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# Prym varieties, curves with automorphisms and the Sato Grassmannian

## E. Gómez González · J.M. Muñoz Porras · F.J. Plaza Martín

**Abstract.** The aim of the paper is twofold. First, some results of Shiota and Plaza-Martín on Prym varieties of curves with an involution are generalized to the general case of an arbitrary automorphism of prime order. Second, the equations defining the moduli space of curves with an automorphism of prime order as a subscheme of the Sato Grassmannian are given.

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## 1. Introduction

The main objectives of this paper are as follows. First, to extend some results of T. Shiota ([S2]) and F. J. Plaza-Martín ([P]) on Prym varieties of curves with an involution to the general case of Prym varieties associated with curves with an arbitrary automorphism of prime order. Second, to give an explicit description of the equations defining the moduli space of curves with an automorphism of prime order (together with a formal trivialization at some points) as a subscheme of the Sato Grassmannian.

The results of this paper are intended to be a first step to solving a Schottky-type problem for curves with automorphisms. The basic concepts used in our approach are the formal spectral covers and their formal Jacobians, which were introduced in [MP2] when studying the Hurwitz schemes.

We define the notion of formal Prym variety associated with a formal Galois cover and prove that this formal Prym satisfies all the expected properties (i.e. the analogues of those of classical Prym varieties, see Proposition 2.2). This formal Prym variety acts on the infinite Grassmannian of $V$, where $V$ is the $\mathbb{C}((z))$-algebra of the formal cover.

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In §3, we study two subschemes of the infinite Grassmannian of $V$ that are relevant for our purposes. The first is the set of points of the Grassmannian invariant under an automorphism of $V$; that is, subspaces $U \subset V$ stable under the automorphism. The second one consists of a generalization of the Grassmannian of maximal totally isotropic subspaces considered in [P, S2] when studying the BKP hierarchy. In the present setting, the bilinear pairing introduced in those papers is replaced by a multilinear form on $V$.

The Krichever map for Pryms is studied in §4 in order to obtain a characterization of Prym varieties (Theorem 4.15) as subvarieties of the Sato Grasmmannian. This statement (together with Theorem 4.13) is to be thought of as an analogue of Mulase's characterization of Jacobians ([M]) and of Li-Mulase of Pryms ([LM]) since it is expressed in terms of the finiteness of the orbits (modulo $\bar{\Pi}^+$) of the action of the formal group $\Pi$ on the Grassmannian.

Finally, in §5 we state the explicit equations of the moduli spaces of curves with automorphisms as subvarieties of the infinite Grassmannians. To this end, we use some ideas and results on the equations of the moduli spaces of curves that were given by two of the authors in a previous paper ([MP]).

Through this paper we will assume that the base field is $\mathbb{C}$, the field of complex numbers.

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## 2. Formal geometry of coverings

This section is concerned with the generalization of the notions of the formal curve, the formal jacobian, etc. (see [AMP, MP2]) for the case of a covering of curves of group $\mathbb{Z}/p$.

Henceforth, $V$ will be a $\mathbb{C}((z))$-algebra with an action of the group $\mathbb{Z}/p$, where $p$ is a prime number such that $V^{\mathbb{Z}/p} = \mathbb{C}((z))$. We shall choose a generator $\sigma$ of $\mathrm{Aut}_{\mathbb{C}((z))} V$. We consider the following two cases:

(a) *Ramified case:* $V = \mathbb{C}((z_1))$ where the $\mathbb{C}((z))$-algebra structure is given by mapping $z$ to $z_1^p$ and $\sigma(z_1) = \xi z_1$ ($\xi$ being a primitive $p$-th root of 1 in $\mathbb{C}$). We will set $V^+ = \mathbb{C}[[z_1]]$ and $V^- = z_1^{-1}\mathbb{C}[z_1^{-1}]$.

(b) *Non-ramified case:* $V = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))$ where the $\mathbb{C}((z))$-algebra structure is given by mapping $z$ to $(z_1, \ldots, z_p)$ and $\sigma(z_i) = z_{i+1}$ (for $i < p$) and $\sigma(z_p) = z_1$. We set $V^+ = \mathbb{C}[[z_1]] \times \cdots \times \mathbb{C}[[z_p]]$ and $V^- = z_1^{-1}\mathbb{C}[z_1^{-1}] \times \cdots \times z_p^{-1}\mathbb{C}[z_p^{-1}]$.

In both cases, we have *distinguished* bases of $V$ as a $\mathbb{C}((z))$-vector space; namely, $\{1, z_1, \ldots, z_1^{p-1}\}$ in the first case, and $\{z_1, \ldots, z_p\}$ in the second one. Observe that $\sigma(V^+) = V^+$ and that $\sigma(V^-) = V^-$.

*Remark 1.* If $p$ is not a prime number a more general $\mathbb{C}((z))$-algebra $V$ must be considered. Then, the results of this paper could be generalized to the case of a cyclic group of automorphisms. However, for the sake of clarity, we will restrict ourselves to the case of $p$ being a prime number.

Following [MP2], Definition 2.1, we define the formal base curve as the formal $\mathbb{C}$-scheme $\widehat{C} := \mathrm{Spf}\ \mathbb{C}[[z]]$ and the formal spectral cover as the formal $\mathbb{C}$-scheme $\widehat{C}_V := \mathrm{Spf}\ V^+$.

Let $\Gamma$ denote the formal group scheme representing the functor:

$$\left\{ \begin{array}{c} \text{category of} \\ \text{formal } \mathbb{C}\text{-schemes} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{category} \\ \text{of groups} \end{array} \right\}$$

$$S \rightsquigarrow \left( \mathbb{C}((z)) \hat{\underset{\mathbb{C}}{\otimes}} H^0(S, \mathcal{O}_S) \right)_0^* = \left\{ \begin{array}{c} \text{invertible elements} \\ \text{of } \mathbb{C}((z)) \hat{\underset{\mathbb{C}}{\otimes}} H^0(S, \mathcal{O}_S) \end{array} \right\}_0$$

where the subscript 0 denotes the connected component of the identity. Let $\Gamma^+$ be the group representing the subfunctor of $\Gamma$, consisting of those elements lying in $\mathbb{C}[[z]] \hat{\otimes} H^0(S, \mathcal{O}_S)$, and let the jacobian of the formal curve, $\mathcal{J}(\widehat{C})$, be the group representing the subfunctor of $\Gamma$, consisting of those elements lying in $\mathbb{C}[z^{-1}] \hat{\otimes} H^0(S, \mathcal{O}_S)$ whose constant term is equal to 1. Observe that its $R$-valued points ($R$ being a $\mathbb{C}$-algebra) is the set:

$$\left\{ \begin{array}{c} \text{polynomials } 1 + \sum_{j=-n}^{-1} a_j z^j \text{ of } R[z^{-1}] \\ \text{where } a_j \in \mathrm{Rad}(R) \text{ for all } j \end{array} \right\}$$

Note that the multiplicative group $\mathbb{G}_m$ is contained in $\Gamma$ and that $\Gamma = \mathcal{J}(\widehat{C}) \times \mathbb{G}_m \times \Gamma^+$.

In the ramified case, to define $\Gamma_V$ (resp. $\Gamma_V^+$) one replaces $\mathbb{C}((z))$ (resp. $\mathbb{C}[[z]]$) by $\mathbb{C}((z_1))$ (resp. $\mathbb{C}[[z_1]]$). The formal jacobian of the formal spectral cover $\mathcal{J}(\widehat{C}_V)$ is defined by replacing $\mathbb{C}[z^{-1}]$ by $\mathbb{C}[z_1^{-1}]$ and it turns out that $\Gamma_V = \mathcal{J}(\widehat{C}_V) \times \mathbb{G}_m \times \Gamma_V^+$.

In the non-ramified case, $\Gamma_V$ (resp. $\Gamma_V^+$) is defined by replacing $\mathbb{C}((z))$ (resp. $\mathbb{C}[[z]]$) by $\mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))$ (resp. $\mathbb{C}[[z_1]] \times \cdots \times \mathbb{C}[[z_p]]$). To define the formal jacobian of the formal spectral cover, $\mathcal{J}(\widehat{C}_V)$, one replaces $\mathbb{C}[z^{-1}]$ by $\mathbb{C}[z_1^{-1}] \times \cdots \times \mathbb{C}[z_p^{-1}]$; one then has that $\Gamma_V = \mathcal{J}(\widehat{C}_V) \times (\mathbb{G}_m \times .^p. \times \mathbb{G}_m) \times \Gamma_V^+$.

*Remark 2.* It is worth noticing that the formal scheme $\Gamma$ is a direct limit of affine group schemes $\Gamma^n$. For $n \geq 0$, one considers:

$$\Gamma^n := \left\{ \begin{array}{c} \text{invertible elements} \\ \text{of } z^{-n}\mathbb{C}[[z]] \hat{\underset{\mathbb{C}}{\otimes}} H^0(S, \mathcal{O}_S) \end{array} \right\}_0$$

and observes that it is the spectrum of $\mathbb{C}[u_n, \ldots, u_1, v_0, v_1, \ldots]/I_n$, where $u_i$ has degree $i$ and $I_n$ is the ideal generated by the monomials on $u$ of total degree greater than $n$. The results of [EGA], I.§10.6 imply that $\Gamma = \varinjlim \Gamma^n$.

Analogous arguments show that $\mathcal{J}(\widehat{C})$, $\Gamma_V$ and $\mathcal{J}(\widehat{C}_V)$ are limits of affine group schemes.

Standard calculation shows that $\mathcal{J}(\widehat{C}_V)$ is the formal spectrum of the ring:

$$\mathbb{C}\{\{t_1, t_2, \ldots\}\}$$

($\mathbb{C}\{\{t_1, t_2, \ldots\}\}$ denotes the inverse limit $\varprojlim_n \mathbb{C}[[t_1, \ldots, t_n]]$) for the ramified case and:

$$\mathbb{C}\{\{t_1^{(1)}, t_2^{(1)}, \ldots\}\} \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}\{\{t_1^{(p)}, t_2^{(p)}, \ldots\}\}$$

for the non-ramified one.

Since $\sigma$ acts on $V^+$ as a homomorphism of $\mathbb{C}[[z]]$-algebras, it gives rise to an order $p$ automorphism:

$$\sigma : \widehat{C}_V \xrightarrow{\sim} \widehat{C}_V$$

such that the quotient $\widehat{C}_V/ <\sigma> = \mathrm{Spf}(V^+)^{<\sigma>} = \mathrm{Spf}\,\mathbb{C}[[z]]$ is the formal base curve.

From the explicit expression of the elements of $\mathcal{J}(\widehat{C}_V)$, it is easy to check that $\sigma$ induces an automorphism $\sigma^* : \mathcal{J}(\widehat{C}_V) \xrightarrow{\sim} \mathcal{J}(\widehat{C}_V)$. The expression in terms of rings is:

$$t_j \longmapsto \sigma^*(t_j) = \xi^{-j} t_j$$

and

$$t_j^{(i)} \longmapsto \sigma^*(t_j^{(i)}) = \begin{cases} t_j^{(i-1)} & i > 1 \\ t_j^{(p)} & i = 1 \end{cases}$$

respectively.

The Abel morphism of degree 1 is the morphism:

$$\widehat{C}_V \longrightarrow \mathcal{J}(\widehat{C}_V)$$

defined by the series:

$$\exp\left(\sum_{j>0} \frac{\bar{z}_1^j}{j z_1^j}\right)$$

in the ramified case (here $\mathcal{O}_{\widehat{C}_V}$ is identified with $\mathbb{C}[[\bar{z}_1]]$) and by the $p$-uple of series:

$$\left(\exp\left(\sum_{j>0} \frac{\bar{z}_1^j}{j z_1^j}\right), \ldots, \exp\left(\sum_{j>0} \frac{\bar{z}_p^j}{j z_p^j}\right)\right)$$

in the non-ramified one ($\mathcal{O}_{\widehat{C}_V}$ is identified with $\mathbb{C}[[\bar{z}_1]] \times \cdots \times \mathbb{C}[[\bar{z}_p]]$).

$V$ being a finite $\mathbb{C}((z))$-algebra, we have the norm, which is a group homomorphism:

$$\mathrm{Nm} : V^* \longrightarrow \mathbb{C}((z))^*$$

that maps an element $v \in V^*$ to the determinant of the homothety defined by itself or, equivalently, the product of its transforms under $\sigma^i$ ($0 \leq i \leq p - 1$). Recalling the definition of formal jacobians of $\widehat{C}_V$ and $\widehat{C}$, one has that the norm gives rise to a morphism of formal groups schemes:

$$\mathrm{Nm} \colon \mathcal{J}(\widehat{C}_V) \longrightarrow \mathcal{J}(\widehat{C})$$

which corresponds to the ring homomorphism:

$$\mathbb{C}\{\{\bar{t}_1, \bar{t}_2, \ldots \}\} \longrightarrow \mathbb{C}\{\{t_1, t_2, \ldots \}\}$$
$$\bar{t}_i \longmapsto \mathrm{Nm}(\bar{t}_i) = p \cdot t_{ip}$$

in the ramified case and:

$$\mathbb{C}\{\{\bar{t}_1, \bar{t}_2, \ldots \}\} \longrightarrow \mathbb{C}\{\{t_1^{(1)}, \ldots \}\} \widehat{\otimes} \ldots \widehat{\otimes} \mathbb{C}\{\{t_1^{(p)}, \ldots \}\}$$
$$\bar{t}_i \longmapsto \mathrm{Nm}(\bar{t}_i) = t_i^{(1)} + \cdots + t_i^{(p)}$$

in the non-ramified case.

