} T \hookrightarrow E_0$ *is the kernel of* T.
(2) $F/(\operatorname{Im} T \cap F) \in \operatorname{Gr}(F_0/\operatorname{Im} T, F_0^+/(\operatorname{Im} T \cap F_0^+))$ *together with the natural morphism* $p : F_0 \to F_0/\operatorname{Im} T$ *is the cokernel of* T.

Proof. Let us prove the first statement. First, let us observe that $\operatorname{Ker}(T|_{E_0^+}) = \operatorname{Ker} T \cap E_0^+$ is a free $k[[z]]$-module of finite rank and that $\operatorname{Ker} T = \operatorname{Ker}(T|_{E_0^+}) \otimes_{k[[z]]} k((z))$. Thus, we have to check that $\operatorname{Ker} T \cap E$ belongs to $\operatorname{Gr}(\operatorname{Ker} T, \operatorname{Ker} T \cap E_0^+)$ using (2.1). Since $E \in \operatorname{Gr}(E_0, E_0^+)$, one has that $E \cap E_0^+$ is finite dimensional, and thus, $\operatorname{Ker} T \cap E \cap E_0^+$ is finite dimensional too. It remains to prove that $\frac{\operatorname{Ker} T}{\operatorname{Ker} T \cap E + \operatorname{Ker} T \cap E_0^+}$ is finite dimensional. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\operatorname{Ker} T \cap E) \oplus (\operatorname{Ker} T \cap E_0^+) & \longrightarrow & E \oplus E_0^+ & \longrightarrow & \frac{E}{\operatorname{Ker} T \cap E} \oplus \frac{E_0^+}{\operatorname{Ker} T \cap E_0^+} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker} T & \longrightarrow & E_0 & \longrightarrow & E_0/\operatorname{Ker} T & \longrightarrow & 0 \end{array}$$

and consider the following piece of the associated long exact sequence

$$\frac{E}{\operatorname{Ker} T \cap E} \cap \frac{E_0^+}{\operatorname{Ker} T \cap E_0^+} \to \frac{\operatorname{Ker} T}{\operatorname{Ker} T \cap E + \operatorname{Ker} T \cap E_0^+} \to \frac{E_0}{E + E_0^+}$$

The first term is isomorphic to $T(E) \cap T(E_0^+)$ and it is finite dimensional because it is contained in $F \cap F_0^+$ and $F \in \operatorname{Gr}(F_0, F_0^+)$. The third term is finite dimensional since $E \in \operatorname{Gr}(E_0, E_0^+)$. Therefore, the middle term is finite dimensional, as we wanted.

The inclusion $i : \operatorname{Ker} T \cap E \to E$ is a morphism in our category whose underlying linear map is $i : \operatorname{Ker} T \hookrightarrow E_0$. Let us prove that it is the kernel of T; namely, for any object H of $\mathcal{G}$ and any morphism $S \in \operatorname{Hom}_{\mathcal{G}}(H, E)$ such that $T \circ S = 0$, there exists a morphism $Q : H \to \operatorname{Ker} T \cap E$ with $S = i \circ Q$.

Since $\operatorname{Ker} T$ is the kernel of T in the category of $k((z))$-vector spaces, there exists a $k((z))$-linear map $Q : H_0 \to \operatorname{Ker} T$ such that $S = i \circ Q$ with $i : \operatorname{Ker} T \hookrightarrow E_0$. It suffices to check that Q defines a morphism of $\mathcal{G}$; i.e. $Q(H) \subset \operatorname{Ker} T \cap E$ and $Q(H_0^+) \subset \operatorname{Ker} T \cap E_0^+$ and this is an easy computation.

For the second claim, note that $\operatorname{Im} T \cap F_0^+$ is a free $k[[z]]$-module such that $\operatorname{Im} T = (\operatorname{Im} T \cap F_0^+) \otimes_{k[[z]]} k((z))$ and that, therefore, $F_0^+/(\operatorname{Im} T \cap F_0^+)$ is a free $k[[z]]$-module such that $F_0/\operatorname{Im} T = F_0^+/(\operatorname{Im} T \cap F_0^+) \otimes_{k[[z]]} k((z))$. The rest of the proof can be carried out using ideas similar to those of the first part. □

It is easy to prove the following:

Lemma 3.18. *Let E, F be objects of $\mathcal{G}$ and let $T \in \operatorname{Hom}_{\mathcal{G}}(E, F)$. Then*

(1) $T^{-1}(F)/\operatorname{Ker} T \in \operatorname{Gr}(E_0/\operatorname{Ker} T, T^{-1}(F_0^+)/\operatorname{Ker} T)$ *together with the morphism* $T : E_0/\operatorname{Ker} T \to F_0$ *is the image of* T. *It is isomorphic to the object given by* $\operatorname{Im} T \cap F \in \operatorname{Gr}(\operatorname{Im} T, \operatorname{Im} T \cap F_0^+)$ *and the morphism* $i : \operatorname{Im} T \hookrightarrow F_0$ *(via the isomorphism* $T : E_0/\operatorname{Ker} T \to \operatorname{Im} T$*).*
(2) $T(E) \in \operatorname{Gr}(\operatorname{Im} T, T(E_0^+))$ *together with the morphism* $T : E_0 \to \operatorname{Im} T$ *is the coimage of* T. *It is isomorphic to* $E/(\operatorname{Ker} T \cap E) \in \operatorname{Gr}(E_0/\operatorname{Ker} T, E_0^+/(\operatorname{Ker} T \cap E_0^+))$, *with the natural morphism* $p : E_0 \to E_0/\operatorname{Ker} T$.

