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PRYM VARIETIES AND THE INFINITE GRASSMANNIAN

FRANCISCO J. PLAZA MARTÍN

DEPARTAMENTO DE MATEMÁTICA PURA Y APLICADA
UNIVERSIDAD DE SALAMANCA

Abstract. In this paper we study Prym varieties and their moduli space using the well known techniques of the infinite Grassmannian. There are three main results of this paper: a new definition of the BKP hierarchy over an arbitrary base field (that generalizes the classical one over $\mathbb{C}$); a characterization of Prym varieties in terms of dynamical systems, and explicit equations for the moduli space of (certain) Prym varieties. For all of these problems the language of the infinite Grassmannian, in its algebro-geometric version, allows us to deal with these problems from the same point of view.

Contents

1. Introduction

The aim of this paper is to generalize some results concerning the BKP hierarchy and geometric characterizations of Jacobians and Pryms proved in [LM, M1, S, S2] and to study the moduli space of (certain) Prym varieties following similar ideas to those of [MP]. I should remark that the techniques employed here are those of algebraic geometry, and

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most statements are therefore valid over an arbitrary base field. The organization of the paper is as follows:

In §2 some basic definitions, results and tools needed for the next sections are introduced. Some of them are known (e.g. infinite Grassmannian, Krichever functor, Γ group, etc) and have their origin in the study of the moduli space of Riemann surfaces, Jacobian varieties and conformal field theory ([BS, DKJM, KSU, M1, N, PS, SW] . To develop analogues for the theory of Prym varieties, we will define certain subschemes of the infinite Grassmannian and will avoid the introduction of the formalism of the n-component KP hierarchy. However, the analogue of the determinant bundle is a non-trivial problem. An important result (2.5) of this section is the existence of a square root of the determinant bundle (over a certain subscheme), which will be called Pfaffian, as well as an explicit construction of global sections of this Pfaffian bundle, improving previous results of [B, PS].

Section §3 contains a new definition of the BKP hierarchy as the defining equations of a suitable subscheme of the infinite Grassmannian (3.5); namely, the space of maximal totally isotropic subspaces. This new definition is quite natural since we should recall that the KP hierarchy is in fact equivalent to the Plücker equations, which are the defining equations of the infinite Grassmannian ([SS]). The definition is therefore valid over an arbitrary base field, but is shown to be equivalent to the classical one when the base field is $\mathbb{C}$. We wish to point out that this new definition of the BKP hierarchy, together with the previously defined Pfaffian line bundle, will give a justification for some results relating solutions of the KP and BKP hierarchies ([DKJM, DKJM2]) as well as for techniques involving pfaffians when computing τ-functions for the BKP hierarchy ([H, O]). Note that this (algebraic) approach allows us to introduce the BKP hierarchy with no mention of pseduodifferential operators.

The standard way to relate the study of Jacobians to that of infinite Grassmannian is the Krichever functor ([M2]). Two remarkable papers dealing with the case of Pryms are [LM, S2]. The characterization given in [LM] for a point of (a certain quotient of) the infinite Grassmannian to be associated to a Prym (via the Krichever functor) is that its orbit (under a suitable group) is finite dimensional (see Theorem 5.14 of [LM] for the precise statement). Moreover, that action is interpreted as a dynamical system on that space. The idea of Shiota ([S, S2]), related to the above one, is to characterize Jacobian varieties as compact solutions of the KP hierarchy through the (analytic) study of infinitesimal deformations of sheaves.

The characterizations of Jacobians (4.2) and Pryms (4.6) given in §4 profit from both approaches. Recalling that the action of the formal Jacobian (formal group Γ, §2.E) on the Grassmannian is algebraic and studying the structure of the orbits (4.3), one can give an algebraic statement generalizing Theorem 6 of [S] and Theorem 5.14 of [LM]. But in our statement no quotient space is needed. Bearing this in mind, there is no problem in extending this result to the case of Pryms (4.6). Nevertheless, we shall use the language of dynamical systems to express both characterizations.

In the last section (§5), and following the spirit of [MP], explicit equations for the moduli space of Pryms are given; first as Bilinear Identities (5.4), then as partial differential equations (5.6) when $char(k) = 0$. This set of equations should not be confused with the BKP hierarchy. Here we make two considerations; firstly, that a formal trivilization is attached to each datum (e.g. a curve, a bundle, ...) since we wish to work in a uniform frame (e.g. $\mathrm{Gr}(k((z)))$, Γ, ...) and, secondly, that not all Pryms are considered; only those coming from an integral curve together with an involution with at least one fixed smooth point are taken into account since, for technical reasons, the involution should correspond to an automorphism of $k((z))$ preserving $k[[z]]$.

We address the reader to [AMP] for a detailed discussion of the infinite Grassmannian and to [Mu] for the basic facts on Prym varieties needed.

2. Preliminaries on the Infinite Grassmannian

2.A. **Basic Facts.** Recall ([AMP, BS]) that given a pair (V, V_+) consisting of a k-vector space[a] and a subspace of it, there exists a scheme $\mathrm{Gr}(V, V_+)$ over $\mathrm{Spec}(k)$ whose rational points are:

$$\left\{ \begin{array}{c} \text{subspaces } L \subseteq \hat{V}, \text{ such that } L \cap \hat{V}_+ \\ \text{and } \hat{V}/L + \hat{V}_+ \text{ are of finite dimension} \end{array} \right\}$$

where $\hat{}$ denotes the completion with respect to the topology given by the subspaces that are commensurable with V_+. The points of $\mathrm{Gr}(V, V_+)$ will be called discrete subspaces. The essential fact for its existence is that there is a covering by open subfunctors F_A (where $A \sim V_+$ are commensurable) representing the functor $\underline{\mathrm{Hom}}(L_A, \hat{A})$, where L_A is a rational point of $\mathrm{Gr}(V, V_+)$ such that $L_A \oplus \hat{A} \simeq \hat{V}$ ([BS]). (See [AMP] for the definition of the functor of points of $\mathrm{Gr}(V, V_+)$). Let us denote this infinite Grassmannian simply by $\mathrm{Gr}(V)$, and let $\mathcal{L}$ be the universal discrete submodule of $\hat{V}_{\mathrm{Gr}(V)}$. We will assume that V

[a] For simplicity's sake we will assume that k is an algebraically closed field

is complete with respect to the V_+-topology. From this construction it is easily deduced that $\mathrm{Gr}(V)$ is locally integral and separated.