Finally, since the quotient map $\pi \colon \widehat{C}_V \to \widehat{C}$ corresponds to the canonical inclusion $\mathbb{C}[[z]] \hookrightarrow V^+$, it also induces a group morphism:

$$\pi^* \colon \mathcal{J}(\widehat{C}) \hookrightarrow \mathcal{J}(\widehat{C}_V)$$

whose expression in terms of rings is:

$$t_j \longmapsto \pi^*(t_j) = \begin{cases} \bar{t}_{j/p} & \text{if } j = \dot{p} \\ 0 & \text{otherwise} \end{cases}$$

for the ramified case and:

$$t_j^{(i)} \longmapsto \pi^*(t_j^{(i)}) = \bar{t}_j$$

for the non-ramified one.

Note that the set of invariant elements of the formal jacobian of the spectral cover, $\mathcal{J}(\widehat{C}_V)^{\sigma^*}$, is the formal jacobian of the formal base curve, $\mathcal{J}(\widehat{C})$.

## 2.A. Formal Prym varieties

Let us introduce two important subgroups of $\Gamma_V$. Note that the norm can be generalized to the relative case since $V \widehat{\otimes} \mathcal{O}_S$ is a finite $\mathbb{C}((z)) \widehat{\otimes} \mathcal{O}_S$-algebra.

**Definition 2.1.** *The formal group $\Pi$ is the formal group scheme representing the subfunctor of $\Gamma_V$ given by:*

$$S \rightsquigarrow \Pi(S) := \{g \in \Gamma_V(S) \text{ such that } \mathrm{Nm}(g) \in \mathcal{O}_S\}$$

*and $\Pi^+$ is defined by $\Pi \cap \Gamma_V^+$.*

*The formal Prym variety of $(V, V^+, \sigma)$ is the formal group scheme given by:*

$$\mathcal{P}(\widehat{C}_V) := \mathrm{Ker}(\mathrm{Nm}|_{\mathcal{J}(\widehat{C}_V)}) = \{g \in \mathcal{J}(\widehat{C}_V) \mid \mathrm{Nm}(g) = 1\}$$

Let us state some of the properties of these subgroups of $\Gamma_V$.

**Proposition 2.2.** *The following properties hold:*

(1) $\mathcal{P}(\widehat{C}_V) = \Pi \cap \mathcal{J}(\widehat{C}_V)$;

(2) *the map $p.\,\mathrm{Id} - \pi_0^*\mathrm{Nm} : \Gamma_V \to \Gamma_V$ induces a surjection $\mathcal{J}(\widehat{C}_V) \to \mathcal{P}(\widehat{C}_V)$.*
    *Further, it gives a map $\widehat{C}_V \to \mathcal{P}(\widehat{C}_V)$;*

(3) *the multiplication of series gives rise to an isomorphism:*

$$m : \mathcal{P}(\widehat{C}_V) \times \mathcal{J}(\widehat{C}) \to \mathcal{J}(\widehat{C}_V);$$

(4) *there is an isomorphism:*

$$(p.\,\mathrm{Id} - \pi_0^*\mathrm{Nm}, \mathrm{Nm}) : \mathcal{J}(\widehat{C}_V) \to \mathcal{P}(\widehat{C}_V) \times \mathcal{J}(\widehat{C});$$

(5) *the compositions $(p.\,\mathrm{Id} - \pi_0^*\,\mathrm{Nm}, \mathrm{Nm}) \circ m$ and $m \circ (p.\,\mathrm{Id} - \pi_0^*\,\mathrm{Nm}, \mathrm{Nm})$ map*
    *an element to its $p$-th power.*

*Proof.* Note that the set of $R$-valued points of $\mathcal{P}(\widehat{C}_V)$ is the following subset of $\mathcal{J}(\widehat{C}_V)(R)$:

$$\left\{ \exp(\sum_j t_j z_1^{-j}) \in \mathcal{J}(\widehat{C}_V)(R) \text{ such that } t_j = 0 \text{ for } j = \dot{p} \right\}$$

for the ramified case and:

$$\left\{ \left( \exp(\sum_{j>0} t_j^{(1)} z_1^{-j}), \ldots, \exp(\sum_{j>0} t_j^{(p)} z_p^{-j}) \right) \text{ such that } \sum_{i=1}^p t_j^{(i)} = 0 \text{ for } j > 0 \right\}$$

for the non-ramified one.

Now, the claims follow easily from the explicit expressions of the morphisms Nm, $\pi^*$ and $\sigma^*$ given in the previous section. $\qquad\qquad\square$

## 3. Vector-valued infinite Grassmannians

We begin this section by reviewing from [AMP, MP2] some definitions and main properties of infinite Grassmannians.

From [AMP] we learn that the Grassmannian functor of $(V, V^+)$ is representable by a $\mathbb{C}$-scheme whose set of rational points is:

$$\left\{ \begin{array}{l} \text{subspaces } U \subset V \text{ such that } U \to V/V^+ \\ \text{has finite dimensional kernel and cokernel} \end{array} \right\}$$

The connected components of this scheme are indexed by the Poincaré-Euler characteristic of $U \to V/V^+$. The connected component of index $m$ will be denoted by $\mathrm{Gr}^m(V)$. This scheme is equipped with the determinant bundle, which is the determinant of the complex of $\mathcal{O}_{\mathrm{Gr}(V)}$-modules:

$$\mathcal{L} \longrightarrow V/V^+$$

where $\mathcal{L}$ is the universal submodule of $\mathrm{Gr}(V)$ and $\mathcal{L} \to V/V^+$ is the natural projection.

Recall that $\mathrm{Gr}(V)$ is endowed with an action of the group $\Gamma_V$:

$$\Gamma_V \times \mathrm{Gr}(V) \longrightarrow \mathrm{Gr}(V)$$

since the elements of $\Gamma_V$ act by homotheties on $V$.

The Baker-Akhiezer function of a point $U \in \mathrm{Gr}(V)$ has been introduced in §3 of [MP2]. This is a function on $\widehat{C}_V \times \mathcal{J}(\widehat{C}_V)$ that will be denoted by $\psi_U(z_\cdot, t)$, where $z_\cdot$ is $z_1$ (resp. $(z_1, \ldots, z_p)$) in the ramified case (resp. in the non-ramified case). It is appropriate to point out that when defining the Baker-Akhiezer function of $U \in \mathrm{Gr}^m(V)$, one has to choose an element $v_m \in V$ such that the kernel and cokernel of $U \to V/v_m V^+$ will have the same dimension.

Following that paper, we choose $v_m$ to be $z_1^m$ in the ramified case and $(z_1^{q+1}, \ldots, z_r^{q+1}, z_{r+1}^q, \ldots, z_p^q)$ (where $m = pq + r$) in the non-ramified one.

The main property of Baker-Akhiezer functions is that $\frac{v_m}{z_\cdot}\psi_U(z_\cdot, t)$ can be understood as a generating function of the vectors of $U \in \mathrm{Gr}^m(V)$ as a subspace of $V$. That is, every $u \in U$ is obtained by evaluating $\frac{v_m}{z_\cdot}\psi_U(z_\cdot, t)$ at certain values of $t$, and, conversely, that this expression belongs to $U$ for all values of $t$.

Since $V$ is a finite $\mathbb{C}((z))$-algebra, there is a natural linear map:

$$\mathrm{Tr}\colon V \longrightarrow \mathbb{C}((z))$$

which assigns to an element $v \in V$ the trace of the homothety of $V$ defined by $v$. Furthermore, it gives rise to a non-degenerated metric on $V$ as $\mathbb{C}((z))$-vector space, which will be denoted by $\mathrm{Tr}$ again.

This metric allows us to consider the following pairing:

$$\begin{aligned} V \times V &\longrightarrow \mathbb{C} \\ (a, b) &\longmapsto \mathrm{Res}_{z=0} \mathrm{Tr}(a, b) dz \end{aligned} \tag{3.1}$$

Since this pairing is non-degenerate, we have an involution of $\mathrm{Gr}(V)$ which maps a point $U$ to its orthogonal $U^\perp$. This involution sends the connected component $\mathrm{Gr}^m(V)$ to $\mathrm{Gr}^{1-m-p}(V)$ in the ramified case and to $\mathrm{Gr}^{-m}(V)$ in the non-ramified one.

Finally, the adjoint Baker-Akhiezer function of $U$ is defined by:

$$\psi_U^*(z, t) := \psi_{U^\perp}(z, -t)$$

### 3.A. Stable points of $\mathrm{Gr}(V)$ under an automorphism

We shall now take into account the additional structure provided by the automorphism $\sigma$. We shall show that the subset of points of $\mathrm{Gr}(V)$ invariant under the

automorphism is a closed subscheme and shall compute how the Baker-Akhiezer function behaves. In fact, we shall consider a more general setting.

Let us denote by $\mathrm{Aut}^+_{\mathbb{C}((z))} V$ the group of $\mathbb{C}((z))$-algebra automorphisms of $V$ that leave $V^+$ stable. Let $G$ be a finite group and $\rho : G \hookrightarrow \mathrm{Aut}^+_{\mathbb{C}((z))} V$ be a linear representation of $G$.

Observe that if $A \subset V$ is a subspace commensurable with $V^+$, then $\rho(\sigma)A$ is also commensurable with $V^+$ for all $\sigma \in G$. Therefore, the representation $\rho$ yields an action on $\mathrm{Gr}(V)$:

$$
\begin{aligned}
G \times \mathrm{Gr}(V) &\longrightarrow \mathrm{Gr}(V) \\
(\sigma, U) &\longmapsto \rho(\sigma)U
\end{aligned}
\tag{3.2}
$$

Let us set:

$$\mathrm{Gr}(V)^G := \{U \in \mathrm{Gr}(V) \text{ such that } \rho(\sigma)U = U \text{ for all } \sigma \in G\}$$

Then, one has that:

$$\mathrm{Gr}(V)^G = \bigcap_{\sigma \in G} \mathrm{Gr}(V)^\sigma$$

where:

$$\mathrm{Gr}(V)^\sigma := \{U \in \mathrm{Gr}(V) \text{ such that } \rho(\sigma)U = U\}$$

Note that $\mathrm{Gr}(V)^\sigma$ is a closed subscheme of $\mathrm{Gr}(V)$ because $\mathrm{Gr}(V)$ is separated ([AMP], Theorem 2.15) and, therefore, $\mathrm{Gr}(V)^G$ is also a closed subscheme.

Since the Baker-Akhiezer and adjoint Baker-Akhiezer functions might be thought of (up to a factor) as generating functions, we obtain the following characterization of $\mathrm{Gr}(V)^\sigma$ for $\sigma^p = \mathrm{Id}$:

**Theorem 3.3.** *Let* $\sigma \in \mathrm{Aut}^+_{\mathbb{C}((z))} V$ *with* $\sigma^p = \mathrm{Id}$. *A closed point* $U \in \mathrm{Gr}(V)$ *is a point of* $\mathrm{Gr}(V)^\sigma$ *if and only if its Baker-Akhiezer function satisfies the equation:*

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{1}{z_1} \psi_{\rho(\sigma)U}(z_1, t) \psi^*_U(z_1, s) \right) \frac{dz}{z} = 0$$

*in the ramified case, and:*

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{1}{z_\bullet^2} \psi_{\rho(\sigma)U}(z_\bullet, t) \psi^*_U(z_\bullet, s) \right) dz = 0$$

*in the non-ramified one.*

*Proof.* Let us begin with the ramified case. Given a point $U \in \mathrm{Gr}^m(V)$, we know that $U^\perp$ belongs to $\mathrm{Gr}^{1-m-p}(V)$. In particular, we know that $\frac{v_m}{z_1} \psi_U(z_1, t)$ (resp. $\frac{v_{1-m-p}}{z_1} \psi^*_U(z_1, t)$) is a generating function of $U$ (resp. the orthogonal of $U$ w.r.t. the pairing 3.1). Since $v_m = z_1^m$, the condition $\rho(\sigma)U = U$ is equivalent to:

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{z_1^m}{z_1} \psi_{\rho(\sigma)U}(z_1, t), \frac{z_1^{1-m-p}}{z_1} \psi^*_U(z_1, s) \right) dz = 0$$

Bearing in mind that $z_1^p = z$, together with the properties of the trace map, the claim follows.