One checks that the monomorphisms (resp. epimorphisms) of $\mathcal{G}$ are those morphisms whose underlying linear maps are injective (resp. surjective). Given an object $F \in \mathcal{G}$ a subobject of F is a diagram

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ E_0 & \overset{i}{\hookrightarrow} & F_0 \end{array},$$

where i the natural inclusion, which is a monomorphism.

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Definition 3.19. Let E be a fixed object of $\mathcal{G}$, we denote by $\mathcal{G}(E)$ the full subcategory of $\mathcal{G}$ formed by the subobjects of E.

If we have an object F and a subobject $i\colon F \hookrightarrow E$, the cokernel of i is $\operatorname{Coker} i = (E/F_0 \cap E, E_0/F_0, E_0^+/F_0 \cap E_0^+)$, and its image is $\operatorname{Im} i = (F_0 \cap E, F_0, F_0 \cap E_0^+)$. From now on, we shall say that the quotient E/F exists if $\operatorname{Coker} i$ is $(E/F, E_0/F_0, E_0^+/F_0^+)$. This happens precisely when $F_0 \cap E = F$ and $F_0 \cap E_0^+ = F_0^+$, i.e., $\operatorname{Im} i = F$.

Lemma 3.20. *Let $G \in \mathcal{G}(E)$ be a subobject such that E/G exists. Let $F \in \mathrm{Gr}(F_0, F_0^+)$ be in $\mathcal{G}(E/G)$ and let π be the natural projection $E \to E/G$.*

It holds that $\pi^{-1}(F_0) \cap E \in \mathcal{G}(E)$. Further, if $(E/G)/F$ exists, then $\pi^{-1}(F_0) \cap E = \pi^{-1}(F) \cap E$ and the quotient $E/(\pi^{-1}(F) \cap E)$ exists.

Proof. In order to prove that $\pi^{-1}(F_0) \cap E$ is an object of $\mathcal{G}(E)$ it suffices to check that $\pi^{-1}(F_0) \cap E \in \mathrm{Gr}(\pi^{-1}(F_0), \pi^{-1}(F_0) \cap E_0^+)$ using expression (2.1). The subspace $\pi^{-1}(F_0) \cap E \cap E_0^+$ is finite dimensional because it is contained in $E \cap E_0^+$ and $E \in \mathrm{Gr}(E_0, E_0^+)$.

It remains to check that $\pi^{-1}(F_0)/(\pi^{-1}(F_0) \cap E + \pi^{-1}(F_0) \cap E_0^+)$ is of finite dimension. Consider the surjective map induced by π

$$\frac{\pi^{-1}(F_0)}{\pi^{-1}(F_0) \cap E + \pi^{-1}(F_0) \cap E_0^+} \to \frac{F_0}{F_0 \cap E/G + F_0 \cap E_0^+/G_0^+}$$

The image has finite dimension because $F_0 \cap E/G$ is an object of $\mathcal{G}(E/G)$. The kernel, on the other hand, is

$$\frac{(\pi^{-1}(F_0) \cap E) + (\pi^{-1}(F_0) \cap E_0^+) + G_0}{(\pi^{-1}(F_0) \cap E) + (\pi^{-1}(F_0) \cap E_0^+)}$$

which is a quotient of $G_0/G + G_0^+$, because $G = G_0 \cap E \subset (\pi^{-1}(F_0) \cap E)$ and $G_0^+ = G_0 \cap E_0^+ \subset (\pi^{-1}(F_0^+) \cap E_0^+)$. We conclude that the kernel and image have finite dimension, and hence $\pi^{-1}(F_0)/(\pi^{-1}(F_0) \cap E + \pi^{-1}(F_0) \cap E_0^+)$ has finite dimension too.

Let us prove the second claim. Observe that if $(E/G)/F$ exists, then $F = F_0 \cap E/G$. Furthermore, $\pi^{-1}(F) \cap E = \pi^{-1}(F_0) \cap E$ is of the form $H_0 \cap E$ and the quotient $E/(\pi^{-1}(F) \cap E)$ exists. □

Lemma 3.21. *Let us fix E an object of $\mathcal{G}$ and let F be an object of $\mathcal{G}(E)$. Then, for any objet $\bar{F}$ of $\mathcal{G}(E)$ such that $F \hookrightarrow \bar{F} \hookrightarrow E$ and $r(F) = r(\bar{F})$ it holds that $\mu(F) \le \mu(\bar{F})$, and the equality holds if and only if $F = \bar{F}$. Among the objects $\bar{F}$ as above, the object $F_0 \cap E \in \mathrm{Gr}(F_0, F_0 \cap E_0^+)$ has maximal slope.*