The connected components of $\mathrm{Gr}(V)$ are given by the Euler characteristic (index) of the complex:

$$\mathcal{L} \to (\hat{V}/\hat{V}_+)_{\mathrm{Gr}(V)} \tag{2.1}$$

and that of index n will be denoted by $\mathrm{Gr}^n(V, V_+)$. It is also shown, that $\mathrm{Gr}(V)$ carries a line bundle, Det_V, given by the determinant of the complex 2.1 (see [KM] for a general theory of determinants) whose stalk at a rational point L is:

$$\overset{max}{\wedge}(L \cap \hat{V}_+) \otimes \overset{max}{\wedge}(\hat{V}/(L + \hat{V}_+))^*$$

This line bundle has no global sections but its dual does. Moreover, for each $A \sim V_+$ one can define a global section, Ω_A of Det_V^* such that it vanishes outside F_A. For every subspace Ω of $H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*)$ one has a sheaf homomorphism:

$$\Omega \otimes_k \mathcal{O}_{\mathrm{Gr}(V)} \to \mathrm{Det}_V^*$$

If it is surjective (Ω is “big enough”), it induces a scheme homomorphism:

$$\mathfrak{p}_V : \mathrm{Gr}^0(V) \to \check{\mathbb{P}}\Omega^* \overset{\text{def}}{=} \mathrm{Proj}\, S^\bullet\Omega$$

which is known as the Plücker morphism.

Although $k((z))$ is not complete with respect to the $k[[z]]$-topology, it is easy to see that the functor:

$$S \rightsquigarrow \{L \in \mathrm{Gr}(k((z)), k[[z]])(S) \mid L \subseteq \mathcal{O}_S((z))\} \tag{2.2}$$

is locally closed, and it is therefore representable by a closed subscheme, which will be denoted by $\mathrm{Gr}(k((z)))$ again, when no confussion arises ([P]). It can now be shown that $H^0(\mathrm{Gr}^0(k((z))), \mathrm{Det}_V^*)$ has a dense subspace, Ω, consisting of sections of the kind Ω_A and labelled by Young diagrams (see [P, SS]), and that the Plücker morphism is a closed immersion. The general results given along this section are valid for subscheme 2.2.

2.B. **Important subschemes of the Infinite Grassmannian.** Let us now introduce two closed subschemes of $\mathrm{Gr}(V)$ that will be useful in the next sections. Assume now that an automorphism σ of V (as k-vector space) is given, and that $\sigma(\hat{V}_+) = \hat{V}_+$; it then induces an automorphism of the scheme $\mathrm{Gr}(V)$. Since $\mathrm{Gr}(V)$ is separated, one has that:

$$\mathrm{Gr}_\sigma(V) = \{L \in \mathrm{Gr}(V) \,|\, \sigma(L) = L\}$$

is a closed subscheme of $\mathrm{Gr}(V)$.

For the second, recall the isomorphism $\mathrm{Gr}(V, V_+) \xrightarrow{\sim} \mathrm{Gr}(V^*, V_+^\circ)$ given by incidence (see [P]); that is, it sends a discrete subspace L to L°, the space of continuous linear forms that vanish on L. Let $p : V \xrightarrow{\sim} V^*$ be the isomorphism of V with its dual vector space V^* induced by an irreducible hemisymmetric metric on V. Assume further that $p(\hat{V}_+) = \hat{V}_+^\circ$. It then induces an isomorphism $\mathrm{Gr}(V^*, V_+^\circ) \xrightarrow{\sim} \mathrm{Gr}(V, V_+)$, which composed with the one given by incidence, gives rise to the following automorphism of $\mathrm{Gr}(V)$:

$$\begin{aligned} R : \mathrm{Gr}(V) &\longrightarrow \mathrm{Gr}(V) \\ L &\longmapsto L^\perp \end{aligned}$$

(where $\perp$ denotes the orthogonal with respect to the metric).

Straightforward calculation shows that $R^* \mathrm{Det}_V \simeq \mathrm{Det}_V$, and that the index of a point $L \in \mathrm{Gr}(V)$ is exactly the opposite of the index of $R(L) = L^\perp$.

Given $\sigma \in \mathrm{Aut}_{k-alg}\, k((z))$ such that $\sigma^2 = Id$, consider the following irreducible hemisymmetric metric:

$$\begin{aligned} &V \times V \to k \\ &(f, g) \mapsto \mathrm{Res}_{z=0}\, f(z) \cdot (\sigma^* g(z)) dz \end{aligned} \tag{2.3}$$

It is now clear that there exists a closed subscheme $\mathrm{Gr}_\sigma^I(V)$ of $\mathrm{Gr}^0(V)$ such that:

$$\mathrm{Gr}_\sigma^I(V)^\bullet(k) = \left\{ L \in \mathrm{Gr}(V) \,|\, \begin{array}{c} L \text{ is maximal totally isotropic} \\ \text{with respect to the metric 2.3} \end{array} \right\} \tag{2.4}$$

From now on, a subspace of V will be called m.t.i. when it is maximal totally isotropic (compare with §2.2 of [S2]).

Remark 1. Whenever we use a hemisymmetric metric or consider the automorphism of $k((z))$ induced by $z \mapsto -z$, is assumed $char(k) \neq 2$.

Example 1. This is a fundamental example because it will be the situation when studying Prym varieties in terms of $\mathrm{Gr}_\sigma^I(V)$. Let $V = k((z))$, $V_+ = k[[z]]$ and σ_0 that given by $z \mapsto -z$. The metric is now: $< f(z), g(z) >= \mathrm{Res}_{z=0}\, f(z)g(-z)dz$. It is then easy to prove that:

$$\mathrm{Gr}_0(k((z)), k[[z]]) \simeq \mathrm{Gr}(k((z^2)), k[[z^2]]) \underset{\mathrm{Spec}(k)}{\times} \mathrm{Gr}(z \cdot k((z^2)), z \cdot k[[z^2]])$$

(we simply write 0 instead of σ_0). It is worth comparing the 2 component BKP hierarchy (as given in §3 of [S2]) with $\mathrm{Gr}_0(k((z)), k[[z]])$.