For the non-ramified case, recall that $U^\perp \in \mathrm{Gr}^{-m}(V)$ for $U \in \mathrm{Gr}^m(V)$. Then, the condition $\rho(\sigma)U = U$ is written as:

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{v_m}{z_\bullet} \psi_{\rho(\sigma)U}(z_\bullet, t), \frac{v_{-m}}{z_\bullet} \psi_U^*(z_\bullet, s) \right) dz = 0$$

The formula follows from the fact that $v_m v_{-m} = 1$. $\qquad\qquad\square$

Observe that the action considered in 3.2, preserves the determinant bundle. To see this, it suffices to note that for $\sigma \in G$ the commutative diagram:

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & V/V^+ \\ \rho(\sigma) \downarrow & & \downarrow \rho(\sigma) \\ \rho(\sigma)\mathcal{L} & \longrightarrow & V/V^+ \end{array}$$

implies the existence of an isomorphism $\mathrm{Det} \simeq \rho(\sigma)^* \mathrm{Det}$.

This will allow us to find an explicit expression relating the Baker-Akhiezer functions of $U$ and $\rho(\sigma)U$ in the following general setting.

Assume that $V$ is the $\mathbb{C}((z))$-vector space $V_1 \times \cdots \times V_r$ where $V_i = \mathbb{C}((z^{1/e_i}))$ and $e_i$ is a positive integer and that the action can be described as follows. There exist natural numbers $\{k_1, \ldots, k_l\}$ with $m := k_1 + \cdots + k_l \leq r$ and a set $\{\xi_{m+1}, \ldots, \xi_r\}$ with $\xi_i$ a primitive $e_i$-th root of 1 in $\mathbb{C}$ such that:

- $\rho(\sigma)$ acts as a cyclic permutation on the products:

$$V_1 \times \cdots \times V_{k_1}$$

$$\cdots$$

$$V_{k_1+\cdots+k_{l-1}+1} \times \cdots \times V_{k_1+\cdots+k_l}$$

- $\rho(\sigma)$ leaves the factor $V_j$ (for $m < j \leq r$) stable and maps $f(z^{1/e_j}))$ to $f(\xi_j z^{1/e_j})$.

**Proposition 3.4.** *The Baker-Akhiezer function of $\rho(\sigma)U$ is:*

$$\psi_{\rho(\sigma)U}(z_\bullet, t) = \psi_U \Big( \rho(\sigma)z_\bullet, t^{(2)}, \ldots, t^{(k_1)}, t^{(1)}, \ldots$$
$$t^{(k_1+\cdots+k_{l-1}+2)}, \ldots, t^{(k_1+\cdots+k_l)}, t^{(k_1+\cdots+k_{l-1}+1)},$$
$$\xi_{m+1}^{-1} t_1^{(m+1)}, \xi_{m+1}^{-2} t_2^{(m+1)}, \ldots, \xi_r^{-1} t_1^{(r)}, \xi_r^{-2} t_2^{(r)}, \ldots \Big)$$

*Proof.* It suffices to see that the automorphism $\rho(\sigma)$ gives rise to a natural automorphism on the local Jacobian of the formal spectral cover:

$$\rho(\sigma)^* \colon \mathcal{J}(\widehat{C}_V) \longrightarrow \mathcal{J}(\widehat{C}_V)$$

whose expression in terms of the underlying rings is:

$$\rho(\sigma)^*(t_i^{(1)}) = t_i^{(2)}$$

$$\cdots$$

$$\rho(\sigma)^*(t_i^{(k_1)}) = t_i^{(1)}$$

$$\cdots$$

$$\rho(\sigma)^*(t_i^{(k_1+\cdots+k_{l-1}+1)}) = t_i^{(k_1+\cdots+k_{l-1}+2)}$$

$$\cdots$$

$$\rho(\sigma)^*(t_i^{(k_1+\cdots+k_l)}) = t_i^{(k_1+\cdots+k_{l-1}+1)}$$

$$\rho(\sigma)^*(t_i^{(j)}) = \xi_j^{-i} t_i^{(j)} \qquad \text{for } m < j \leq r$$

$$\square$$

### 3.B. Grassmannian of isotropic subspaces

For later purposes it is convenient to introduce a second subscheme of the Grassmannian. Under our hypothesis one can consider a skewsymmetric $p$-form on $V$. To this end, recall that $V$ is a $p$-dimensional $\mathbb{C}((z))$-vector space; hence, mapping $z_1 \wedge \cdots \wedge z_p$ (resp. $1 \wedge z_1 \wedge \cdots \wedge z_1^{p-1}$) to $dz$ we obtain an identification of $\wedge_{\mathbb{C}((z))}^p V$ with $\mathbb{C}((z))dz$ for the non-ramified case (resp. ramified case). Let us define a skewsymmetric form on $V$ by:

$$V \times .\overset{p}{.}. \times V \longrightarrow \mathbb{C}$$

$$(f_1, \ldots, f_p) \longmapsto \text{Res}_{z=0}(f_1 \wedge \cdots \wedge f_p)$$

Let $\wedge_k^p W$ be the $p$-th exterior algebra of a $k$-vector space $W$. For a $\mathbb{C}$-subspace $U$ of $V$, let us denote by $\wedge^p U \subseteq \mathbb{C}((z))dz$ the image of the map:

$$\wedge_{\mathbb{C}}^p U \lhook\joinrel\longrightarrow \wedge_{\mathbb{C}}^p V \longrightarrow\!\!\!\!\!\rightarrow \wedge_{\mathbb{C}((z))}^p V \overset{\sim}{\longrightarrow} \mathbb{C}((z))dz \qquad (3.5)$$

A subspace $U$ of $V$ will be called isotropic if the restriction of this skewlinear form to $U \times \cdots \times U$ is zero or, what amounts to the same, the map:

$$\text{Res}_{z=0} : \wedge^p U \longrightarrow \mathbb{C}$$

vanishes.

Consider the following subfunctor of the Grassmannian consisting of the isotropic subspaces of $V$; that is:

$$S \rightsquigarrow \underline{\text{Gr}_I(V)}(S) := \left\{ \begin{array}{l} U \in \underline{\text{Gr}(V)}(S) \text{ such that the} \\ \text{map } \wedge^p U \overset{\text{Res}}{\longrightarrow} \mathcal{O}_S \text{ is zero} \end{array} \right\}$$

**Proposition 3.6.** *The functor* $\underline{\mathrm{Gr_I}}(V)$ *is representable by a closed subscheme* $\mathrm{Gr_I}(V)$ *of* $\mathrm{Gr}(V)$, *which will be called the Grassmannian of isotropic subspaces.*

*Proof.* Let $\mathcal{O}$ denote the sheaf of $\mathrm{Gr}(V)$ and let $\mathcal{L} \subset V \hat{\otimes} \mathcal{O}$ denote the universal submodule. Then, the composition of the multilinear form and the residue yields a morphism of locally free $\mathcal{O}$-sheaves:

$$\mathcal{L} \otimes_{\mathcal{O}} \overset{p}{\ldots} \otimes_{\mathcal{O}} \mathcal{L} \longrightarrow \mathcal{O}$$

Since $\underline{\mathrm{Gr_I}}(V)$ consists of the points where this map vanishes, the conclusion follows. □

Let us write down equations for this subscheme of $\mathrm{Gr}(V)$.

**Lemma 3.7.** *Let $U$ be a point of* $\mathrm{Gr}^m(V)$ *and let $\psi_U(z., t)$ be its Baker-Akhiezer function. Let:*

$$\left( \psi_U^1(z, t), \ldots, \psi_U^p(z, t) \right)$$

*be the coordinates of $\frac{v_m}{z.}\psi_U(z., t)$ w.r.t. the distinguished basis of $V$ as a $\mathbb{C}((z))$-vector space.*

   *Then, the $\mathbb{C}$-subspace $\wedge^p U$ of $\mathbb{C}((z))dz$ is generated by the series of $\mathbb{C}((z))$ of the type:*

$$\begin{vmatrix} \psi_U^1(z, t_1) & \ldots & \psi_U^p(z, t_1) \\ \vdots & & \vdots \\ \psi_U^1(z, t_p) & \ldots & \psi_U^p(z, t_p) \end{vmatrix}$$

*where $t_1, \ldots, t_p$ denotes $p$ independent sets of variables $t$.*

*Proof.* Since $\frac{v_m}{z.}\psi_U(z., t)$ is a generating function of $U$, it follows that $\wedge^p U$ is generated by evaluating at certain values of $t_1, \ldots, t_p$ the image of $\frac{v_m}{z.}\psi_U(z., t_1) \wedge \cdots \wedge \frac{v_m}{z.}\psi_U(z., t_p)$ by the map (3.5). To conclude, it suffices to express the Baker-Akhiezer function of $U$ in terms of the distinguished basis of $V$. □

**Theorem 3.8.** *Let $U$ be a closed point of* $\mathrm{Gr}^m(V)$. *The point $U$ belongs to* $\mathrm{Gr_I}(V)$ *if and only if the following identity holds:*

$$\mathrm{Res}_{z=0} \begin{vmatrix} \psi_U^1(z, t_1) & \ldots & \psi_U^p(z, t_1) \\ \vdots & & \vdots \\ \psi_U^1(z, t_p) & \ldots & \psi_U^p(z, t_p) \end{vmatrix} dz = 0 \tag{3.9}$$

*Proof.* This is trivial from the previous lemma. □

*Remark 3.* It is worth noticing that the residue condition of the previous theorem can be translated into a set of differential equations. To this end, one should proceed as in Theorem 5.13 of [MP2].

*Remark 4.* If $p = 2$ and $V = \mathbb{C}((z_1))$ (ramified case), then the index 0 connected component of $\mathrm{Gr}_I(V)$ coincides with the subscheme of $\mathrm{Gr}(V)$ consisting of those subspaces that are maximal totally isotropic:

$$\{U \in \mathrm{Gr}^0(V) \text{ such that } \mathrm{Res}_{z_1=0} \, f(z_1)g(-z_1)\frac{dz_1}{z_1^2} = 0 \text{ for all } f, g \in U\}$$

and equation (3.9) is equivalent to the BKP hierarchy. That is, our approach generalizes those of [P, S2]. Observe that the slight difference between equation (3.9) and the bilinear equation of Lemma 4 in [S2] stems from the fact that Shiota works with the connected component of index 1 while we do so with that of index 0.

The relationship between the formal Prym variety and the Grassmannian of isotropic subspaces is unveiled in the following:

**Theorem 3.10.** *The largest subgroup of $\Gamma_V$ acting on $\mathrm{Gr}_I(V)$ is $\Pi$.*

*Proof.* Observe that an element $g \in \Gamma_V$ acts on $\mathrm{Gr}(V)$, mapping a point $U$ to $g \cdot U$, and that the homothety defined by $g$ is an automorphism of $V$ as a $\mathbb{C}((z))$-vector space. It follows that:

$$\wedge^p(g \cdot U) = \mathrm{Nm}(g) \cdot \wedge^p U \tag{3.11}$$

Then, it is trivial that $\Pi$ should act on the $\mathrm{Gr}_I(V)$ because the norm of its elements belongs to $\mathbb{C}$.

Let us prove the converse. Let us consider an element $g \in \Gamma_V$ acting on $\mathrm{Gr}_I(V)$. Since the norm takes values in $\mathbb{C}((z))$, it suffices to check that the coefficient of $z^n$ in $\mathrm{Nm}(g)$ is zero for all $n \neq 0$.

Given an integer $n \neq 0$, let $U_n$ be a subspace of $V$ of the type:

$$<z^{-n-1}e_1, \ldots, e_p> \oplus z^N V^- \qquad N \in \mathbb{Z}$$

where $\{e_1, \ldots, e_p\}$ is the distinguished basis of $V$ and $N \in \mathbb{Z}$ is such that $U_n$ belongs to the $\mathrm{Gr}_I(V)$ (this condition is attained for small enough $N$).

The construction of $U_n$ implies that $z^{-n-1}e_1 \wedge \cdots \wedge e_p \in \wedge^p U_n$. Since $g \cdot U_n \in \mathrm{Gr}_I(V)$, it follows by identity (3.11) that the coefficient of $z^n$ in $\mathrm{Nm}(g)$ must be 0 for all $n \neq 0$. $\qquad\square$

## 4. Prym varieties

### 4.A. Preliminaries

Let $\pi : Y \to X$ be a cyclic covering of degree $p$ ($p$ being a prime number) between smooth irreducible projective curves over $\mathbb{C}$ of genus $g$, $\bar{g}$, respectively. Assume that $\pi$ is given by an automorphism $\sigma_Y$ of $Y$ of order $p$; in particular, $Y/ <\sigma_Y> = X$. Let $R_\pi \subset Y$ denote the ramification divisor of $\pi$ and consider the line bundle on $X$ of degree $1/2 \deg R_\pi$ given by $\delta = \wedge \pi_* \mathcal{O}(R_\pi)$.