Proof. We have $F \in \mathrm{Gr}(F_0, F_0^+)$, $\bar{F} \in \mathrm{Gr}(\bar{F}_0, \bar{F}_0^+)$, $E \in \mathrm{Gr}(E_0, E_0^+)$. By hypothesis, we have $F_0 \subset \bar{F}_0 \subset E_0$, $F \subset \bar{F} \subset E$ and $F_0^+ \subset \bar{F}_0^+ \subset E_0^+$. Since the ranks are equal, we conclude that $F_0 = \bar{F}_0$. We now apply the Snake Lemma to the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F \oplus F_0^+ & \longrightarrow & \bar{F} \oplus \bar{F}_0^+ & \longrightarrow & \frac{\bar{F}}{F} \oplus \frac{\bar{F}_0^+}{F_0^+} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F_0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

obtaining the long exact sequence

$$0 \to F \cap F_0^+ \to \bar{F} \cap \bar{F}_0^+ \to \frac{\bar{F}}{F} \oplus \frac{\bar{F}_0^+}{F_0^+} \to \frac{F_0}{F + F_0^+} \to \frac{F_0}{\bar{F} + \bar{F}_0^+} \to 0$$

From here we have that $\chi(\bar{F}) = \chi(F) + \dim\left(\frac{\bar{F}}{F} \oplus \frac{\bar{F}_0^+}{F_0^+}\right)$ and, dividing by $r(F) = r(\bar{F}) = \dim F_0$, we conclude the first part.

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Moreover, the object $F_0 \cap E \in \mathrm{Gr}(F_0, F_0 \cap E_0^+)$ belongs to $\mathcal{G}(E)$, because it is the image of $i: F \hookrightarrow E$ (Lemma 3.18) and it contains F as subobject. Bearing in mind that if $\bar{F}$ is an object of $\mathcal{G}(E)$ satisfying the hypothesis then it holds that $\bar{F}_0 = F_0$, $\bar{F} \subset F_0 \cap E$, $\bar{F}_0^+ \subset F_0 \cap E_0^+$ and one concludes from the first part. □

In particular, Lemma 3.21 proves that E/F exists if and only if F is maximal among those having the same rank.

Lemma 3.22. *Let F be an object of $\mathcal{G}(E)$ for which E/F exists, then*

(1) *$r(E) = r(F) + r(E/F)$ and $\chi(E) = \chi(F) + \chi(E/F)$.*
(2) *If F is different from zero and from E*

$$\mu(F) < \mu(E) \Leftrightarrow \mu(F) < \mu(E/F) \Leftrightarrow \mu(E) < \mu(E/F)$$
$$\mu(F) > \mu(E) \Leftrightarrow \mu(F) > \mu(E/F) \Leftrightarrow \mu(E) > \mu(E/F)$$
$$\mu(F) = \mu(E) \Leftrightarrow \mu(F) = \mu(E/F) \Leftrightarrow \mu(E) = \mu(E/F).$$

Proof. (1) The equality in the case of the ranks is trivial. For the characteristics we apply its additivity property to the exact sequence of complexes (written vertically)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F \oplus F_0^+ & \longrightarrow & E \oplus E_0^+ & \longrightarrow & \frac{E}{F} \oplus \frac{E_0^+}{F_0^+} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_0 & \longrightarrow & E_0 & \longrightarrow & \frac{E_0}{F_0} \simeq k((z))^{r(\frac{E}{F})} & \longrightarrow & 0
\end{array}$$

(2) This is an easy computation. □

Theorem 3.23. *Let E be an object of $\mathcal{G}$. Then E is (semi)stable for the action of* $\mathrm{Sl}(r, k[[z]])$ *if and only if any proper subobject F of E satisfies $\mu(F)\,(\leq) < \mu(E)$.*

Proof. Note that the group $\mathrm{Sl}(r, k[[z]])$ acts on $\mathrm{Ob}(\mathcal{G})$ and that it preserves (semi)stability. Therefore, it suffices to prove the claim for the case $E_0^+ = k[[z]]^{\oplus r}$. Let $E \in \mathrm{Gr}(k((z))^{\oplus r}, k[[z]]^{\oplus r})$ be semistable and let F be a proper subobject. By Lemma 3.21 and the stability of E (see Theorem 3.14), we have $\mu(F) \leq \mu(F_0 \cap E)\,(\leq) < \mu(E)$.

Conversely, let us assume that $\mu(F)\,(\leq) < \mu(E)$ holds for every subobject F of E. Let $H^l \subset k((z))^{\oplus r}$ be an arbitrary $k((z))$-subspace. If $E \cap H^l \notin \mathrm{Gr}(H^l, H^l \cap V_+)$, then $\mu(E \cap H^l) = -\infty < \mu(E)$. Moreover, if $E \cap H^l \in \mathrm{Gr}(H^l, H^l \cap V_+)$, then it is a suboject of E and one concludes by the definition of (semi)stability. □

Definition 3.24. For $E \neq 0$, we denote by $\mu_m(E)$ the maximum among the slopes of the nonzero objects of $\mathcal{G}(E)$.

By $\mathcal{G}(\mu_m(E))$ we denote the set of objects, F, of $\mathcal{G}(E)$ such that $F = 0$ or $\mu(F) = \mu_m(E)$. For these objects, since they are of maximum slope, the quotient E/F exists.