2.C. **Pffafian Line Bundle.**

Theorem 2.5. *There exists a line bundle,* Pf *(called Pfaffian), over* $\mathrm{Gr}^I_\sigma(V)$ *such that:*

$$\mathrm{Pf}^{\otimes 2} \simeq \mathrm{Det}^*_V |_{\mathrm{Gr}^I_\sigma(V)}$$

Proof. Note that Det^*_V is isomorphic to the determinant of the dual of the complex over $\mathrm{Gr}^I_\sigma(V)$: $\hat{V}_+ \to \hat{V}/\mathcal{L}$ ($\mathcal{L}$ being the universal m.t.i. submodule). Let us compute its cohomology. Let $p : V \to V^*$ be the isomorphism induced by the metric 2.3. Then, in the commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathcal{L} & \longrightarrow & \hat{V}/\hat{V}_+ & \longrightarrow & \frac{\hat{V}}{\hat{V}_+ + \mathcal{L}} & \longrightarrow & 0 \\
 & & {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \big\downarrow & & \\
0 \to (\frac{\hat{V}}{\hat{V}_+ + \mathcal{L}})^* & \longrightarrow & (\hat{V}/\mathcal{L})^* & \longrightarrow & \hat{V}_+^* & \longrightarrow & \frac{\hat{V}_+^*}{(\hat{V}/\mathcal{L})^*} & \longrightarrow & 0
\end{array}$$

the two middle vertical arrows are isomorphisms, since $\mathcal{L}$ and $\hat{V}_+$ are m.t.i. ; and thus the right one is an isomorphism.

One now has:

$$\mathrm{Det}^*_V \xrightarrow{\sim} \Big(\bigwedge \hat{V}/(\hat{V}_+ + \mathcal{L})\Big)^{*\otimes 2}$$

By the local structure of $\mathrm{Gr}^I_\sigma(V)$ it is not difficult to show that $\hat{V}/(\hat{V}_+ + \mathcal{L}$ is locally free of finite type, and by [KM] it makes sense to define:

$$\mathrm{Pf} \overset{\text{def}}{=} (\bigwedge \hat{V}/(\hat{V}_+ + \mathcal{L}))^*$$

□

An analytic construction of the Pfaffian line bundle can be found in [B], but nevertheless we prefer to continue with the algebraic machinery. Another construction is given in [PS].

We are now interested in building sections of this line bundle. First, observe that the covering $\{F_A\}$ ($A \sim V_+$) of $\mathrm{Gr}(V)$ induces another one of $\mathrm{Gr}^I_\sigma(V)$ by open subsets of the form $\{\bar{F}_A = F_A \cap \mathrm{Gr}^I_\sigma(V)\}$ where $F_A \subset \mathrm{Gr}(V)$ ($A \sim V_+$) and A is m.t.i. . The second ingredient is the following:

Lemma 2.6. *Let A and B be two m.t.i. subspaces of V. One then has a canonical isomorphism:*

$$B/(B \cap A) \xrightarrow{\sim} (A/(B \cap A))^*$$

induced by the metric.

Proof. Note that the morphism $p : V \to V^*$ gives an isomorphism from A (resp. B) to A° (resp. B°). Therefore, from the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A\cap B & \longrightarrow & A & \longrightarrow & A/(A\cap B) & \longrightarrow & 0 \\ & & {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & B^\circ & \longrightarrow & V^* & \longrightarrow & B^* & \longrightarrow & 0 \end{array}$$

one has an injection $A/(A\cap B) \to B^*$. The linear forms belonging to the image vanish on $A\cap B$, and hence:

$$A/(A\cap B) \to (A\cap B)^\circ \cap B^* \simeq (B/(A\cap B))^*$$

Analogously, one obtains another injection $B/(A\cap B) \to (A/(A\cap B))^*$, and they are transposed of each other, and are therefore both isomorphisms. □

Theorem 2.7. *To each $A \sim V_+$ m.t.i. one associates a section $\bar{\Omega}_A$ of* Pf *that vanishes outside $\bar{F}_A$, and hence:*

$$\Omega_A|_{\mathrm{Gr}^I_\sigma} = \lambda \cdot \bar{\Omega}_A^2 \qquad \lambda \in k^*$$

Proof. Observe now that by similar arguments to those of [AMP] one can construct sections of Pf for each $A \sim V_+$ m.t.i. . The only remarkable aspect is that this is possible because $B/(B\cap A)$ and $(A/(B\cap A))^*$ are canonically isomorphic (for A, B m.t.i. and commensurable with V_+).

From this last property, and from the fact $\mathrm{Pf}^{\otimes 2} \simeq \mathrm{Det}^*_V$, the claim follows. □

2.D. **Krichever Functor.**

Definition 2.8. *Define the moduli functor of pointed curves $\mathcal{M}_\infty$ over the category of k-schemes by the sheafication of:*

$$S \rightsquigarrow \{\text{ families } (C, D, z) \text{ over } S\ \}/\text{equivalence}$$

where these families satisfy:

1. *$\pi : C \to S$ is a proper flat morphism, whose geometric fibres are integral curves,*
2. *$s : S \to C$ is a section of π, such that when considered as a Cartier Divisor D over C it is smooth and of relative degree 1, and flat over S. (We understand that $D \subset C$ is smooth over S, iff for every closed point $x \in D$ there exists an open neighborhood U of x in C such that the morphism $U \to S$ is smooth).*

3. *z is a formal trivialization of C along D; that is, a family of epimorphisms of rings:*

$$\mathcal{O}_C \longrightarrow s_* \left(\mathcal{O}_S[t]/t^m\, \mathcal{O}_S[t]\right) \qquad m \in \mathbb{N}$$

compatible with respect to the canonical projections:

$$\mathcal{O}_S[t]/t^m\, \mathcal{O}_S[t] \to \mathcal{O}_S[t]/t^{m'}\, \mathcal{O}_S[t] \qquad m \geq m'$$

and such that that corresponding to $m = 1$ equals s.

and the families (C, D, z) and (C', D', z') are said to be equivalent, if there exists an isomorphism $C \to C'$ (over S) such that the first family goes to the second under the naturally induced morphisms.