Let $J_d(Y)$ (resp. $J_d(X)$) denote the Jacobian of $Y$ (resp. $X$) of degree $d$ and let Nm: $J_d(Y) \to J_d(X)$ be the Albanese morphism induced by $\pi$. Note that $\wedge \pi_* L \simeq \mathrm{Nm}(L) \otimes \delta^*$ for every line bundle $L$ on $Y$. In particular, for a line bundle $L \in J_d(Y)$ of degree:

$$d = (g - 1) - (p - 2)(\bar{g} - 1)$$

it follows that $\deg(\wedge \pi_* L) = 2(\bar{g} - 1)$. Then, for this value of $d$, we define the Prym variety of degree $d$ associated with the covering $\pi$ as:

$$P_d(Y, \sigma_Y) := \{ L \in J_d(Y) : \wedge \pi_* L \simeq \omega_X \} \subset J_d(Y)$$

where $\omega_X$ is the canonical sheaf on $X$. In other words:

$$L \in P_d(Y, \sigma_Y) \quad \Longleftrightarrow \quad \mathrm{Nm}(L) \simeq \omega_X \otimes \delta$$

We should be noted that the functor of points of $P_d(Y, \sigma_Y)$ is given by

$$P_d(Y, \sigma_Y)^\bullet(S) = \left\{ \begin{array}{c} [L] \in J_d(Y)^\bullet(S) : \wedge (\pi \times \mathrm{Id})_*(L) \simeq q_1^* \omega_X \otimes q_2^* N \\ \text{for some line bundle } N \text{ on } S \end{array} \right\}$$

for each $\mathbb{C}$-scheme $S$, where $[L]$ denotes the equivalence class of a line bundle $L$ on $Y \times S \to S$ and $q_1, q_2$ are the natural projections of $X \times S$.

*Remark 5.* It follows that the Prym variety of degree $d$ is the translation of Ker Nm $\subset J_0(Y)$ by an arbitrary $L_0 \in P_d(Y, \sigma_Y)$. Observe that the classical Prym variety associated with $\pi$ is the connected component of $\mathcal{O}_Y$ in Ker Nm.

In this section the following data will be fixed $(Y, \sigma_Y, \bar{y}, t_{\bar{y}})$ where:

(1) $Y$ is a smooth irreducible projective curve over $\mathbb{C}$ of genus $g$;
(2) $\sigma_Y$ is an order $p$ automorphism of $Y$;
(3) $\bar{y}$ is an orbit of $\sigma_Y$; and,
(4) $t_{\bar{y}} : \widehat{\mathcal{O}}_{Y,\bar{y}} \xrightarrow{\sim} V^+$ is a formal parameter along $\bar{y}$ that is equivariant with respect to the actions of $\sigma_Y$ and $\sigma$.

Let $\mathrm{Pic}^\infty(Y, \bar{y})$ be the scheme whose functor of points is given by:

$$S \rightsquigarrow \mathrm{Pic}^\infty(Y, \bar{y})^\bullet(S) = \left\{ \begin{array}{c} \text{pairs } (L, \phi) \text{ where } L \text{ is a line bundle} \\ \text{on } Y \times S \text{ and } \phi : \widehat{L}_{\bar{y} \times S} \xrightarrow{\sim} \mathcal{O}_S \widehat{\otimes} V^+ \end{array} \right\}$$

We denote by $\mathrm{Pic}_d^\infty(Y, \bar{y})$ the subscheme of $\mathrm{Pic}^\infty(Y, \bar{y})$ consisting of those pairs $(L, \phi)$, where the line bundle $L$ has degree $d$.

The Krichever map is the injective morphism of functors given by:

$$K : \mathrm{Pic}^\infty(Y, \bar{y})^\bullet(S) \longrightarrow \mathrm{Gr}(V)^\bullet(S)$$
$$(L, \phi) \longmapsto \phi\left( \varinjlim_{n \geq 0} (p_Y)_* L(n \cdot (\bar{y} \times S)) \right)$$

where $p_Y \colon Y \times S \to S$ is the natural projection (for a more detailed study of $\text{Pic}^\infty(Y, \bar{y})$ and $K$, see [A]).

Consider the point in $X := Y/<\sigma_Y>$ given by $x = \pi(\bar{y})$. Since $t_{\bar{y}}$ is equivariant, it gives rise to an isomorphism between the subrings of invariant elements; that is, a formal parameter $t_x \colon \widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathbb{C}[[z]]$ at $x$ such that the diagram:

$$
\begin{array}{ccc}
\widehat{\mathcal{O}}_{Y,\bar{y}} & \xrightarrow[\sim]{t_{\bar{y}}} & V^+ \\
\big\uparrow & & \big\uparrow \\
\widehat{\mathcal{O}}_{X,x} & \xrightarrow[\sim]{t_x} & \mathbb{C}[[z]]
\end{array}
$$

is commutative. Moreover, note that $t_x$ induces a map:

$$
dt_x \colon (\omega_X)\widehat{\phantom{l}}_x \xrightarrow{\sim} \Omega_{\widehat{\mathcal{O}}_{X,x}/\mathbb{C}} \xrightarrow{\sim} \Omega_{\mathbb{C}[[z]]/\mathbb{C}} \xrightarrow{\sim} \mathbb{C}[[z]]dz
$$

If we are given a pair $(L, \phi)$ in $\text{Pic}_d^\infty(Y, \bar{y})$, then the determinant of $\phi$ yields an isomorphism:

$$
(\det \phi) \colon (\wedge \pi_* L)\widehat{\phantom{l}}_x \xrightarrow{\sim} \wedge_{\mathbb{C}[[z]]} V^+ \xrightarrow{\sim} \mathbb{C}[[z]]z_1 \wedge \cdots \wedge z_p
$$

When $\omega_X \simeq \wedge \pi_* L$, we have an isomorphism (up to a non-zero constant): $(\omega_X)\widehat{\phantom{l}}_x \xrightarrow{\sim} (\wedge \pi_* L)\widehat{\phantom{l}}_x$. We require these isomorphism to be compatible:

**Definition 4.1.** *If $\omega_X \simeq \wedge \pi_* L$, the formal parameter $t_{\bar{y}}$ and the formal trivialization $\phi$ of $L$ along $\bar{y}$ are said to be compatible if the diagram:*

$$
\begin{array}{ccc}
(\omega_X)\widehat{\phantom{l}}_x & \xrightarrow{\sim} & (\wedge \pi_* L)\widehat{\phantom{l}}_x \\
{\scriptstyle dt_x}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \det \phi} \\
\mathbb{C}[[z]]dz & \xrightarrow{\sim} & \mathbb{C}[[z]]z_1 \wedge \cdots \wedge z_p
\end{array}
$$

*commutes (up to a non-zero constant), where the bottom arrow maps $dz$ to $z_1 \wedge \cdots \wedge z_p$ and $t_x$ is the formal parameter at $x$ defined by $t_{\bar{y}}$.*

Similarly, one defines compatibility for $S$-valued points. Indeed, let $t_{\bar{y}}$ be a formal trivialization of $\mathcal{O}_Y$ along $\bar{y}$ and let $(L, \phi)$ be a point in $\text{Pic}^\infty(Y, \bar{y})^\bullet(S)$ such that $L \in P_d(Y, \sigma_Y)^\bullet(S)$. Then, $t_{\bar{y}}$ and $\phi$ are said to be compatible, if the diagram:

$$
\begin{array}{ccc}
(\omega_X)\widehat{\phantom{l}}_x \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S & \xrightarrow{\sim} & (\wedge(\pi \times \text{Id})_* L)\widehat{\phantom{l}}_{x \times S} \\
{\scriptstyle dt_x \otimes 1}\big\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \det \phi} \\
\mathcal{O}_S[[z]]dz & \xrightarrow{\sim} & \mathcal{O}_S[[z]]z_1 \wedge \cdots \wedge z_p
\end{array}
$$

commutes (up to an element of $H^0(S, \mathcal{O}_S^*)$).

**Definition 4.2.** *The functor* $\underline{\mathrm{Prym}}^{\infty}(Y, \sigma_Y, \bar{y})$ *associated with the data* $(Y, \sigma_Y, \bar{y}, t_{\bar{y}})$ *is the subfunctor of* $\overline{\mathrm{Pic}_d^{\infty}(Y, \bar{y})}^{\bullet}$ *defined by:*

$$S \rightsquigarrow \underline{\mathrm{Prym}}^{\infty}(Y, \sigma_Y, \bar{y})(S) := \left\{ \begin{array}{c} (L, \phi) \in \mathrm{Pic}_d^{\infty}(Y, \bar{y})^{\bullet}(S) \ such \\ that \ \wedge (\pi \times \mathrm{Id})_* L \simeq q_1^* \omega_X \\ and \ t_{\bar{y}} \ and \ \phi \ are \ compatible \end{array} \right\}$$

**Theorem 4.3.** *The functor* $\underline{\mathrm{Prym}}^{\infty}(Y, \sigma_Y, \bar{y})$ *is representable by a closed subscheme* $\mathrm{Prym}^{\infty}(Y, \sigma_Y, \bar{y})$ *of* $\overline{\mathrm{Pic}_d^{\infty}(Y, \bar{y})}$.

*Proof.* It suffices to check that the injective morphism of functors:

$$\underline{\mathrm{Prym}}^{\infty}(Y, \sigma_Y, \bar{y}) \hookrightarrow \mathrm{Pic}_d^{\infty}(Y, \bar{y})^{\bullet} \underset{J_d(Y)^{\bullet}}{\times} P_d(Y, \sigma_Y)^{\bullet}$$

$$(L, \phi) \mapsto \big((L, \phi), [L]\big)$$

(where $[L] \in J_d(Y)^{\bullet}(S)$ denotes the equivalence class of $L$) is a closed immersion; in other words, that given an $S$-valued point:

$$\big((L, \phi), [L]\big) \in \mathrm{Pic}_d^{\infty}(Y, \bar{y})^{\bullet}(S) \underset{J_d(Y)^{\bullet}(S)}{\times} P_d(Y, \sigma_Y)^{\bullet}(S)$$

the condition that this point belongs to $\underline{\mathrm{Prym}}^{\infty}(Y, \sigma_Y, \bar{y})(S)$ is a closed condition on $S$.

One checks that $\wedge (\pi \times \mathrm{Id})_* L \simeq q_1^* \omega_X \otimes q_2^* N$ for some line bundle $N$ on $S$. Therefore, considering formal completions along $x \times S$, the formal parameter $t_x$ at $x$ induces an isomorphism (up to a non-zero constant):

$$(\wedge (\pi \times \mathrm{Id})_* L)\widehat{}_{x \times S} \simeq ((\omega_X)\widehat{}_x \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S) \widehat{\otimes}_{\mathcal{O}_S} N$$
$$\simeq (\Omega_{\mathbb{C}[[z]]/\mathbb{C}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S) \widehat{\otimes}_{\mathcal{O}_S} N \simeq \mathcal{O}_S[[z]] dz \widehat{\otimes}_{\mathcal{O}_S} N$$

On the other hand, the determinant of $\phi$ yields an isomorphism:

$$(\det \phi) : (\wedge (\pi \times \mathrm{Id})_* L)\widehat{}_{x \times S} \overset{\sim}{\to} \wedge_{\mathcal{O}_S[[z]]}(V^+ \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S) \overset{\sim}{\to} \mathcal{O}_S[[z]] z_1 \wedge \cdots \wedge z_p$$

Comparing both expressions, it follows that $N \simeq \mathcal{O}_S$. Fixing an isomorphism $q_1^* \omega_X \simeq \wedge (\pi \times \mathrm{Id})_* L$, the condition that $t_{\bar{y}}$ and $\phi$ are compatible is equivalent to saying that the $\mathcal{O}_S[[z]]$-module isomorphism:

$$\mathcal{O}_S[[z]] dz \overset{(dt_x \otimes 1)^{-1}}{\underset{\sim}{\longrightarrow}} (\omega_X)\widehat{}_x \widehat{\underset{\mathbb{C}}{\otimes}} \mathcal{O}_S \underset{\sim}{\longrightarrow} (\wedge (\pi \times \mathrm{Id})_* L)\widehat{}_{x \times S} \overset{\det \phi}{\underset{\sim}{\longrightarrow}} \mathcal{O}_S[[z]] z_1 \wedge \cdots \wedge z_p$$

sends $dz \mapsto \lambda \cdot z_1 \wedge \cdots \wedge z_p$ with $\lambda \in H^0(S, \mathcal{O}_S)$, which is a closed condition. Hence, the statement follows. $\qquad \square$

**Proposition 4.4.** *One has the following cartesian diagram:*

$$
\begin{array}{ccc}
\mathrm{Pic}_d^\infty(Y, \bar{y}) & \xhookrightarrow{\;K\;} & \mathrm{Gr}(V) \\
\big\uparrow & & \big\uparrow \\
\mathrm{Prym}^\infty(Y, \sigma_Y, \bar{y}) & \xhookrightarrow{\;K\;} & \mathrm{Gr}_\mathrm{I}(V)
\end{array}
$$

*Proof.* It suffices to check the claim for geometric points. Let $(L, \phi) \in \mathrm{Prym}^\infty$ $(Y, \sigma_Y, \bar{y})$ and $U = K(L, \phi) = \phi\big(H^0(Y - \bar{y}, L)\big) \subset V$.