Lemma 3.25. *Let E be an object of $\mathcal{G}$ with $E \neq 0$. The following properties hold*

(1) *$\mu_m(E)$ is a rational number.*
(2) *Let G be an object of $\mathcal{G}(\mu_m(E))$ with $r(G) < r(E)$ maximal. Then*

$$\mu_m(E/G) \lneqq \mu_m(E). \tag{3.6}$$

(3) *There is a unique maximal object in $\mathcal{G}(\mu_m(E))$ with respect to the inclusions, $G(E)$.*
(4) *$E/G(E)$ is an object of $\mathcal{G}$.*
(5) *$G(E)$ is semistable.*
(6) *Let F be a nonzero semistable object of $\mathcal{G}(E)$ such that the quotient E/F exists. We then have*

$$F = G(E) \Leftrightarrow \mu_m(E/F) < \mu_m(E).$$

(7) *Let F be a semistable object of $\mathcal{G}$. Then*

$$\mathrm{Hom}_{\mathcal{G}}(F, E) = 0 \quad \text{if } \mu(F) > \mu_m(E)$$
$$\mathrm{Hom}_{\mathcal{G}}(F, E) = \mathrm{Hom}_{\mathcal{G}}(F, G(E)) \quad \text{if } \mu(F) = \mu_m(E).$$

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Proof. (1) Note that the set of rational numbers

$$\left\{\mu(F) = \frac{\chi(F)}{r(F)} \text{ s.t. } F \in \mathcal{G}(E) \text{ and } F \neq (0)\right\}$$

is upper bounded. Thus, it does have a supremum. This supremum is achieved because the set of ranks $\{r(F)\}_{F\in\mathcal{G}(E)}$ is finite and $\chi(F) \leq \dim(E \cap V_+)$.

(2) Let $\bar{F}$ be an arbitrary non-zero object of $\mathcal{G}(\mu_m(E/G))$. Let $\pi: E_0 \to E_0/G_0$ be the natural projection. Lemma 3.21 states that $\bar{F} = \bar{F}_0 \cap E/G$ and Lemma 3.20 that $F := \pi^{-1}(\bar{F}) \cap E = \pi^{-1}(\bar{F}_0) \cap E$ is an object of $\mathcal{G}(E)$. Since G has maximal slope, it follows that $\mu(F) \leq \mu(G) = \mu_m(E)$. If the equality holds, we would have that F is an object of $\mathcal{G}(\mu_m(E))$ with $r(F) = \dim_{k((z))} \bar{F}_0 + \dim_{k((z))} G_0 > r(G)$. However, this is not possible because $r(G)$ is maximal in $\mathcal{G}(\mu_m(E))$. Accordingly, we conclude that $\mu(F) < \mu(G)$. Applying Lemma 3.22 part 2, to F and G (observe that the quotient exists and $F/G = \bar{F}$), we conclude that $\mu(\bar{F}) < \mu(G)$ for all $\bar{F}$, hence $\mu_m(E/G) < \mu_m(E)$.

(3) If $r(E) = 1$, then Lemma 3.21 implies that E is itself the maximum. Let us now assume that $r(E) > 1$. Let G be an object in $\mathcal{G}(\mu_m(E))$ with $r(G)$ maximal. Since it has maximal slope, we have that $G = G_0 \cap E$, $G_0^+ = G_0 \cap E_0^+$ and that the quotient E/G exists. If $r(G) = r(E)$, then $E_0 = G_0$, $G = E$, $G_0^+ = E_0^+$ from Lemma 3.21. Thus, the maximum is E.

It remains to study the situation $1 \leq r(G) < r(E)$. If this is the case, then E/G exists and does not vanish. Let us now prove that G is the maximum. Let F be an object of $\mathcal{G}(\mu_m(E))$. By Lemma 3.18, we have that $\pi(F)$ is an object of $\mathcal{G}(E/G)$. If $\pi(F)$ does not vanish, the inequality of part 2. yields

$$\mu(F) = \mu_m(E) > \mu_m(E/G) \geq \mu(\pi(F)) = \mu(F/\operatorname{Ker}\pi \cap F).$$

The space $\operatorname{Ker}\pi \cap F$ exists as object in $\mathcal{G}(F)$ because is the kernel of the morphism $F \xrightarrow{\pi} E/G$. Then, Lemma 3.22 part 2. implies that $\mu(\operatorname{Ker}\pi \cap F) > \mu(F) = \mu_m(E)$, which is not possible since $\operatorname{Ker}\pi \cap F$ is also a subobject of E. We conclude that $\pi(F) = (0)$ and hence $F \subset G$.

(4) We have proved that G has maximum slope, and hence the quotient E/G exists.

(5) Let F be a subobject of $G(E)$. Since F is also a subobject of E, one has that $\mu(G(E)) = \mu_m(E) \geq \mu(H)$, and the conclusion follows from Theorem 3.23.

(6) If $E \neq 0$, formula (3.6) tells us that $\mu_m(E/F) < \mu_m(E)$ for $F = G(E)$. Conversely, let F be a nonzero semistable object of $\mathcal{G}(E)$ such that $\mu_m(E/F) < \mu_m(E)$. Let us consider the map $G(E) \hookrightarrow E \to E/F$ and observe that $\mu(G(E)) = \mu_m(E) > \mu_m(E/F)$. Thus, with arguments analogous to those of part 3, we conclude that $G(E) \subset F$. Since F is semistable and $G(E)$ is a subobject, we conclude that $\mu(G(E)) \leq \mu(F)$. The maximality of $\mu(G(E))$ implies that $\mu(G(E)) = \mu(F)$ and F thus belongs to $\mathcal{G}(\mu_m(E))$. The maximality of the rank of $G(E)$ and Lemma 3.21 imply that $G(E) = F$.