By [MP] it is known that the so called "Krichever map" is in fact the following morphism of functors:

$$\begin{aligned} K : \mathcal{M}_\infty &\longrightarrow \operatorname{Gr}(k((z)), k[[z]]) \\ (C, D, z) &\longmapsto \varinjlim_n \pi_* \mathcal{O}_C(n) \end{aligned}$$

It is also known that K is an immersion and that there exists a locally closed subscheme of $\operatorname{Gr}(V)$ representing $\mathcal{M}_\infty$ (which we will denote again by $\mathcal{M}_\infty$).

Let us recall another construction very similar to the above one. Set $m = (C, p, z) \in \mathcal{M}_\infty(\operatorname{Spec}(k))$, and consider the functor:

$$S \rightsquigarrow \widetilde{\operatorname{Pic}}(C, p) = \left\{ (L, \phi) \mid \begin{array}{c} L \in \operatorname{Pic}(C)^\bullet(S) \text{ and } \phi \text{ is a} \\ \text{formal trivialization of } L \text{ around } p \end{array} \right\}$$

Define the morphism:

$$\begin{aligned} K_m : \widetilde{\operatorname{Pic}}(C, p) &\longrightarrow \operatorname{Gr}(k((z)), k[[z]]) \\ (L, \phi) &\longmapsto \varinjlim_n \pi_* L(n) \end{aligned}$$

which is also usually called "Krichever functor". (For a more detailed study of $\widetilde{\operatorname{Pic}}(C, p)$ and K_m see [Al]).

Note that while K is very well adapted to study of the moduli space of curves ([MP, N]), the other one, K_m, is good for the study of Jacobian varieties and their subvarieties ([S]). See also [M2].

2.E. **The Formal Group Γ.** Let us now recall some basic facts about the formal group Γ (for a complete study and definitions see [AMP]). Γ is defined as the formal group scheme $\Gamma_- \times \mathbb{G}_m \times \Gamma_+$ over $\operatorname{Spec}(k)$, where Γ_- is the formal scheme representing the functor on groups:

$$S \rightsquigarrow \Gamma_-(S) = \left\{ \begin{array}{c} \text{series } \ a_n\, z^{-n} + \dots + a_1\, z^{-1} + 1 \\ \text{where } a_i \in H^0(S, \mathcal{O}_S) \text{ are} \\ \text{nilpotents and } n \text{ is arbitrary} \end{array} \right\}$$

$\mathbb{G}_m$ is the multiplicative group, and the scheme Γ_+ represents:

$$S \rightsquigarrow \Gamma_+(S) = \left\{ \begin{array}{c} \text{series } 1 + a_1\, z + a_2\, z^2 + \dots \\ \text{where } a_i \in H^0(S, \mathcal{O}_S) \end{array} \right\}$$

The group laws of Γ_- and Γ_+ are those induced by the multiplication of series. Note also that there exists a natural inclusion of Γ in the identity connected component of:

$$S \rightsquigarrow H^0(S, \mathcal{O}_S)((z))^* \overset{\text{def}}{=} H^0(S, \mathcal{O}_S)[[z]][z^{-1}]^*$$

which is an isomorphism when $char(k) = 0$.

Further, Γ_- is the inductive limit of the schemes:

$$S \rightsquigarrow \Gamma_-^n(S) = \left\{ \begin{array}{c} \text{series } \ a_n\, z^{-n} + \dots + a_1\, z^{-1} + 1 \\ \text{where } a_i \in H^0(S, \mathcal{O}_S) \text{ and the } n^{\text{th}} \\ \text{power of the ideal } (a_1, \dots, a_n) \text{ is zero} \end{array} \right\}$$

in the category of formal schemes.

Observe now that there exist two actions of $g(z) \in \Gamma$ in V; namely, the one given by homotheties:

$$\begin{aligned} H_g : V &\to V \\ h(z) &\mapsto g(z) \cdot h(z) \end{aligned}$$

and the one defined by the automorphism of k-algebras:

$$\begin{aligned} U_g : V &\to V \\ z &\mapsto z \cdot g(z) \end{aligned}$$

Remark 2. It is known that $U : g \mapsto U_g$ establishes a bijection $k[[z]]^* = \Gamma_+(k) \xrightarrow{\sim} \operatorname{Aut}_{k\text{-alg}}(k((z)))$. Recall from [Bo] (chapter III, §4.4) that given a k-algebra automorphism σ of $k((z))$, there exists a unique $g(z) \in \Gamma_+(k)$ such that $U_g \circ \sigma = \sigma_0$ (where σ_0 is the k-algebra automorphism of $k((z))$ given by $z \mapsto -z$; that is, it is possible to "normalize" σ such that $\sigma^*(g(z)) = g(-z)$.

Remark 3. Now, $g \in \Gamma_+$ acts on $\widetilde{\operatorname{Pic}}(C, p)$ sending (L, ϕ) to $(L, H_g \circ \phi)$. And hence the projection morphism:

$$\widetilde{\operatorname{Pic}}(C, p) \xrightarrow{p_1} \operatorname{Pic}(C)$$

may be interpreted as a principal bundle of group Γ_+. Now comparing the zero locus of sections of Det_V^* and $\mathcal{O}(\Theta)$, one deduces:

$$\operatorname{Det}_V^* |_{\widetilde{\operatorname{Pic}}(C,p)} \overset{\sim}{\longrightarrow} p_1^*\mathcal{O}_{\operatorname{Pic}(C)}(\Theta)$$

which allows one to write the τ-function of the point $U = K(C, p, z)$ (restricted to $\widetilde{\operatorname{Pic}}(C, p)$ via K) in terms of the theta function of the Jacobian of C.

For explicit formulas relating τ-functions and theta functions of Riemann surfaces, see [K, S] (see also [S2] for the case of Pryms).

Remark 4. For other constructions and properties of the group Γ see [AMP, C, KSU, PS, SW].

3. Formal Prym variety and BKP hierarchy

Observe now that Γ, and hence all the above-mentioned subgroups, acts on $\operatorname{Gr}(V)$ by homotheties and that on the set of rational points $\mathbb{G}_m$ acts trivially, and Γ_+ freely. Recall from [AMP, KSU] that Γ behaves like the Jacobian of the formal curve. Our goal is then to achieve an anologous result for the case of Pryms. This arises from the answer of the following question: which is the maximal subgroup of Γ acting on $\operatorname{Gr}^I_\sigma(V)$?.