Since $\wedge \pi_* L \simeq \omega_X$ and $t_{\bar{y}}$ and $\phi$ are compatible, we have the commutativity (up to a constant) of:

$$
\begin{array}{ccc}
(\omega_X)^{\widehat{}}_x & \xrightarrow{\;\sim\;} & (\wedge \pi_* L)^{\widehat{}}_x \\
dt_x \big\downarrow \simeq & & \simeq \big\downarrow \det \phi \\
\mathbb{C}[[z]]dz & \xrightarrow{\;\sim\;} & \mathbb{C}[[z]]z_1 \wedge \cdots \wedge z_p
\end{array}
$$

This diagram implies that the composition map:

$$
\wedge_{\mathbb{C}}^p H^0(X - x, \pi_* L) \twoheadrightarrow H^0(X - x, \wedge \pi_* L) \simeq H^0(X - x, \omega_X) \xrightarrow{dt_x} \mathbb{C}((z))dz
$$

coincides (up a multiplicative constant) with morphism (3.5) given in the definition of $\mathrm{Gr}_\mathrm{I}(V)$. In particular, it turns out that:

$$
\wedge^p U = dt_x\big(H^0(X - x, \omega_X)\big)
$$

as subspaces of $\mathbb{C}((z))dz$.

Since $H^0(X - x, \omega_X)$ are meromorphic differentials with one only pole, the condition of the vanishing of the residue (at $z = 0$) is satisfied; that is, $U = K(L, \phi) = \phi(H^0(Y - \bar{y}, L)) \in \mathrm{Gr}_\mathrm{I}(V)$.

Conversely, let $U \in \mathrm{Gr}_\mathrm{I}(V)$ such that there exists $(L, \phi) \in \mathrm{Pic}_d^\infty(Y, \bar{y})$ with $K(L, \phi) = U$. The above arguments show that:

$$
\wedge^p U = (\det \phi)\big(H^0(X - x, \wedge \pi_* L)\big)
$$

as subspaces of $\mathbb{C}((z))dz$.

The non-degenerated pairing:

$$
\begin{aligned}
\mathbb{C}((z)) \times \mathbb{C}((z))dz &\longrightarrow \mathbb{C} \\
(f, \omega) &\longmapsto \mathrm{Res}_{z=0}(f \cdot \omega)
\end{aligned}
$$

induces isomorphisms:

$$
\begin{aligned}
R &: \mathrm{Gr}\,\mathbb{C}((z)) \longrightarrow \mathrm{Gr}\big(\mathbb{C}((z))dz\big) \\
R' &: \mathrm{Gr}\big(\mathbb{C}((z))dz\big) \longrightarrow \mathrm{Gr}\,\mathbb{C}((z))
\end{aligned}
$$

such that $R \circ R' = \mathrm{Id}_{\mathrm{Gr}\,\mathbb{C}((z))dz}$ and $R' \circ R = \mathrm{Id}_{\mathrm{Gr}\,\mathbb{C}((z))}$.

Let $A_{\wedge^p U} = t_x(H^0(X-x, \mathcal{O}_X)) \subset \mathbb{C}((z))$. The condition $U \in \mathrm{Gr}_I(V)$ implies that the map:

$$\wedge^p U = (\det \phi)(H^0(X-x, \wedge \pi_* L)) \subset \mathbb{C}((z))dz \xrightarrow{\mathrm{Res}} \mathbb{C}$$

vanishes. Then, it follows that $A_{\wedge^p U} \subseteq R'(\wedge^p U)$ because $f \cdot \wedge^p U \subset \wedge^p U$ for all $f \in A_{\wedge^p U}$.

Using the Krichever construction for the triple $(X, x, t_x)$, one checks that the subspace $R'(\wedge^p U) \subset \mathbb{C}((z))$ is attached to the geometric datum $(\omega_X \otimes (\wedge \pi_* L)^{-1}, dt_x \otimes ((\det \phi)^*)^{-1})$, where $(\det \phi)^*$ denotes the transposed map of $\det \phi$. From this, one deduces that $A_{\wedge^p U}$ and $R'(\wedge^p U)$ lie in the same connected component of $\mathrm{Gr}\,\mathbb{C}((z))$. Therefore, $A_{\wedge^p U} = R'(\wedge^p U)$.

Bearing in mind that $R \circ R' = \mathrm{Id}_{\mathrm{Gr}\,\mathbb{C}((z))dz}$, it holds that:

$$dt_x(H^0(X-x, \omega_X)) = R(A_{\wedge^p U}) = \wedge^p U = (\det \phi)(H^0(X-x, \wedge \pi_* L))$$

By the injectivity of the Krichever morphism, this identity implies that $\omega_X \simeq \wedge \pi_* L$ and that $t_{\bar{y}}$ and $\phi$ are compatible. That is, $(L, \phi) \in \mathrm{Prym}^\infty(Y, \sigma_Y, \bar{y})$. $\square$

**Theorem 4.5.** *The formal group $\Pi$ acts on the image of the Krichever map $K$:* $\mathrm{Prym}^\infty(Y, \sigma_Y, \bar{y}) \hookrightarrow \mathrm{Gr}_I(V)$.

*Proof.* Bearing in mind that $\Pi \subset \Gamma_V$ acts on the image of the Krichever map $K : \mathrm{Pic}_d^\infty(Y, \bar{y}) \hookrightarrow \mathrm{Gr}(V)$, the proof follows from Theorem 3.10 and the above proposition. $\square$

### 4.B. Geometric characterization of Pryms

In this part we shall show an analogue for Prym varieties of Mulase's characterization of Jacobians varieties ([M]). Roughly speaking, he shows that Jacobian varieties are precisely the finite dimensional orbits of a certain group acting on a quotient Grassmannian. Our approach will work within the frameset of formal schemes ([EGA], I.§10) and follow [P] closely. This characterization is strongly related to that of Shiota, which claims that Jacobians are the finite dimensional solutions of the KP hierarchy ([S, S2]).

The action of $\Gamma_V$ on $V$ by homotheties gives rise to an action:

$$\mu : \Pi \times \mathrm{Gr}(V)^p \longrightarrow \mathrm{Gr}(V)^p$$
$$(g, U.) \longmapsto g \cdot U.$$

where $\mathrm{Gr}(V)^p := \mathrm{Gr}(V) \times .^p. \times \mathrm{Gr}(V)$, $U. := (U_1, \ldots, U_p) \in \mathrm{Gr}(V)^p$ and $g \cdot U. := (g \cdot U_1, \ldots, g \cdot U_p)$.

On the other hand, the following map:

$$\mathrm{Gr}(V) \hookrightarrow \mathrm{Gr}(V)^p$$
$$U \mapsto U_\sigma := (U, \sigma(U), \ldots, \sigma^{p-1}(U))$$

turns out to be a closed immersion.

**Definition 4.6.** *Given a closed point $U \in \mathrm{Gr}(V)$ we define the morphism $\mu_U$ by:*

$$\mu_U \colon \Pi \longrightarrow \mathrm{Gr}(V)^p$$
$$g \mapsto g \cdot U_\sigma = (g \cdot U, g \cdot \sigma(U), \dots, g \cdot \sigma^{p-1}(U))$$

Our goal consists in characterizing those points of $\mathrm{Gr}(V)$ coming from geometric data in terms of the above morphism. To this end, we begin with some general properties.

**Lemma 4.7.** *Let $U$ be a closed point of $\mathrm{Gr}(V)$. The orbit of $U_\sigma \in \mathrm{Gr}(V)^p$ under $\mu$ is the schematic image of $\mu_U$ which is a formal scheme and will be denoted by $\Pi(U_\sigma)$.*

*Proof.* Following Remark 2, we write the group $\Pi$ as a direct limit of affine group schemes $\Pi^n$.
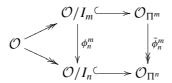
Let us denote by $\mathcal{O}$ the sheaf of rings of $\mathrm{Gr}(V)^p$, by $\mu^*_{U,n} \colon \mathcal{O} \to \mathcal{O}_{\Pi^n}$ the morphism induced by $\mu_U|_{\Pi^n}$, and by $I_n$ the ideal $\mathrm{Ker}\,\mu^*_{U,n}$. Finally, define $\Pi^n(U_\sigma)$ to be the schematic image of the morphism:

$$\mu_U|_{\Pi^n} \colon \Pi^n \longrightarrow \mathrm{Gr}(V)^p$$

that is:

$$\Pi^n(U_\sigma) := \mathrm{Spec}\left(\mathcal{O}/I_n\right)$$

For $m \geq n \geq 0$, consider the morphisms $\phi^m_n$ and $\bar{\phi}^m_n$ defined by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{O}/I_m & \hookrightarrow & \mathcal{O}_{\Pi^m} \\
\phi^m_n \downarrow & & \downarrow \bar{\phi}^m_n \\
\mathcal{O}/I_n & \hookrightarrow & \mathcal{O}_{\Pi^n}
\end{array}
$$

(with $\mathcal{O}$ mapping to $\mathcal{O}/I_m$ and $\mathcal{O}/I_n$)

Since $\Pi(U_\sigma) = \cup_{n \geq 0} \Pi^n(U_\sigma)$, in order to show that $\varinjlim_n \Pi^n(U_\sigma)$ exists in the category of formal schemes it suffices to show that $\phi^m_n$ is surjective and that its kernel is a nilpotent ideal ([EGA], I.§10.6.3).

Bearing in mind that $I_k = \mathrm{Ker}(\mu^*_{U,k})$ and that $\mu^*_{U,n} = \bar{\phi}^m_n \circ \mu^*_{U,m}$, the surjectivity of $\phi^m_n$ follows easily. Finally, note that the ideal $\mathrm{Ker}(\phi^m_n)$ is contained in $\mathrm{Ker}(\bar{\phi}^m_n)$, which is nilpotent. □

**Lemma 4.8.** *The stabilizer of $U_\sigma$ is a closed subgroup of $\Pi$, which will be denoted by $H_{U_\sigma}$, and there is a canonical isomorphism of formal schemes:*

$$\Pi/H_{U_\sigma} \simeq \Pi(U_\sigma)$$

*(We understand by a closed subgroup of $\Pi$ a subgroup defined by a closed ideal of $\mathcal{O}_\Pi$).*

*Proof.* Recall that $\Pi = \varinjlim \Pi^n$, where $\Pi^n$ are affine group schemes. For $n \geq 0$, consider the subfunctor $\underline{H}^n_{U_\sigma}$ whose set of $R$-valued points ($R$ being a $\mathbb{C}$-algebra) is:

$$\underline{H}^n_{U_\sigma}(R) := \{g \in \Pi^n(R) \mid g \cdot U_\sigma = U_\sigma\}$$

Since $\underline{H}^n_{U_\sigma}$ is defined by a closed condition, it is representable by a closed subscheme $H^n_{U_\sigma}$ of $\Pi^n$, which is precisely the stabilizer of $U_\sigma$ under the action of $\Pi^n$ on $\mathrm{Gr}(V)^p$.

Note that the following cartesian diagram (for $m \geq n \geq 0$):

$$\begin{array}{ccc} H^n_{U_\sigma} & \hookrightarrow & \Pi^n \\ \downarrow & & \downarrow \\ H^m_{U_\sigma} & \hookrightarrow & \Pi^m \end{array} \qquad (4.9)$$

implies that the morphism $\mathcal{O}_{H^m_{U_\sigma}} \to \mathcal{O}_{H^n_{U_\sigma}}$ is surjective and its kernel is a nilpotent ideal. It follows that $H_{U_\sigma} := \varinjlim H^n_{U_\sigma}$ is a formal scheme and a closed subgroup of $\Pi$ that coincides with the stabilizer of $U_\sigma$.

Let us now prove the second part. Bearing in mind that $H^n_{U_\sigma}$ is the stabilizer subgroup of the action of $\Pi^n$ and that $\Pi^n(U_\sigma)$ is the schematic image of $\mu_U|_{\Pi^n}$, one has that:

$$\Pi(U_\sigma) = \varinjlim \Pi^n(U_\sigma) = \varinjlim \Pi^n/H^n_{U_\sigma}$$

We conclude if we can show that $\varinjlim \Pi^n/H^n_{U_\sigma} = \Pi/H_{U_\sigma}$. However, this follows easily from the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^n_{U_\sigma} & \longrightarrow & \Pi^n & \longrightarrow & \Pi^n/H^n_{U_\sigma} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^m_{U_\sigma} & \longrightarrow & \Pi^m & \longrightarrow & \Pi^m/H^m_{U_\sigma} & \longrightarrow & 0 \end{array}$$

where the injectivity of the third vertical arrow follows from the fact that the diagram 4.9 is cartesian. $\qquad\square$

For the sake of notation, it is convenient to consider the following subgroup of $\Pi$:

$$\bar{\Pi}^+ = \begin{cases} \mathbb{G}_m \times \Pi^+ & \text{in the ramified case,} \\ (\mathbb{G}_m \times .\overset{p}{.}. \times \mathbb{G}_m) \times \Pi^+ & \text{in the non-ramified case.} \end{cases}$$

**Lemma 4.10.** *The quotient $\Pi(U_\sigma)/\bar{\Pi}^+$ is a formal scheme canonically isomorphic to $\Pi/(H_{U_\sigma} + \bar{\Pi}^+)$. In particular, its tangent space at $U_\sigma$ is isomorphic to $T_1\Pi/(\mathrm{Ker}\, d\mu_U + T_1\bar{\Pi}^+)$ where:*

$$d\mu_U : T_1\Pi \longrightarrow T_{U_\sigma}\mathrm{Gr}(V)^p$$

*is the map induced by $\mu$ on the tangent spaces.*

*Proof.* In order to show that $\Pi(U_\sigma)/\bar{\Pi}^+$ is a formal scheme, it suffices to check that it is a direct limit of affine schemes in the conditions of [EGA], I.§10.6.3. Since $\Pi^n(U_\sigma) = \Pi^n/H_{U_\sigma}^n$, one has that $\Pi^n(U_\sigma)/\bar{\Pi}^+ = \Pi^n/(H_{U_\sigma}^n + \bar{\Pi}^+)$.