(7) Let F be a semistable object of $\mathcal{G}$ and let $T : F \to E$ be a nonzero morphism. Since F is semistable, it holds that $\mu(F) \geq \mu(\operatorname{Ker} T \cap F)$ and, by Lemma 3.22 part 2, one has that

$$\mu(F) \leq \mu(F/\operatorname{Ker} T \cap F) = \mu(T(F)).$$

Furthermore, $\mu(T(F)) \leq \mu_m(E)$ since $T(F)$ is a subobject of E. And the first equality follows. In order to prove the second claim, one uses arguments similar to those of part 3. □

Definition 3.26. Let E be an object of $\mathcal{G}$. A Harder–Narasimhan filtration of E is an ascending chain of subobjects

$$E^0 := 0 \subset E^1 \subset E^2 \subset \cdots \subset E^l = E$$

such that $E^i \in \mathcal{G}(E)$, the quotients E^i/E^{i-1} are semistable, and the sequence of slopes, $\{\mu(E^l/E^{l-1}), \mu(E^{l-1}/E^{l-2}), \ldots, \mu(E^1/E^0)\}$, is strictly decreasing.

Bearing in mind all the previous results, the following Theorem follows from standard arguments (for instance, see the proof of Theorem 1 of [3]).

Theorem 3.27. *Every object, E, of $\mathcal{G}$ has a unique Harder–Narasimhan filtration.*

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3.4. Jordan–Hölder filtration

We shall prove the existence of a Jordan–Hölder filtration for the semistable points $\mathcal{G}_\mu^{ss}$ for which the graded object does not depend on the filtration (Theorem 3.39).

Let $\mathcal{G}_\mu^{ss}$ ($\mathcal{G}_\mu^{s}$) be the full subcategory of $\mathcal{G}$ whose objects are $E \in \mathcal{G}$ such that E is (semi)stable and E is zero or $\mu(E) = \frac{\chi(E)}{r(E)} = \mu$.

Lemma 3.28. *Let E, F be objects of $\mathcal{G}_\mu^{ss}$ where F is a subobject of E. Then, the quotient E/F exists and belongs to $\mathcal{G}_\mu^{ss}$.*

Proof. Note that Lemma 3.21 states that $\mu = \mu(F) \leq \mu(F_0 \cap E)$ and from the semistability of E it follows that $\mu(F_0 \cap E) \leq \mu(E) = \mu$. Accordingly, $\mu(F) = \mu(F_0 \cap E)$ and, by Lemma 3.21, $F = F_0 \cap E$ and the quotient E/F exists.

The slope of E/F is now computed from Lemma 3.22, part 2.

Finally, let us prove that E/F is semistable. Let $\bar{H}$ be a proper subobject of E/F. Let us consider the projection $\pi : E \to E/F$. Accordingly, Lemma 3.20 claims that $\pi^{-1}(\bar{H}_0) \cap E$ is a subobject of E and, due to the semistability of E, it holds that $\mu(\pi^{-1}(\bar{H}_0) \cap E) \leq \mu(E)$. From Lemma 3.22, part 2, we deduce that

$$\mu(E/F) = \mu(F) \geq \mu\left(\frac{\pi^{-1}(\bar{H}_0) \cap E}{F}\right) = \mu(\bar{H}_0 \cap E/F) \geq \mu(\bar{H}).$$

E/F is therefore semistable. □

Lemma 3.29. *Let F be a proper subobject of E such that E/F exists. Let us assume that two of the k-spaces E, F and E/F have the same slope.*

Then, all three have the same slope. Moreover, E is semistable if and only if F and E/F are semistable.

Proof. The first part is obtained from Lemma 3.22, part 2.

Let us see the second one. Let E be semistable. Since $\mu(F) = \mu(E)$, it is trivial that F is also semistable. The previous Lemma shows that E/F is semistable.

Conversely, let F and E/F be semistable with the same slope μ. Let G be a proper subobject of E. Let us consider the exact sequence

$$0 \to F \cap G_0 \to E \cap G_0 \to (E \cap G_0)/(F \cap G_0) \to 0.$$

One checks that the three terms are objects of $\mathcal{G}$; $\mu(F \cap G_0) \leq \mu(F) = \mu$ because of the semistability of F; and, $\mu((E \cap G_0)/(F \cap G_0)) \leq \mu(E/F) = \mu$ because of the semistability of E/F. Let us assume that $\mu < \mu(G)$. It then holds that

$$\mu(F \cap G_0) \leq \mu < \mu(G) \leq \mu(G_0 \cap E).$$

This is equivalent, by Lemma 3.22, part 2, to $\mu((E \cap G_0)/(F \cap G_0)) > \mu$, which contradicts the above-mentioned inequality. Thus, $\mu \geq \mu(G)$ and E is semistable. □