Theorem 3.1. *The maximal subgroup of* Γ *acting on* $\operatorname{Gr}^I_\sigma(V)$ *is:*

$$\Pi_\sigma \;=\; \{g(z) \in \Gamma \,|\, g(z)\cdot \sigma^* g(z) = 1\,\}$$

which is a subscheme of Γ.

Proof. Observe that the homothety by $g(z) \in \Gamma$ restricts to a automorphism of $\operatorname{Gr}^I_\sigma(V)$ if and only if:

$$g(z)\cdot U \in \operatorname{Gr}^I_\sigma(V) \qquad \text{for all } U \in \operatorname{Gr}^I_\sigma(V)$$

or, what amounts to the same:

$$g(z)\cdot U \;=\; (g(z)\cdot U)^\perp \qquad \text{for all } U \in \operatorname{Gr}^I_\sigma(V)$$

Recalling the definition of the metric: $(f, g) \mapsto \operatorname{Res}_{z=0} f(z)\cdot\sigma^* g(z)dz$, one has:

$$(g(z)\cdot U)^\perp \;=\; (\sigma^* g(z))^{-1}\cdot U^\perp$$

Note that $U = U^\perp$, since $U \in \operatorname{Gr}^I_\sigma(V)$. And one concludes that $g(z)\cdot \sigma^* g(z)\cdot U = U$ for all $U \in \operatorname{Gr}^I_\sigma(V)$ and hence $g(z)\cdot\sigma^* g(z)$ must be equal to 1. □

Definition 3.2. *The formal Prym variety is the formal group scheme:*

$$\Pi^\sigma_- \overset{\text{def}}{=} \Pi_\sigma \cap \Gamma_-$$

It is therefore natural to consider Π^σ_- instead of Γ_- in the study of $\mathrm{Gr}^I_\sigma(V)$, and hence in the study of Pryms. Recall that the action of Γ_- on $\mathrm{Gr}(V)$ is essential in the definition of the τ-function and the Baker-Akhiezer function of a point $U \in \mathrm{Gr}(V)$. But for $\mathrm{Gr}^I_\sigma(V)$ we must restrict this action to Π^σ_-. Denote by μ^σ_U the restriction of μ_U to Π^σ_-; that is:

$$\begin{array}{ccccc} \mu_U : \Gamma_- \times \{U\} & \hookrightarrow & \Gamma_- \times \mathrm{Gr}(V) & \to & \mathrm{Gr}(V) \\ \cup & & \cup & & \cup \\ \mu^\sigma_U : \Pi^\sigma_- \times \{U\} & \hookrightarrow & \Pi^\sigma_- \times \mathrm{Gr}^I_\sigma(V) & \to & \mathrm{Gr}^I_\sigma(V) \end{array}$$

These actions are the cornerstone of §4, where they will be studied at the tangent space level.

Definition 3.3. *The $\bar\tau$-function of a point $U \in \mathrm{Gr}^I_\sigma(V)$ is the section $(\mu^\sigma_U)^*\bar\Omega_+$ of $(\mu^\sigma_U)^*\,\mathrm{Pf}$.*

Theorem 3.4.

$$\tau_U|_{\Pi^\sigma_-} = \lambda \cdot \bar\tau_U^2 \qquad \lambda \in k^*$$

It is known that the KP hierarchy is a system of partial differential equations for the τ-function of a point $U \in \check{\mathbb{P}}\Omega$. These are in fact equivalent to the Plücker equations for the coordinates of U. It is thus quite natural to give the following:

Definition 3.5.

- *char*(k) arbitrary: *The BKP hierarchy is the set of algebraic equations defining $\mathrm{Gr}^I_\sigma(V)$ inside $\check{\mathbb{P}}\Omega$; in particular it gives,*
- *char*$(k) = 0$: *The BKP hierarchy is the system of partial differential equations that characterizes when a function is a $\bar\tau$-function of a point of $\mathrm{Gr}^I_\sigma(V)$.*

The relationship between the BKP hierarchy and Pryms will be clear in 4.6.

Remark 5. Let us relate all the above claims to the classical results when *char*$(k) = 0$. Classically, the BKP hierarchy is introduced as the system of equations obtained from the KP system making $t_i = 0$ for all even i.

Take the formal trivilization around p equal to σ_0; that is: $\sigma_0(g(z)) = g(-z)$ for all $g(z) \in \Gamma$. Then, using the isomorphism of Γ with an additive group given by the exponential map, one has that the set of

A-valued points of Π^0_- is (we write only 0 instead of σ_0):

$$\Pi^0_-(\mathrm{Spec}(A)) = \left\{ \begin{array}{c} \text{series } \exp\big(\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} a_i z^{-i}\big) \text{ where } a_i \in A \\ \text{is nilpotent and } n>0 \text{ arbitrary} \end{array} \right\}$$

Some well known results, such as formula 1.9.8 of [DKJM], are now a consequence of the relationship between the τ-function of $U \in \mathrm{Gr}^I_0(V)$ as a point of $\mathrm{Gr}(V)$ and the $\bar{\tau}$-function as a point of $\mathrm{Gr}^I_0(V)$ given in Theorem 3.4.

These connections of KP and BKP, of $\mathrm{Gr}(V)$ and $\mathrm{Gr}^I_0(V)$, of Det^*_V and Pf, and of τ and $\bar{\tau}$ (given above) justify the expression given in [DKJM2] of a tau-function for the BKP in terms of theta functions of a Prym variety[b], as well as the methods of [H] and [O] based on pfaffians of matrices in order to construct solutions for the BKP.

4. Geometric Characterizations

Geometric characterizations of Jacobians and Pryms offered in several papers ([M1, S, LM, S2]) are based on the study of an action of a group on a space at the tangent space level and are therefore suitable for being expressed in terms of dynamical systems. Roughly, the group is a subgroup of the linear group and the space is the Grassmannian (or a quotient of it), and the way to relate Jacobians and Pryms with the Grassmannian is through the Krichever functor.

Since we aim to give a scheme-theoretic generalization of Theorem 6 of [S], §2.5 of [S2] and Theorem 5.14 of [LM], we should use only algebraic methods. In Shiota's paper, the action of the group is given by analytic techniques, while the characterization of Pryms given by Li and Mulase involves quotients of the Grassmannian that do not need to be algebraic.