Finally, the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_{U_\sigma}^n + \bar{\Pi}^+ & \longrightarrow & \Pi^n & \longrightarrow & \Pi^n/(H_{U_\sigma}^n + \bar{\Pi}^+) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \downarrow & & \\
0 & \longrightarrow & H_{U_\sigma}^m + \bar{\Pi}^+ & \longrightarrow & \Pi^m & \longrightarrow & \Pi^m/(H_{U_\sigma}^m + \bar{\Pi}^+) & \longrightarrow & 0
\end{array}
$$

implies that the direct limit of $\Pi^n/(H_{U_\sigma}^n + \bar{\Pi}^+)$ exists as a formal scheme; it will be denoted by $\Pi/(H_{U_\sigma} + \bar{\Pi}^+)$.

The claim about the tangent space is now straightforward since $T_1 H_{U_\sigma} \simeq \operatorname{Ker} d\mu_U$. $\qquad\square$

*Remark 6.* Observe that $d\mu_U$ is explicitly given by:

$$
d\mu_U : T_1\Pi \longrightarrow T_{U_\sigma}\operatorname{Gr}(V)^p \simeq \prod_{i=0}^{p-1} \operatorname{Hom}(\sigma^i(U), V/\sigma^i(U)) \tag{4.11}
$$

$$
g \longmapsto \left\{ \sigma^i(U) \hookrightarrow V \xrightarrow{\cdot g} V \twoheadrightarrow V/\sigma^i(U) \right\}_{i=0,\ldots,p-1}
$$

and that $T_1\Pi = \{g \in V \mid \operatorname{Tr}(g) \in \mathbb{C}\}$ because the norm $\operatorname{Nm} : \Gamma_V \to \Gamma$ induces the trace at their tangent spaces. Therefore, one has that:

$$
\begin{aligned}
\operatorname{Ker} d\mu_U &= \{g \in V \mid \operatorname{Tr}(g) \in \mathbb{C} \text{ and } g \cdot \sigma^i(U) \subseteq \sigma^i(U) \text{ for all } i\} \\
&= \{g \in V \mid \operatorname{Tr}(g) \in \mathbb{C} \text{ and } \sigma^i(g) \cdot U \subseteq U \text{ for all } i\}
\end{aligned}
$$

**Theorem 4.12.** *Let $U$ be a closed point of $\operatorname{Gr}(V)$. Then, the following conditions are equivalent:*

(1) *$\Pi(U_\sigma)/\bar{\Pi}^+$ is algebraizable,*
(2) *$\dim_\mathbb{C} T_{U_\sigma}(\Pi(U_\sigma)/\bar{\Pi}^+) < \infty$,*

*Further, if one of these conditions holds true, then the ring of $\Pi(U_\sigma)/\bar{\Pi}^+$ is isomorphic to a ring of series in finitely many variables.*

*Proof.* $1 \implies 2$. Let us write $\Pi(U_\sigma)/\bar{\Pi}^+$ as $\operatorname{Spf} A$ and let us note that $A$ is an admissible linearly topologized ring. Recall that algebraizable ([H] II.9.3.2) means that the formal scheme is isomorphic to the completion of a noetherian scheme along a closed subscheme. Since the completion of a noetherian ring with respect to an ideal is noetherian, one concludes that $A$ is noetherian.

Let $J := \varprojlim J_n$ be a definition ideal of $A$. Since $A$ is noetherian and $A/J \simeq k$, it follows from [EGA] 0.7.2.6 that $(J/J^2)^*$ is a finite dimensional vector space.

We conclude from the following inclusion:

$$T_{U_\sigma}\left(\Pi(U_\sigma)/\bar\Pi^+\right) = \mathrm{Hom}_{\text{for-sch}}\left(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)), \mathrm{Spf}\, A\right)$$

$$= \mathrm{Hom}_{\substack{\text{topological}\\ k\text{-algebras}}}(A, k[\epsilon]/(\epsilon^2))$$

$$\subseteq \mathrm{Hom}_{k\text{-algebras}}(A, k[\epsilon]/(\epsilon^2)) \subseteq \left(J/J^2\right)^*$$

$2 \implies 1$. To prove the claim, it will be enough to show that $A$ is isomorphic to a ring of power series in finitely many variables, $\mathbb{C}[[u_1, \ldots, u_d]]$, endowed with the $(u_1, \ldots, u_n)$-adic topology.

Recall that for all $n \gg 0$ there is a surjection of vector spaces:

$$T_1\Pi^n/\bar\Pi^+ \longrightarrow T_{U_\sigma}\left(\Pi^n(U_\sigma)/\bar\Pi^+\right)$$

Let us introduce the following notation: $\mathrm{Spec}(A_n) := \Pi^n(U_\sigma)/\bar\Pi^+$, $\mathrm{Spf}\, A := \varinjlim \mathrm{Spec}(A_n)$ and $\mathrm{Spec}(\mathcal{O}_n) := \Pi^n/\bar\Pi^+$. Bearing in mind that $A_n$ and $\mathcal{O}_n$ are artinian rings, one has that:

$$\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2 \hookrightarrow \mathfrak{m}_{\mathcal{O}_n}/\mathfrak{m}_{\mathcal{O}_n}^2 \qquad \forall n \gg 0$$

where $\mathfrak{m}_{A_n}$ (resp. $\mathfrak{m}_{\mathcal{O}_n}$) denotes the maximal ideal of $A_n$ (resp. $\mathcal{O}_n$).

Recall from Remark 2 that $\mathcal{O}_n$ is of the type $\mathbb{C}[u_{n'}, \ldots, u_1]/I_{n'}$, with $n'$ depending on $n$, and where $(u_{n'}, \ldots, u_1)^n \subseteq I_{n'}$.

On the other hand, the hypothesis claims that the vector space:

$$T_{U_\sigma}\left(\Pi(U_\sigma)/\bar\Pi^+\right) = \varinjlim T_{U_\sigma}\left(\Pi^n(U_\sigma)/\bar\Pi^+\right)$$

is of finite dimension, say $d$. That is:

$$\dim\left(T_{U_\sigma}\left(\Pi^n(U_\sigma)/\bar\Pi^+\right)\right) = d \qquad \forall n \gg 0$$

Since $T_{U_\sigma}\mathrm{Spec}(A_n) \simeq (\mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2)^*$ and the maps:

$$J := \varprojlim \mathfrak{m}_{A_n} \xrightarrow{\pi_n} \mathfrak{m}_{A_n}$$

$$\mathfrak{m}_{A_m} \longrightarrow \mathfrak{m}_{A_n}$$

are surjective, there exist elements $\bar v_1, \ldots, \bar v_d \in J$ such that:

$$<\{\pi_n(\bar v_1), \ldots, \pi_n(\bar v_d)\}> = \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2 \qquad \text{for all } n \gg 0$$

By Nakayama's lemma one has epimorphisms:

$$\begin{aligned} p_n : k[v_1, \ldots, v_d] &\to A_n \\ v_i &\mapsto \pi_n(\bar v_i) \end{aligned} \qquad \forall n \gg 0$$

compatible with the surjections $A_m \to A_n$ for $m \geq n \gg 0$.

Summing up, we have obtained a commutative diagram:

$$
\begin{array}{ccccc}
<v_1,\ldots,v_d> & \xrightarrow{\;\sim\;} & \mathfrak{m}_{A_n}/\mathfrak{m}_{A_n}^2 & \hookrightarrow & <u_1,\ldots,u_{n'}> \\
\| & & \| & & \downarrow \\
<v_1,\ldots,v_d> & \xrightarrow{\;\sim\;} & \mathfrak{m}_{A_m}/\mathfrak{m}_{A_m}^2 & \hookrightarrow & <u_1,\ldots,u_{m'}>
\end{array}
$$

for $n \geq m \gg 0$.

Up to a change of coordinates, it can be assumed that the image of $v_i$ is $u_i$ for all $1 \leq i \leq d$. Looking at the corresponding morphisms of rings, we find the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{C}[v_1,\ldots,v_d]/(v_1,\ldots,v_d)^n & \longrightarrow\!\!\!\rightarrow & A_n & \hookrightarrow & \mathbb{C}[u_1,\ldots,u_{n'}]/I_{n'} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}[v_1,\ldots,v_d]/(v_1,\ldots,v_d)^m & \longrightarrow\!\!\!\rightarrow & A_m & \hookrightarrow & \mathbb{C}[u_1,\ldots,u_{m'}]/I_{m'}
\end{array}
$$

Taking inverse limits, we have that the following morphism of linearly topologized rings:

$$
\mathbb{C}[[v_1,\ldots,v_d]] \longrightarrow A = \varprojlim A_n \hookrightarrow \mathbb{C}\{\{u_1,u_2\ldots\}\}
$$

maps $v_i$ to $u_i$. In particular, this implies that:

$$
A = \mathbb{C}[[u_1,\ldots,u_d]]
$$

with the $(u_1,\ldots,u_d)$-adic topology and the conclusion follows.                           $\square$

**Theorem 4.13.** *Let $U$ be a closed point of $\mathrm{Gr}(V)$. Then, the following conditions are equivalent:*

(1) *$\Pi(U_\sigma)/\bar{\Pi}^+$ is algebraizable,*
(2) *there exist data $(Y,\sigma_Y,\bar{y},t_{\bar{y}})$ and a line bundle with a formal trivialization $(L,\phi)$ such that $\bar{y}$ is an orbit of $\sigma_Y$ and:*

$$
U = (t_{\bar{y}} \circ \phi)(H^0(Y - \bar{y}, L))
$$

*Proof.* $1 \implies 2$. For $U \in \mathrm{Gr}(V)$, consider the vector spaces:

$$
\begin{aligned}
\mathrm{Ker}\, d\mu_U &= \mathrm{Ker}\left(T_1\Pi \to T_{U_\sigma}\mathrm{Gr}(V)^p\right) \\
&= \{g \in V \mid \mathrm{Tr}(g) \in \mathbb{C} \text{ and } g \cdot \sigma^i(U) \subseteq \sigma^i(U) \text{ for all } i\} \\
B &:= \mathrm{Ker}\left(T_1\Gamma_V \to T_{U_\sigma}\mathrm{Gr}(V)^p\right) \\
&= \{g \in V \mid g \cdot \sigma^i(U) \subseteq \sigma^i(U) \text{ for all } i\}
\end{aligned}
$$

Note that the exact sequence of formal group schemes:

$$
0 \longrightarrow \Pi \longrightarrow \Gamma_V \xrightarrow{\;\mathrm{Nm}\;} \Gamma/\mathbb{C}^* \longrightarrow 0
$$

induces the exact sequence of vector spaces:

$$0 \longrightarrow T_1\Pi \longrightarrow T_1\Gamma_V = V \xrightarrow{\ \text{Tr}\ } T_1\Gamma = \mathbb{C}((z))/\mathbb{C} \longrightarrow 0$$

This sequence allows us to consider the following diagram of $\mathbb{C}$-vector spaces:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Ker}\, d\mu_U & \longrightarrow & B & \xrightarrow{\ \text{Tr}\ } & \text{Tr}(B) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_1\Pi/T_1\bar{\Pi}^+ & \longrightarrow & V/V^+ & \longrightarrow & \mathbb{C}((z))/\mathbb{C}[[z]] & \longrightarrow & 0
\end{array}
$$

Snake's Lemma yields the following exact sequence:

$$
\begin{aligned}
0 \to \text{Ker}\, d\mu_U \cap T_1\bar{\Pi}^+ &\to B \cap V^+ \to \text{Tr}(B) \cap \mathbb{C}[[z]] \\
&\to T_1\Pi/(\text{Ker}\, d\mu_U + T_1\bar{\Pi}^+) \to V/(B + V^+) \\
&\to \mathbb{C}((z))/(\text{Tr}(B) + \mathbb{C}[[z]]) \to 0
\end{aligned}
$$

From the expressions of $\text{Ker}\, d\mu_U$ and $B$, one easily checks that the first two terms of the above sequence are finite dimensional vector spaces.

Observe that the subalgebra of $V$ generated by $\text{Ker}\, d\mu_U$ is a subalgebra of $B$ of finite codimension (because $p$ is a prime number). From hypothesis (1) and Theorem 4.12, we know that $T_1\Pi/(\text{Ker}\, d\mu_U + T_1\bar{\Pi}^+)$ is finite dimensional, so $V/(B + V^+)$ is finite dimensional too. Summing up, we have obtained a pair $(U, B) \in \text{Gr}(V) \times \text{Gr}(V)$ such that $B$ is a sub-$\mathbb{C}$-algebra of $V$ and $U$ is a $B$-module.