Corollary 3.30. *Let $E = F \oplus G$, where F and G are two proper subobjects of E. Then, E is semistable if and only if F and G are semistable and $\mu(F) = \mu(G)$. In particular, a stable element of $\mathcal{G}$ is indecomposable.*

Lemma 3.31. *Let E be a semistable object of $\mathcal{G}$ and G be the maximum object of $\mathcal{G}(\mu_m(E))$. Then, E/G is stable.*

Proof. Let $\pi : E \to E/G$ be the natural projection. If E/G were semistable but nonstable, there would be a maximal subobject $\bar{F} \in \mathcal{G}(\mu_m(E/G))$. Consider $F := \pi^{-1}(\bar{F}) \cap E$ and observe that $G \subsetneq F$. Since G is maximal, it must hold that $\mu(F) < \mu(G) = \mu$. Now, Lemma 3.22 part 2 implies that $\mu(E/G) = \mu(G) > \mu(F/G) = \mu(\bar{F})$, which contradicts the construction of $\bar{F}$. We conclude that E/G is stable. □

Lemma 3.32. *Let E, F be objects of $\mathcal{G}_\mu^{ss}$ and let $T : E \to F$ be a morphism. Then,* $\operatorname{Ker} T \cap E$, $\operatorname{Im} T \cap F$, $F/(\operatorname{Im} T \cap F)$ *and* $E/(\operatorname{Ker} T \cap E)$ *are objects of $\mathcal{G}_\mu^{ss}$.*

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Proof. We already know by Lemmas 3.17 and 3.18 that all four objects belong to $\mathcal{G}$. The rest is straightforward. □

Theorem 3.33. *The category $\mathcal{G}_\mu^{ss}$ as subcategory of $\mathcal{G}$ is closed under extensions and direct factors. Moreover, it is an abelian category.*

Proof. The first statement is obtained by Lemma 3.29, Corollary 3.30 and Lemma 3.32. To prove that it is abelian, it is enough to prove that given $T \in \operatorname{Hom}_{\mathcal{G}}(E, F)$, we have

(1) if T is a monomorphism, then $\operatorname{Ker}(\operatorname{Coker}(T)) = T$;
(2) if T is a epimorphism, then $\operatorname{Coker}(\operatorname{Ker}(T)) = T$.

Let us prove only the first case, the second one being analogous. Let $T \in \operatorname{Hom}_{\mathcal{G}}(E, F)$ be a monomorphism. Then the underlying linear map $E_0 \to F_0$ is injective. Moreover, we have seen that $F/\operatorname{Im} T \cap F \in \operatorname{Gr}(F_0/\operatorname{Im} T)$ is the cokernel of T. It now remains to show that the kernel of

$$\begin{array}{ccc} F & \twoheadrightarrow & F/\operatorname{Im} T \cap F \\ \downarrow & & \downarrow \\ F_0 & \xrightarrow{\pi} & F_0/\operatorname{Im} T \end{array}$$

(where π is the quotient morphism) coincides with T. The kernel is given by

$$\operatorname{Ker}\pi \cap F \in \operatorname{Gr}(\operatorname{Ker}\pi, \operatorname{Ker}\pi \cap F_0^+).$$

Since the category of $k((z))$-vector spaces is abelian, $\operatorname{Ker}(\operatorname{Coker} T) = T$, i.e. $\operatorname{Ker}\pi = \operatorname{Im} T$ and, hence

$$\operatorname{Ker}\pi \cap F = \operatorname{Im} T \cap F \in \operatorname{Gr}(\operatorname{Im} T).$$

It suffices to prove that this object is equal to E; i.e. that the morphisms

$$\begin{array}{ccc} E & \hookrightarrow & F \\ \downarrow & & \downarrow \\ E_0 & \xrightarrow{T} & F_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{Im} T \cap F & \hookrightarrow & F \\ \downarrow & & \downarrow \\ \operatorname{Im} T & \xrightarrow{i} & F_0 \end{array}$$

define the same subobject of F; i.e., that there is an isomorphism on the category between the objects $E \hookrightarrow E_0$ and $\operatorname{Im} T \cap F \hookrightarrow \operatorname{Im} T$. We leave the details to the reader. □

Corollary 3.34. *If a morphism is a monomorphism and an epimorphism, then it is an isomorphism.*

Lemma 3.35. *The category $\mathcal{G}_\mu^{ss}$ is artinian and is noetherian.*

Proof. Let us prove that it is artinian the other case being similar. Let E be an object of $\mathcal{G}_\mu^{ss}$ and let

$$\cdots \subset E^n \subset \cdots \subset E^2 \subset E^1$$

be a decreasing chain of objects of $\mathcal{G}_\mu^{ss}$ where $E^n \in \operatorname{Gr}(E_0^n)$ are subobjects of $E \in \operatorname{Gr}(E_0)$. Let $r = \dim_{k((z))} E_0$ and $r_n = \dim_{k((z))} E_0^n$ where $r \ge r_n \ge r_{n+1} \ge 0$. Therefore, there is a l such that $r_n = r_{n+1}$ for every $n \ge l$. Thus $E_0^n = E_0^l$ for every $n \ge l$ and, since $E^n = E \cap E_0^n = E^{n+1} \cap E_0^n$, it follows that $E^n = E^l$ for every $n \ge l$. □

Definition 3.36. An object F of $\mathcal{G}_\mu^{ss}$ is said to be simple if every monomorphism $\alpha : E \to F$ of $\mathcal{G}_\mu^{ss}$ is zero or is an isomorphism.