Our approach therefore needs to use the notion that the action of Γ in $\mathrm{Gr}(V)$ (Π^σ_- on $\mathrm{Gr}^I_\sigma(V)$) is algebraic and that there is no need to use quotient spaces because of Lemma 4.3. Essentially, this Lemma implies that the orbit of a point of $\mathrm{Gr}(V)/\Gamma$ under Γ coincides with that of a preimage in $\mathrm{Gr}(V)$ under Γ_-. Although the methods are algebraic, it seems quite natural to use the language of dynamical systems when working at the tangent space level.

[b]It is known that the restriction of a theta function of a Jacobian variety is the square of a theta function of the Prym when the involution has two fixed points ([Mu]).

4.A. **Dynamical systems and the Grassmannian.** Observe that the action:

$$\begin{gathered}\Gamma \times \operatorname{Gr}(V) \xrightarrow{\mu} \operatorname{Gr}(V) \\ (g, U) \mapsto g \cdot U\end{gathered}$$

canonically induces a system of partial differential equations (p.d.e.) on $\operatorname{Gr}(V)$. Taking $\operatorname{Spec}(k[\epsilon]/(\epsilon^2))$-valued points, and using the canonical identification:

$$T\operatorname{Gr}(V) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{L}, \hat{V}/\mathcal{L})$$

we obtain a morphism of functors on groups:

$$\begin{gathered}T_{\{1\}}\Gamma \xrightarrow{d\mu} \operatorname{Hom}(\mathcal{L}, \hat{V}/\mathcal{L}) \\ 1 + \epsilon g \longmapsto (\mathcal{L} \hookrightarrow \hat{V} \xrightarrow{\cdot g} \hat{V} \to \hat{V}/\mathcal{L})\end{gathered}$$

Denote by μ_- ($d\mu_-$) the restriction $\mu|_{\Gamma_-}$ ($d\mu|_{\Gamma_-}$ respectively).

Moreover, note that the map g to $1 + \epsilon g$ gives an isomorphism of functors $\hat{V} \simeq T_{\{1\}}\Gamma$, where $\hat{V}(S) = \varprojlim_{\substack{A \\ A \sim V_+}} (V/A \underset{k}{\otimes} \mathcal{O}_S)$. Also, the kernel of $d\mu_-$ at a point U is the maximal sub-k-algebra of V acting (by homotheties) on U.

Definition 4.1. *Given a subbundle $E \subseteq T\operatorname{Gr}(V)$, a finite dimensional solution of the p.d.e. associated with E at a point U will be a finite dimensional subscheme $X \subseteq \operatorname{Gr}(V)$ containing U and such that $E_U \simeq T_U X$.*

Remark 6. It is convenient to consider not only finite dimensional subschemes as solutions but also algebraizable formal schemes.

4.B. **Characterization of Jacobian varieties.** The goal of this subsection is to prove the following generalization of the Theorem 6 of [S]:

Theorem 4.2. *A necessary and sufficient condition for a rational point $U \in \operatorname{Gr}(V)$ to lie in the image of the Krichever map K_m (for a point $m \in \mathcal{M}_\infty$) is that there exists a finite dimensional solution of the p.d.e. $\operatorname{Im} d\mu_-$ at the point U.*

Proof of the theorem is a direct consequence of the following two lemmas, that are quite akin to Mulase's and Shiota's ideas ([M1, S]).

Lemma 4.3. *Let U be rational point of $\operatorname{Gr}(V)$, and let $G(U)$ denote the orbit of U under the action of a group G. Then:*

$$\Gamma(U) \simeq \Gamma_-(U) \times \Gamma_+$$

Proof. Since $\Gamma = \varinjlim(\Gamma_-^n \times \mathbb{G}_m \times \Gamma_+)$ (as formal schemes), it is then enough to show that the natural map:

$$\begin{aligned} \Gamma_-^n(U) \times \Gamma_+ &\to (\Gamma_-^n \times \mathbb{G}_m \times \Gamma_+)(U) \\ (f_-U, f_+) &\mapsto (f_- \cdot f_+)U \end{aligned}$$

is an isomorphism; or, what amounts to the same: if $f_-U = f_+U$, then $f_- = f_+ = 1$. Since U is a rational point, there exists an element $g = \sum_{i\geq m} c_i\, z^i \in U$ ($c_m \neq 0$) with m maximal (this m will be called order of g); note also that we can assume $c_m = 1$. Note that homotheties map elements of maximal order into elements of maximal order, and therefore $f_- \cdot g = f_+ \cdot g$, from which one deduces $f_- = f_+$, and hence $f_-, f_+ \in \Gamma_- \cap \Gamma_+ = \{1\}$.

Finally, the natural structure of the formal scheme of $\Gamma_-(U)$ is equal to $\cup_{n>0}\Gamma_-^n(U)$ (where $\Gamma_-^n(U)$ denotes the schematic image of $\Gamma_-^n \times \{U\} \to \mathrm{Gr}(V)$).

Let Γ_-^n be $\mathrm{Spec}\,(k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n)$, $\mathcal{O}$ the structural sheaf $\mathcal{O}_{\mathrm{Gr}(V)}$, I_U^n the kernel of $\mathcal{O} \to k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n$, and $u_n^{n'}$ the morphism $\mathcal{O}/I_U^{n'} \to \mathcal{O}/I_U^n$ ($n' > n$).

In order to show that $\Gamma_-(U) = \mathrm{Spf}(A)$ (A being $\varprojlim \mathcal{O}/I_U^n$), it remains only to check that ([EGA] I.10.6.3) $u_n^{n'}$ is surjective and $\ker(u_n^{n'})$ is nilpotent. However, both are obvious.

Note that as a by-product we have that the topology of A is given by the ideals $J_n = \ker(A \to \mathcal{O}/I_U^n)$ and that the definition ideal is $J = \varprojlim_n (x_1, \dots, x_n)$. □

Lemma 4.4. *Let U be a rational point of $\mathrm{Gr}(V)$. Then the following conditions are equivalent:*

1. $\Gamma_-(U)$ *is algebraizable,*
2. $\dim_k(J/J^2) < \infty$,
3. $\dim_k(T_U\Gamma_-(U)) < \infty$,
4. $\dim_k \mathrm{Im}(d\mu_-) < \infty$,
5. *there exists a rational point $m = (C, p, z)$ of $\mathcal{M}_\infty$ and a pair (L, ϕ) of $\widetilde{\mathrm{Pic}}(C, p)(\mathrm{Spec}(k))$ such that $K_m(L, \phi) = U$.*

Proof. $1 \implies 2$: Recall that algebraizable ([H] II.9.3.2) means that the formal scheme is isomorphic to the completion of a noetherian scheme along a closed subscheme, but the completion of a noetherian ring with respect to an ideal is noetherian ([AM] 10.26); hence A is noetherian.