Standard arguments of the Krichever construction show that the pair $(U, B)$ correspond to a curve $Y$, a divisor $\bar{y}$, a formal parameter along the divisor $t_{\bar{y}}$ : $\widehat{\mathcal{O}}_{Y,\bar{y}} \simeq V^+$, a line bundle $L$ and a formal trivialization $\phi : \widehat{L}_{\bar{y}} \simeq \widehat{\mathcal{O}}_{Y,\bar{y}}$ ([MP2] Theorem 5.3, [M2,Q]). Further, it is also known that, under the Krichever correspondence, we have:

$$
\begin{aligned}
U &= (t_{\bar{y}} \circ \phi)(H^0(Y - \bar{y}, L)) \\
B &= t_{\bar{y}}(H^0(Y - \bar{y}, \mathcal{O}_Y))
\end{aligned}
$$

Note that $B$ carries an action of $\sigma$ and, therefore, the curve $Y$ is canonically endowed with an order $p$ automorphism $\sigma_Y$.

$2 \implies 1$. Given $(Y, \sigma_Y, \bar{y}, t_{\bar{y}})$ and $(L, \phi)$ as in the statement, one considers the quotient $\pi : Y \to X := Y/<\sigma_Y>$. Let $B$ be the $\mathbb{C}$-algebra $t_{\bar{y}}(H^0(Y - \bar{y}, \mathcal{O}_Y)) \in \text{Gr}(V)$ which is endowed with an action of $\sigma$ induced by $\sigma_Y$. Since $\sigma$ acts on $B$ and $\text{Tr}(b) = \sum_{i=0}^{p-1} \sigma^i(b)$, it follows that:

$$\text{Tr}(B) \subseteq B$$

and (from Theorem 5.10 of [MP2]):

$$\text{Tr}(B) = t_x(H^0(X - \pi(\bar{y}), \mathcal{O}_X)) \in \text{Gr}\,\mathbb{C}((z))$$

Summing up, the maps $B \to V/V^+$ and $\text{Tr}(B) \to \mathbb{C}((z))/\mathbb{C}[[z]]$ have finite dimensional kernels and cokernels. Then, using the long exact sequence of the first part of the proof, we conclude that $T_1 \Pi/(\text{Ker}\, d\mu_U + T_1 \bar{\Pi}^+)$ is finite dimensional or, what amounts to the same, $\Pi(U_\sigma)/\bar{\Pi}^+$ is algebraizable (Theorem 4.12). $\qquad\square$

**Corollary 4.14.** *If the conditions of Theorem 4.13 are satisfied, then there is an isomorphism:*

$$T_{U_\sigma}(\Pi(U_\sigma)/\bar{\Pi}^+) \simeq T_{\mathcal{O}_Y} P_d(Y, \sigma_Y)$$

*Proof.* Recall that in the proof of Theorem 4.13, we obtained that $\text{Tr}(B) \subseteq B$, which implies that the connecting morphism of the long exact sequence of that proof is zero; that is, the sequence splits into two exact short sequences. The second one reads as follows:

$$0 \to T_{U_\sigma}(\Pi(U_\sigma)/\bar{\Pi}^+) \to H^1(Y, \mathcal{O}_Y) \xrightarrow{\text{Tr}} H^1(X, \mathcal{O}_X) \to 0$$

which implies the claim. $\qquad\square$

**Theorem 4.15.** *Let $U$ be a closed point of $\text{Gr}(V)$ such that $\Pi(U_\sigma)/\bar{\Pi}^+$ is algebraizable, or equivalently, there exist data $(Y, \sigma_Y, \bar{y}, t_{\bar{y}})$ and $(L, \phi)$ with $U = (t_{\bar{y}} \circ \phi)(H^0(Y - \bar{y}, L))$. Then, it holds that:*

$$U \in \text{Gr}_I(V) \iff L \in P_d(Y, \sigma_Y) \text{ and } \phi \text{ and } t_{\bar{y}} \text{ are compatible.}$$

*Proof.* This follows from Proposition 4.4 and Theorem 4.13. $\qquad\square$

*Remark 7.* The corresponding Prym flows have been computed by several authors in terms of Lax pairs ([AB, K]) or in terms of the Heisenberg algebras ([DM]).

## 5. Equations of the moduli space of curves with automorphisms

The aim of this section is to explicitly describe the moduli space of pointed curves with a non-trivial automorphism group as a subscheme of the Sato Grassmannian. Let us recall that given a smooth irreducible projective curve $Y$ of genus $g$ over $\mathbb{C}$ and $p$ a prime number such that $p$ divides the order of $\text{Aut}(Y)$, one has that $p \leq 2g + 1$. Then, the problem of characterizing the moduli space of curves of genus $g$ with a non-trivial automorphism group is reduced to the characterization of the moduli space of curves with an order $p$ automorphism, where $p$ is a prime number and $p \leq 2g + 1$. Henceforth, $p$ will denote a prime number.

Let $Y$ be a smooth irreducible projective curve over $\mathbb{C}$, let $\sigma_Y \in \text{Aut}(Y)$ be an order $p$ automorphism and let $X = Y/<\sigma_Y>$ be the quotient under the action of $<\sigma_Y>$. The projection $\pi: Y \to X$ is a cyclic covering of degree $p$. Given a geometric point $x \in X$, there are two posibilities for its fibre:

(a) *Ramified case: x* is a ramification point of $\pi$; that is, the fibre contains only one point $y \in Y$:

$$\pi^{-1}(x) = p \cdot y$$

(b) *Non-ramified case: x* is not a ramification point of $\pi$; that is, the fibre contains $p$ pairwise different points $y_1, \ldots, y_p \in Y$:

$$\pi^{-1}(x) = y_1 + \cdots + y_p$$

We shall study the moduli of pairs $(Y, \sigma_Y)$ distinguishing these two cases: ramified and non-ramified. It is therefore convenient to use different notations for the two cases considered in section §2:

(a) *Ramified case:* $V_R = \mathbb{C}((z^{1/p})) = \mathbb{C}((z_1))$ and $V_R^+ = \mathbb{C}[[z_1]]$.
(b) *Non-ramified case:* $V_{NR} = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))$ and $V_{NR}^+ = \mathbb{C}[[z_1]] \times \cdots \times \mathbb{C}[[z_p]]$.

Following the notations of §3.A, we shall denote by $\mathrm{Gr}(V_R)^\sigma$ and $\mathrm{Gr}(V_{NR})^\sigma$ the closed subschemes whose points are the invariant points under the action of $<\sigma>$.

**Definition 5.1.** *Let $\mathcal{M}^{\infty,1}$ be the closed subscheme of $\mathrm{Gr}(V_R)$ representing the moduli of integral complete curves with one marked smooth point and a formal parameter at this point (see [MP], Theorem 6.5).*

*Let $\mathcal{M}^{\infty,p}$ be the closed subscheme of $\mathrm{Gr}(V_{NR})$ representing the moduli of reduced complete curves with $p$ marked pairwise distinct smooth points and formal parameters at these points (see [MP2], Theorem 5.3).*

## 5.A. The ramified case and the Krichever map

**Definition 5.2.** *The moduli functor $\underline{\mathcal{M}}^\infty(p, R)$ of curves with an automorphism of order $p$ and a formal parameter at a fixed point is the contravariant functor on the category of $\mathbb{C}$-schemes:*

$$\underline{\mathcal{M}}^\infty(p, R): \left\{ \begin{matrix} \text{category of} \\ \mathbb{C} - \text{schemes} \end{matrix} \right\} \rightsquigarrow \left\{ \begin{matrix} \text{category} \\ \text{of sets} \end{matrix} \right\}$$

*that associates the set of equivalence classes of data $\{Y, \sigma_Y, y, t_y\}$ with a $\mathbb{C}$-scheme $S$, where:*

(1) *$p_Y : Y \rightarrow S$ is a proper and flat morphism whose fibres are geometrically integral curves.*
(2) *$\sigma_Y$ is a non-trivial automorphism of the curve $Y \rightarrow S$ of order $p$ such that the quotient $X := Y/<\sigma_Y>$ exists and $p_X : X \rightarrow S$ is a proper and flat morphism whose fibres are geometrically integral curves.*

(3) $y : S \to Y$ is a smooth section of $p_Y$ which is a Cartier divisor of $Y$ invariant under the action of $\sigma_Y$ and such that, for each closed point $s \in S$, $y(s)$ is a smooth point of the fibre $Y_s = p_Y^{-1}(s)$.

(4) *There exists a formal parameter $t_y$ along the section $y$:*

$$t_y \colon \widehat{\mathcal{O}}_{Y, y(S)} \xrightarrow{\sim} V_{\mathrm{R}}^+ \widehat{\otimes} \mathcal{O}_S$$

*that is compatible with respect to the actions of $\sigma_Y$ and $\sigma$.*

(5) $\{Y, \sigma_Y, y, t_y\}$ and $\{Y', \sigma_Y', y', t_{y'}\}$ are said to be equivalent when there is a commutative diagram of $S$-schemes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\sim} & Y' \\
\pi \downarrow & & \downarrow \pi' \\
X & \xrightarrow{\sim} & X'
\end{array}
$$

*compatible with all the data, where $\pi$, $\pi'$ are the natural projections.*

Let us define the Krichever map for $\underline{\mathcal{M}}^\infty(p, \mathrm{R})$ as the morphism of functors:

$$K \colon \underline{\mathcal{M}}^\infty(p, \mathrm{R}) \longrightarrow \mathrm{Gr}(V_{\mathrm{R}})^\bullet$$

such that to each $\{Y, \sigma_Y, y, t_y\} \in \underline{\mathcal{M}}^\infty(p, \mathrm{R})(S)$ one attaches the submodule:

$$K(Y, \sigma_Y, y, t_y) = t_y \left( \varinjlim_{n \geq 0} (p_Y)_* \mathcal{O}_Y(n \cdot y(S)) \right) \subset V_{\mathrm{R}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$$

**Theorem 5.3.** *The functor $\underline{\mathcal{M}}^\infty(p, \mathrm{R})$ is represented by a closed subscheme $\mathcal{M}^\infty(p, \mathrm{R})$ of $\mathrm{Gr}(V_{\mathrm{R}})$ that coincides with $\mathcal{M}^{\infty,1} \cap \mathrm{Gr}(V_{\mathrm{R}})^\sigma$.*

*Proof.* One has only to prove the inclusion:

$$(\mathcal{M}^{\infty,1})^\bullet(S) \cap (\mathrm{Gr}(V_{\mathrm{R}})^\sigma)^\bullet(S) \subseteq K\left( \underline{\mathcal{M}}^\infty(p, \mathrm{R})(S) \right)$$

for any $\mathbb{C}$-scheme $S$. If $U$ is a point of $(\mathcal{M}^{\infty,1})^\bullet(S) \cap (\mathrm{Gr}(V_{\mathrm{R}})^\sigma)^\bullet(S)$ representing a geometric datum $(Y, y, t_y)$, then the automorphism:

$$\sigma \otimes \mathrm{Id} \colon V_{\mathrm{R}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S \to V_{\mathrm{R}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$$

leaves the $\mathcal{O}_S$-module $U \subset V_{\mathrm{R}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$ stable.

Since these functors are sheaves, we can assume that $S = \mathrm{Spec}(B)$ is an affine scheme. Therefore, $U \subset B((z_1)) = V_{\mathrm{R}} \widehat{\otimes}_{\mathbb{C}} B$ is a sub-$B$-algebra endowed with the natural filtration induced by the degree of $z_1^{-1}$. The curve $Y$ can be reconstructed from $U$ as $Y = \mathrm{Proj}_B \mathcal{U}$, where $\mathcal{U}$ is the graded algebra associated with the above filtration (see [MP] Theorem 6.4, [Q]). Since the automorphism $\sigma \otimes \mathrm{Id}$ induces an automorphism of the $B$-algebra $U$ compatible with the filtration, then it induces an automorphism $\widetilde{\sigma} \colon \mathcal{U} \to \mathcal{U}$ of graded $B$-algebras. From this fact, one easily concludes that we can construct from $U$ a geometric datum $(Y, \sigma_Y, y, t_Y)$ defining a point of $K\left( \underline{\mathcal{M}}^\infty(p, \mathrm{R})(S) \right)$. $\qquad\square$

In order to be able to write down equations for $\mathcal{M}^\infty(p, \mathrm{R})$, note that the Euler-Poincaré characteristic of the curve must be fixed. Thus, let us denote by $\mathcal{M}_\chi^\infty(p, \mathrm{R})$ the subscheme of $\mathcal{M}^\infty(p, \mathrm{R})$ representing curves whose structure sheaf has Euler-Poincaré characteristic $\chi$. We have that $\mathcal{M}_\chi^\infty(p, \mathrm{R}) \subset \mathrm{Gr}^\chi(V_\mathrm{R})$.