Corollary 3.37. *An object of $\mathcal{G}_\mu^{ss}$ is simple if and only if it is stable.*

Definition 3.38. Let E be a semistable object of $\mathcal{G}$ of slope μ. A Jordan–Hölder filtration of E is a descending chain

$$S \equiv E^0 := 0 \subset E^1 \subset E^2 \subset \cdots \subset E^l = E$$

where E^i are objects of $\mathcal{G}_\mu^{ss}$ such that, for $i \in \{1, \ldots, l\}$, the quotients E^i/E^{i-1} exist in the category $\mathcal{G}_\mu^s$.

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Given a Jordan–Hölder filtration S, the graded object of S is defined as the following object of $\mathcal{G}$

$$gr\,S := E^1 \oplus E^2/E^1 \oplus \cdots \oplus E^l/E^{l-1}.$$

Theorem 3.39. *The graded object does not depend on the filtration, up to isomorphism. It will be denoted by $gr\,E$. Every object of $\mathcal{G}_\mu^{ss}$ admits a Jordan–Hölder filtration.*

Proof. Once all the preceding properties have been shown, the claim follows from arguments similar to those used in the case of the category of modules. □

4. Applications to the moduli of vector bundles

This final section is devoted to offering a geometrical application of the previous sections. Indeed, we will show that our results are deeply related to the notion of (semi)stability and the construction of filtrations of vector bundles on algebraic curves. For this goal, the Krichever map will be the bridge between both constructions.

We refer the reader to [1,7] and the references therein for the basic facts about the Krichever map and the moduli scheme of vector bundles (endowed with a formal trivialization at a smooth point). On the other hand, the issue of Harder–Narasimhan and Jordan–Hölder filtrations in the case of vector bundles has been exhaustively studied (we address the reader to [3,4,10,14] and the references therein).

We assume that the base field, k, is algebraically closed of characteristic 0. Henceforth, a triple (C, p, t_p) consisting of a irreducible non-singular projective curve over k, a smooth point and an isomorphism of k-algebras $\hat{\mathcal{O}}_p \tilde{\to} k[[z]]$ will be fixed.

Following [1], we know that there is a k-scheme, $\mathcal{M}_\infty(r)$, whose set of rational points is given by

$$\left\{\text{pairs } (\mathcal{F}, \delta) \text{ s.t. } \mathcal{F} \text{ is a rank } r \text{ vector bundle on } C \text{ and } \delta \text{ is an isomorphism } \hat{\mathcal{F}}_p \tilde{\to} \hat{\mathcal{O}}_p^{\oplus r}\right\} / \sim$$

where we write $(\mathcal{F}, \delta) \sim (\mathcal{F}', \delta')$ if and only if there exists an isomorphism of sheaves, $f : \mathcal{F} \tilde{\to} \mathcal{F}'$, compatible with δ and δ'.

The Krichever map is the scheme homomorphism given by

$$\begin{aligned} \mathcal{K}\colon \mathcal{M}_\infty(r) &\longrightarrow \mathrm{Gr}(V, V^+) \\ (\mathcal{F}, \delta) &\longmapsto (t_p \circ \delta)\left(H^0(C \setminus \{p\}, \mathcal{F})\right) \end{aligned}$$

with $V := k((z))^{\oplus r}$ and $V^+ := k[[z]]^{\oplus r}$. Since this map is a closed immersion, the scheme $\mathcal{M}_\infty(r)$ can be thought of as a closed subscheme of $\mathrm{Gr}(V)$. We also denote by $\mathcal{K}$ the map induced by the Krichever map from the set of rational points of $\bigcup_{r\geq 1} \mathcal{M}_\infty(r)$ to the set of objects of the category $\mathcal{G}$.

If $\mathcal{F}$ has degree d, the image $\mathcal{K}(\mathcal{F}, \delta)$ has characteristic $d + r(1 - g)$, where g is the genus of C. From now on, all points of $\mathcal{M}_\infty(r)$ and $\mathrm{Gr}(V, V^+)$ are assumed to be rational.

Let us generalize Proposition 1 of [9] (which deals with the rank 2 case) for higher rank.

Lemma 4.1. *Let $(\mathcal{F}, \delta)$ be a point in $\mathcal{M}_\infty(r)$ and let l be an integer $0 < l < r$. Let $F = \mathcal{K}(\mathcal{F}, \delta) \in \mathrm{Gr}(V)$.*

There is a 1–1 correspondence between the set of rank l coherent subsheaves of $\mathcal{F}$ such that the quotient is a coherent torsion-free sheaf and the set of the l-dimensional $k((z))$-vector subspaces G_0 of V such that $G_0 \cap F \neq 0$ and the dimension of $\frac{G_0}{G_0 \cap F + G_0 \cap V_+}$ over k is finite.