Recall now from [EGA] 0.7.2.6 that if A is an admissible linearly topologized ring, and J a definition ideal, A is noetherian if and only if A/J is so and J/J^2 is a finite type A/J-module.

Since $A/J \simeq k$, we conclude.

$2 \implies 3$: Note that

$$T_U\Gamma_-(U) = \operatorname{Hom}_{\text{for-esq}}(\operatorname{Spec}(k[\epsilon]/(\epsilon^2), \operatorname{Spf}(A)) =$$
$$= \operatorname{Hom}_{\substack{\text{topological} \\ k\text{-algebras}}}(A, k[\epsilon]/(\epsilon^2)) \subset \operatorname{Hom}_{k\text{-algebras}}(A, k[\epsilon]/(\epsilon^2))$$

which is isomorphic to $(J/J^2)^*$.

$3 \iff 4$ By the very definition, the morphism $\Gamma_- \to \operatorname{Gr}(V)$ factorizes through $\Gamma_-(U)$, and therefore:

$$\dim_k \operatorname{Im}(d\mu_-) \le \dim_k T_U\Gamma_-(U) < \infty$$

$4 \implies 5$: From 4 we have $\dim_k \ker(d\mu_-) = \infty$. Note, moreover, that $B = \ker(d\mu_-) \subseteq V$ is a integral k-algebra of transcendence degree 1, since $B_{(0)} = V$ such that U is a free B-module of rank 1. Standard results (see [SW]) show how from the pair (B, U) one can construct the data $(C, p, z) \in \mathcal{P}(\operatorname{Spec}(k))$ and $(L, \phi) \in \widetilde{\operatorname{Pic}}(C, p)(\operatorname{Spec}(k))$ such that $K_m(L, \phi) = U$.

$5 \implies 4$: Easy.

$3 \implies 1$: Let $\operatorname{Spec}(A_n)$ be the schematic image of

$$\Gamma_-^n = \operatorname{Spec}(k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n) \to \operatorname{Gr}(V)$$

By the properties of formal schemes, one has:

$$T_U\Gamma_-(U) \quad = \quad \bigcup_{n>0} T_U \operatorname{Spec}(A_n)$$

And therefore, if $\dim_k(T_U\Gamma_-(U)) = d$ then $\dim_k(T_U \operatorname{Spec}(A_n)) = d$ for all $n >> 0$.

Denote with $J_{(n)}$ the maximal ideal of A_n, since $T_U \operatorname{Spec}(A_n) \simeq (J_{(n)}/J_{(n)}^2)^*$ and $J = \varprojlim J_{(n)} \overset{\pi_n}{\longrightarrow} J_{(n)}$ is surjective, there exist elements $\bar{y}_1, \dots, \bar{y}_d \in J$ such that $< \{\pi_n(\bar{y}_1), \dots, \pi_n(\bar{y}_d)\} >= J_{(n)}/J_{(n)}^2$. By Nakayama's lemma one has an epimorphism:

$$\begin{aligned} p_n : k[y_1, \dots, y_d] &\to A_n \\ y_i \quad &\mapsto \pi_n(\bar{y}_i) \end{aligned}$$

It is now straightforward to see that $\{p_n\}$ is compatible with the natural morphism $A_m \to A_n$ for $m > n$. One concludes that $A_n \simeq k[y_1, \dots, y_d]/I_n$ (I_n being $\ker p_n$) and that $I_m \subset I_n$ for $m > n$.

Recall now that $A = \varprojlim A_n$ and thus $A \simeq k[[y_1, \dots, y_d]]$; the topology on A induced by $\ker(A \to A_n)$ coincides with the $(y_1, \dots, y_d)$-adic; that is, the formal scheme $\operatorname{Spf}(A)$ is algebraizable. □

Remark 7. Assume that the conditions of Lemma 4.4 are satisfied for a point U. Then the orbit $\Gamma_-(U)$, which is a formal scheme, is the desired finite dimensional solution. Let us give another interpretation. Let $m = (C, p, z) \in \mathcal{M}_\infty$ and $(L, \phi) \in \widetilde{\operatorname{Pic}}(C, p)$ such that $K_m(L, \phi) = U$. It is straightforward to check that the p.d.e. defined by $\operatorname{Im}(d\mu_-)$ is in fact the KP hierarchy and hence $\widetilde{\operatorname{Pic}}(C, p)$ is a finite dimensional solution of the KP hierarchy, modulo the action of Γ_+.

Remark 8. Let $char(k) = 0$, $U \in \operatorname{Gr}(V)$ satisfying one condition of the previous lemma and (C, p, z, L, ϕ) that given by the fifth condition.

Then, the morphism $T_{\{1\}}\Gamma_-^n \to T_U\operatorname{Gr}(V)$ is essentially:

$$H^0(C, \mathcal{O}_C(np)/\mathcal{O}_C) \to H^1(C, \mathcal{O}_C) \simeq T_L \operatorname{Pic}(C)$$

which is deduced from the cohomology sequence of:

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(np) \to \mathcal{O}_C(np)/\mathcal{O}_C \to 0$$

and from the proof of the Lemma it is easy to conclude that $\Gamma_-(U)$ is canonically isomorphic to the formal completion of the Jacobian of C along L.

When $k = \mathbb{C}$ and C is a smooth curve, we can interpret the algebraic variety $\operatorname{Pic}(C)$ as a compact Lie group. Now using the exponential map, it is not hard to see that our condition of "finite dimensional solution of the p.d.e." is equivalent to Shiota's "compact solution for the KP hierarchy" (see Theorem 6 of [S]), and that the morphism $T_{\{1\}}\Gamma_-^n \to T_U\operatorname{Gr}(V)$ is the one studied in depth in its §2, especially in Lemmas 2 and 4 of [S].