Then, we can prove the following theorem:

**Theorem 5.4.** *The closed subscheme $\mathcal{M}_\chi^\infty(p, \mathrm{R})$ of $\mathrm{Gr}^\chi(V_\mathrm{R})$ is defined by the following set of equations:*

(1) $\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{1}{z_1} \psi_U(\xi z_1, \xi^{-1} t_1, \xi^{-2} t_2, \dots) \psi_U^*(z_1, t') \right) \frac{dz}{z} = 0.$

(2) $\mathrm{Res}_{z=0} \mathrm{Tr} \left( z_1^{\chi-2} \psi_U(z_1, t) \psi_U(z_1, t') \psi_U^*(z_1, t'') \right) \frac{dz}{z} = 0.$

(3) $\mathrm{Res}_{z=0} \mathrm{Tr} \left( \frac{1}{z_1^\chi} \psi_U^*(z_1, t) \right) \frac{dz}{z} = 0.$

*Proof.* Bearing in mind the expression for the Baker-Akhiezer function given in Proposition 3.4 and the characterization given in Theorem 3.3, one has that the equations defining the subscheme $\mathrm{Gr}^\chi(V_\mathrm{R})^\sigma$ are given by (1).

Recall that $U \in \mathrm{Gr}^\chi(V)$ lies in $\mathcal{M}^{\infty,1}$ if and only if $U \cdot U \subseteq U$ and $1 \in U$ ([MP], Theorem 6.4). Note that $z_1^{\chi-1} \psi_U(z_1, t)$ (resp. $\frac{1}{z_1^{p+\chi}} \psi_U^*(z_1, t)$) is a generating function of $U$ (resp. $U^\perp$). In terms of the pairing defined by formula 3.1, these two conditions read:

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( z_1^{2(\chi-1)} \psi_U(z_1, t) \psi_U(z_1, t'), \frac{1}{z_1^{p+\chi}} \psi_U^*(z_1, t'') \right) dz = 0$$

and

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left( 1, \frac{1}{z_1^{p+\chi}} \psi_U^*(z_1, t) \right) dz = 0$$

respectively. From the properties of the trace and since $z_1^p = z$, these equations are equivalent to (2) and (3). $\qquad \square$

### 5.B. The non-ramified case and the Krichever map

Study of the non-ramified case is quite similar to the previous one.

**Definition 5.5.** *The moduli functor $\underline{\mathcal{M}}^\infty(p, \mathrm{NR})$ of curves with an automorphism of order $p$ and formal parameters at the points of a non-ramified orbit is the contravariant functor on the category of $\mathbb{C}$-schemes:*

$$\underline{\mathcal{M}}^\infty(p, \mathrm{NR}) : \left\{ \begin{array}{c} \text{category of} \\ \mathbb{C} - \text{schemes} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \text{category} \\ \text{of sets} \end{array} \right\}$$

*that associates the set of equivalence classes of data $\{Y, \sigma_Y, \bar{y}, t_{\bar{y}}\}$ with a $\mathbb{C}$-scheme $S$, where:*

(1) $p_Y : Y \rightarrow S$ is a proper and flat morphism whose fibres are geometrically reduced curves.

(2) $\sigma_Y$ is a non-trivial automorphism of the curve $Y \rightarrow S$ of order $p$ such that the quotient $X := Y/ < \sigma_Y >$ exists and $p_X : X \rightarrow S$ is a proper and flat morphism whose fibres are geometrically integral curves.

(3) There exist $p$ disjoint smooth sections $y_j : S \rightarrow Y (j = 1, \ldots , p)$ of $p_Y$ which are Cartier divisors of $Y$ and such that $\bar{y} = \{y_1, \ldots , y_p\}$ is an orbit under the action of $< \sigma_Y >$.

(4) For each closed point $s \in S$, $y_j(s)$ is a smooth point of the fibre $p_Y^{-1}(s) = Y_s$ such that each irreducible component of $Y_s$ contains at least a point $y_j(s)$ for a certain $j = 1, \ldots , p$.

(5) There exists formal parameters $t_{y_j}$ along the sections $y_j$ such that they induce an isomorphism:

$$t_{\bar{y}} : \widehat{\mathcal{O}}_{Y,\bar{y}(S)} \xrightarrow{\sim} V_{\mathrm{NR}}^+ \widehat{\otimes} \mathcal{O}_S$$

which is compatible with respect to the actions of $\sigma_Y$ and $\sigma$.

(6) $\{Y, \sigma_Y, \bar{y}, t_{\bar{y}}\}$ and $\{Y', \sigma_Y', \bar{y}', t_{\bar{y}'}\}$ are said to be equivalent when there is a commutative diagram of $S$-schemes:

$$
\begin{array}{ccc}
Y & \xrightarrow{\sim} & Y' \\
\pi \downarrow & & \downarrow \pi' \\
X & \xrightarrow{\sim} & X'
\end{array}
$$

compatible with all the data where $\pi, \pi'$ are the natural projections.

We define the Krichever map for $\underline{\mathcal{M}}^\infty(p, \mathrm{NR})$ as the morphism of functors:

$$K : \underline{\mathcal{M}}^\infty(p, \mathrm{NR}) \longrightarrow \mathrm{Gr}(V_{\mathrm{NR}})^\bullet$$

such that for each $\{Y, \sigma_Y, \bar{y}, t_{\bar{y}}\} \in \underline{\mathcal{M}}^\infty(p, \mathrm{NR})(S)$ one has:

$$K(Y, \sigma_Y, \bar{y}, t_{\bar{y}}) = t_{\bar{y}} \left( \varinjlim_{n \geq 0} (p_Y)_* \mathcal{O}_Y(n \cdot \bar{y}(S)) \right) \subset V_{\mathrm{NR}} \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$$

**Theorem 5.6.** The functor $\underline{\mathcal{M}}^\infty(p, \mathrm{NR})$ is represented by the closed subscheme $\mathcal{M}^{\infty,p} \cap \mathrm{Gr}(V_{\mathrm{NR}})^\sigma = \mathcal{M}^\infty(p, \mathrm{NR})$ of $\mathrm{Gr}(V_{\mathrm{NR}})$.

*Proof.* This is similar to the proof of Theorem 5.3.                                  □

It is worth pointing out that in our definition of $\mathcal{M}^\infty(p, \mathrm{NR})$, non-connected curves have been considered. Since $p$ is a prime number, we have two posibilities: either the curve $Y$ is connected or has $p$ connected components such that each of them is isomorphic to the quotient curve $X$.

Our aim is to write down equations for $\mathcal{M}^\infty(p, \mathrm{NR})$. As in the previous case, we must restrict ourselves to a given Euler-Poincaré characteristic. Thus,

let $\mathcal{M}_\chi^\infty(p, \mathrm{NR})$ denote the subscheme of $\mathcal{M}^\infty(p, \mathrm{NR})$ representing curves such that the Euler-Poincaré characteristic of the structure sheaf is $\chi$.

If the Euler-Poincaré characteristic is positive, then there are only two cases; namely, $\chi = 1$ and $\chi = p$. The first case corresponds to the projective line endowed with an automorphism of order $p$. The second one consists of the disjoint union of $p$ copies of the projective line where the automorphism is given by a cyclic permutation of the copies. Therefore, the non-trivial cases of curves with automorphisms appear when $\chi \leq 0$.

**Theorem 5.7.** *Let $\chi \leq 0$. The subscheme $\mathcal{M}_\chi^\infty(p, \mathrm{NR})$ of $\mathrm{Gr}^\chi(V_{\mathrm{NR}})$ is defined by the following set of equations:*

(1) $\sum_{i=1}^p \mathrm{Res}_{z=0} \, \psi_{\sigma(U)}^{(i)}(z, t) \cdot \psi_U^{*(i)}(z, t') \frac{dz}{z^2} = 0$, *or equivalently:*

$$\mathrm{Res}_{z=0} \, \psi_U^{(p)}(z, t) \cdot \psi_U^{*(1)}(z, t') \frac{dz}{z^2} + \cdots + \mathrm{Res}_{z=0} \, \psi_U^{(1)}(z, t) \cdot \psi_U^{*(p)}(z, t') \frac{dz}{z^2} = 0$$

(2) $\sum_{i=1}^r \mathrm{Res}_{z=0} \left( \psi_U^{(i)}(z, t) \psi_U^{(i)}(z, t') \psi_U^{*(i)}(z, t'') \right) \frac{dz}{z^{q+4}}$

$\qquad + \sum_{i=r+1}^p \mathrm{Res}_{z=0} \left( \psi_U^{(i)}(z, t) \psi_U^{(i)}(z, t') \psi_U^{*(i)}(z, t'') \right) \frac{dz}{z^{q+3}} = 0$

(3) $\sum_{i=1}^r \mathrm{Res}_{z=0} \, \psi_U^{*(i)}(z, t) z^q dz + \sum_{i=r+1}^p \mathrm{Res}_{z=0} \, \psi_U^{*(i)}(z, t) z^{q-1} dz = 0$

*where $q, r$ are given by $-\chi = q \cdot p + r$ with $0 \leq r < p$.*

*Proof.* Recalling Proposition 3.4 and Theorem 3.3, one sees that the equations defining the subscheme $\mathrm{Gr}^\chi(V_{\mathrm{NR}})^\sigma$ are given by (1).

It remains to compute the equations of $\mathcal{M}_\chi^{\infty,p}$. These equations are derived from the known fact that a point $U \in \mathrm{Gr}^\chi(V_{\mathrm{NR}})$ lies in $\mathcal{M}_\chi^{\infty,p}$ if and only if it is a $\mathbb{C}$-algebra with unity.

Since $\frac{v_\chi}{z_\cdot} \psi_U(z_\cdot, t)$ (resp. $\frac{v_{-\chi}}{z_\cdot} \psi_U(z_\cdot, t)$) is a generating function of $U$ (resp. $U^\perp$), the point $U$ is an $\mathbb{C}$-algebra with unity precisely when:

$$\begin{cases} \mathrm{Res}_{z=0} \, \mathrm{Tr} \left( \frac{v_\chi}{z_\cdot} \psi_U(z_\cdot, t) \cdot \frac{v_\chi}{z_\cdot} \psi_U(z_\cdot, t') \cdot \frac{v_{-\chi}}{z_\cdot} \psi_U^*(z_\cdot, t'') \right) dz = 0 \\ \mathrm{Res}_{z=0} \, \mathrm{Tr} \left( \frac{v_{-\chi}}{z_\cdot} \psi_U^*(z_\cdot, t) \right) dz = 0 \end{cases}$$

If $-\chi = qp + r$ with $0 \leq r < p$, then $v_{-\chi} = z_1^{q+1} \ldots z_r^{q+1} z_{r+1}^q \ldots z_p^q$ and $v_\chi := v_{-\chi}^{-1}$. Recalling that $z_\cdot = (z_1, \ldots, z_p)$ and that:

$$\mathrm{Tr} \left( (f_1(z_1), \ldots, f_p(z_p)) \right) = \sum_{i=1}^p f_i(z)$$

one concludes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

The points associated with non-connected curves define a closed subscheme of $\mathcal{M}_\chi^\infty(p, \mathrm{NR})$ (by [MP2], Proposition 3.20). Let $\mathcal{M}_\chi^\infty(p, \mathrm{NR})^0$ denote the open subscheme parametrizing points such that the corresponding curve is connected.

Bearing in mind that the Euler-Poincaré characteristic of a non-connected curve with an order $p$ automorphism such that the quotient curve is connected is a multiple of $p$, it follows that $\mathcal{M}_\chi^\infty(p, \mathrm{NR})^0$ coincides with $\mathcal{M}_\chi^\infty(p, \mathrm{NR})$ provided that $\chi \neq \dot{p}$.

Let us characterize $\mathcal{M}_\chi^\infty(p, \mathrm{NR})^0$ for $\chi = \dot{p}$ and $\chi \leq 0$. Consider the elements:

$$e_i := (0, \dots, \overset{(i)}{1}, \dots, 0) \in V_{\mathrm{NR}} = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))$$

Then, a point $U \in \mathcal{M}_\chi^\infty(p, \mathrm{NR})$ is associated with a non-connected curve if and only if $e_i \in U$ for some $i \in \{1, \dots, p\}$. This condition is equivalent to saying that one term of the third sum of the statement of Theorem 5.7 does vanish; that is:

$$\mathrm{Res}_{z=0} \, \psi_U^{*(i)}(z, t) z^q dz = 0 \qquad \text{for some } i \in \{1, \dots, p\}$$

(observe that $r = 0$ since $\chi = \dot{p}$).

Since $\sigma$ acts transitively on the subset $\{e_1, \dots, e_p\}$ and $U$ is invariant under $\sigma$, it follows that if $e_i \in U$ for some $i$, then $e_i \in U$ for all $i \in \{1, \dots, p\}$. Therefore, we deduce the following corollary:

**Corollary 5.8.** *The open subscheme $\mathcal{M}_\chi^\infty(p, \mathrm{NR})^0$ of $\mathcal{M}_\chi^\infty(p, \mathrm{NR})$ is defined by the condition:*

$$\mathrm{Res}_{z=0} \, \psi_U^{*(i)}(z, t) z^q dz \neq 0 \qquad \text{for some } i \in \{1, \dots, p\}$$

*for $-\chi = q \cdot p \geq 0$, and $\mathcal{M}_\chi^\infty(p, \mathrm{NR})^0 = \mathcal{M}_\chi^\infty(p, \mathrm{NR})$ for $\chi \neq \dot{p}$.*

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