Proof. The proof is a straightforward generalization of Proposition 1 of [9]. Let us simply sketch how subsheaves and subspaces are related. Given a subsheaf $\mathcal{G}$ as in the statement, the corresponding subspace is the image of

$$H^0(C \setminus \{p\}, \mathcal{G}) \hat{\otimes}_{k[[z]]} k((z)) \subseteq H^0(C \setminus \{p\}, \mathcal{F}) \hat{\otimes}_{k[[z]]} k((z)) \simeq V.$$

Conversely, let G_0 be a subspace in the above conditions. Then, the properties of the Krichever map imply that $G_0 \cap F \in \mathrm{Gr}(G_0, G_0 \cap V^+)$ and, since $G_0 \cap F \subseteq F$, it defines a subbundle $\mathcal{G}$ of $\mathcal{F}$. Moreover, the composition

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$\hat{\mathcal{G}}_p \subseteq \hat{\mathcal{F}}_p \xrightarrow{\sim} V^+$ factorizes by $G_0 \cap V^+$ or, in other words, the formal trivialization δ does induce a formal trivialization, making the following diagram commutative:

$$\begin{array}{ccc} \hat{\mathcal{G}}_p & \hookrightarrow & \hat{\mathcal{F}}_p \\ \wr\downarrow & & \wr\downarrow\delta \\ G_0 \cap V_+ & \hookrightarrow & V_+ \end{array} \qquad \square \tag{4.1}$$

Lemma 4.2. *Let $(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r)$, F be $\mathcal{K}(\mathcal{F}, \delta) \in \mathrm{Gr}(V)$ and*

$$F^0 = 0 \subset F^1 \subset F^2 \subset \cdots \subset F^{l-1} \subset F^l = F$$

be the Harder–Narasimhan filtration of F.

There therefore exist $(\mathcal{F}^i, \delta^i)$ vector bundles endowed with formal trivialization such that $F^i = \mathcal{K}(\mathcal{F}^i, \delta^i)$. Moreover, there is a canonical formal trivialization $\bar{\delta}^i$ of $\mathcal{F}^{i+1}/\mathcal{F}^i$ such that the sequence

$$0 \to F^i \to F^{i+1} \to \mathcal{K}(\mathcal{F}^{i+1}/\mathcal{F}^i, \bar{\delta}^i) \to 0 \tag{4.2}$$

is exact.

Proof. From Lemma 3.21, we know that $F^i = F \cap F_0^i$ and, by Lemma 4.1, there are subbundles, $\mathcal{F}^i$, carrying formal trivializations, δ^i, such that $\mathcal{K}(\mathcal{F}^i, \delta^i) = F^i$. Then, $\bar{\delta}^i$ is the map induced between the cokernels of the monomorphisms of diagram (4.1) and the claim follows. $\square$

The main result of this section unveils how (semi)stableness and filtrations behave under the Krichever map.

Theorem 4.3. *Let $(\mathcal{F}, \delta) \in \mathcal{M}_\infty(r)$ and let F be its image by the Krichever map. It holds that*

(1) *$\mathcal{F}$ is (semi)stable if and only if F is (semi)stable for the action of $\mathrm{Sl}(r, k[[z]])$;*
(2) *the Krichever map transforms the Harder–Narasimhan filtration of $\mathcal{F}$ into the Harder–Narasimhan filtration of F and conversely;*
(3) *the Krichever map transforms the Jordan–Hölder filtration of $\mathcal{F}$ into the Jordan–Hölder filtration of F, and conversely. Moreover, the graded object of $\mathcal{F}$ is transformed into the graded object of $\mathcal{K}(\mathcal{F}, \delta)$.*

Proof. (1) The proof is a straightforward consequence of the results of Section 3.2 and the Lemma 4.1.

(2) Let

$$F^0 = 0 \subset F^1 \subset F^2 \subset \cdots \subset F^{l-1} \subset F^l = F$$

be the Harder–Narasimhan filtration of F. The previous lemma shows that there is a corresponding filtration of $\mathcal{F}$. We claim that it is the Harder–Narasimhan filtration of $\mathcal{F}$. By (4.2), we have that

$$\mathcal{K}(\mathcal{F}^i/\mathcal{F}^{i-1}, \bar{\delta}^i) \simeq \mathcal{K}(\mathcal{F}^i, \delta^i)/\mathcal{K}(\mathcal{F}^{i-1}, \delta^{i-1}) = F^i/F^{i-1}.$$

Now, the semistability of $\mathcal{F}^i/\mathcal{F}^{i-1}$ follows from the fact that F^i/F^{i-1} is semistable (Theorem 4.3). Moreover, the quotients $\mathcal{F}^i/\mathcal{F}^{i-1}$ and F^i/F^{i-1} have the same characteristic and rank, thus

$$\mu(F^i/F^{i-1}) = \mu(\mathcal{F}^i/\mathcal{F}^{i-1}) - (g-1).$$

Accordingly the sequence of slopes of the quotients $\{\mathcal{F}^i/\mathcal{F}^{i-1}\}$ is strictly decreasing.

The converse is proved similarly.

(3) The first part is deduced with similar arguments as those need for the previous theorem with the help of the results of Section 3.4. To prove the second part, it suffices to note that the exactness of the sequence (4.2) implies that

$$gr(F) = \oplus F^i/F^{i-1} = \mathcal{K}(\mathcal{F}^i/\mathcal{F}^{i-1}, \bar{\delta}^{i-1}). \quad \square$$

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