Remark 9. Note further that the point $m = (C, p, z) \in \mathcal{M}_\infty$ obtained by Lemma 4.4 is not unique. However, it has a characterizing property. From the construction, it is easily seen that the ring B is the maximal subring of $\hat{V}$ such that $B \cdot U = U$. This implies that there is no morphism $f : C' \to C$ and line bundle L' such that $f_*L' = L$ (unless f is an isomorphism). See [M2] for more details.

4.C. **Characterization of Prym varieties.** We shall say that a point $(C, p, z) \in \mathcal{M}_\infty$ admits the automorphism σ if $\sigma : k((z)) \xrightarrow{\sim} k((z))$ (as a k-algebra) restricts to $K(C, p, z) \xrightarrow{\sim} K(C, p, z)$. Note that in this case σ induces an automophism of C with p as a fixed point (it will also be denoted by σ).

Definition 4.5. *Let $\mathcal{P}_\sigma$ denote the subfunctor of $\mathcal{M}_\infty$ consisting of the data that admit the automorphism σ. (In the next section we shall see that it is in fact a subscheme).*

In this setting, the Prym variety associated with the data $m = (C, p, z) \in \mathcal{M}_\infty$ that admits an involution σ is a subscheme of $\widetilde{\mathrm{Pic}}(C, p)$ whose rational points are:

$$\widetilde{\mathrm{Prym}}(C, p, \sigma) = \left\{ (L, \phi) \in \widetilde{\mathrm{Pic}}(C, p) \,|\, \sigma^*(L) \xrightarrow{\sim} \omega_C \otimes L^{-1} \right\}$$

In this subsection we shall restrict ourselves to the situation addressed in Example 1; that is: $\sigma = \sigma_0$. Note, however, that this can always be achieved (see Remark 2). Thus, we shall assume here that $\sigma_0^*(g(z)) = g(-z))$ for all $g(z) \in \Gamma$, and shall remove the super/subscript σ in the notations. From the above discussion and recalling 2.4, one has the following cartesian diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Pic}}(C, p) & \overset{K_m}{\hookrightarrow} & \mathrm{Gr}(V) \\ \cup & & \cup \\ \widetilde{\mathrm{Prym}}(C, p, \sigma_0) & \overset{K_m}{\hookrightarrow} & \mathrm{Gr}_0^I(V) \end{array}$$

Let μ_-^0 be the restriction of μ_- to Π_-^0, and let $d\mu_-^0$ be that induced in the tangent spaces. Our version of the Theorem 5.14 of [LM] is the following:

Theorem 4.6. *A necessary and sufficient condition for a rational point $U \in \mathrm{Gr}_0^I(V)$ to lie in the image of the Krichever map K_m (for a point of $m \in \mathcal{P}_0$) is that there exists a finite dimensional solution of the p.d.e. $\mathrm{Im}\, d\mu_-^0$ at the point U.*

Proof. Observe that $\dim_k \mathrm{Im}\, d\mu_-^0 < \infty$ if and only if $\dim_k \mathrm{Im}\, d\mu_- < \infty$, and therefore that it is only necessary to show that in the last condition of Lemma 4.4, the constructed data $(C, p, z) \in \mathcal{M}_\infty$ admits the involution given by $z \mapsto -z$.

First, note that since U is m.t.i. we have: $< f, u >= 0 \quad \forall u \in U \implies f \in U$. Now for an element $f \in \ker(d\mu_-^0)$ we have $f \cdot U \subseteq U$ and therefore $< f \cdot u, v >= 0$ for all $u, v \in U$.

We want to see that $f(-z) \in \ker(d\mu_-^0)$; that is, $< f(-z)u(z), v(z) >= 0$ for all $u, v \in U$. Note, however, that:

$$< f(-z)u(z), v(z) >=< u(z), f(z)v(z) >= - < f(z)v(z), u(z) >$$

and hence $f(-z) \in \ker(d\mu_-^0)$, as desired. □

Remark 10. Analogously to the case of Jacobian varieties, one has that $\mathrm{Im}(d\mu_-^0)$ is equivalent to the BKP hierarchy and therefore $\Pi_-^0(U)$ is a finite dimensional solution for the BKP hierarchy. As before, one can say that $\widetilde{\mathrm{Prym}}(C, p, z)$ (modulo the action of Γ_+) is a finite dimensional solution too.

Remark 11. Note that in the proof of Lemma 4.4 a k-algebra B is constructed that turns out to be the ring $H^0(C-p, \mathcal{O}_C)$. If we now assume that $U \in \mathrm{Gr}_0^I(V)$, B has an involution. It is easy to check that in this situation $B \in \mathrm{Gr}_0(k((z)), k[[z]])$. Since we know the structure of this Grassmannian (see Example 1), the projection on the first factor is precisely:

$$B' = B \cap k((z^2)) \in \mathrm{Gr}_0(k((z^2)), k[[z^2]])$$

Now, the curve C' constructed from B' is the quotient of C with respect to the involution σ. Note that U is a rank 2 free B'-module, and that the sheaf induced by U over C' is the direct image of L by $C \to C'$.

5. Equations for the Moduli Space of Prym varieties

Although the notion of Prym variety is more general, here we shall restrict our study to those coming from a curve and an involution with at least one fixed point.

Definition 5.1. *The functor of Prym varieties, $\mathcal{P}$, is the sheafication of the following functor on the category of k-schemes:*

$$S \rightsquigarrow \left\{ (C, D, z, \sigma) \mid \begin{array}{c} (C,D,z) \in \mathcal{M}_\infty(S),\ \sigma \text{ is an involution} \\ \text{that induces an automorphism of} \\ \text{the formal completion of } C \text{ along } D \end{array} \right\}$$

(up to isomorphisms).

Given $(C, D, z, \sigma) \in \mathcal{P}(\mathrm{Spec}(k))$, note that σ induces an automorphism of $k((z))$, and that the action of Γ_+ on the group $\mathrm{Aut}_{k\text{-alg}}(k((z)))$ (via U) is transitive and free. One therefore has a bijection (set-theoretic) $\mathcal{P} \simeq \mathcal{P}_0 \times \Gamma_+$.

From Example 1 and Definition 4.5 one now has the following easy but fundamental result:

Theorem 5.2. *Via the Krichever map, one has:*

$$\mathcal{P}_0 \simeq \mathcal{M}_\infty \underset{\mathrm{Gr}(V)}{\times} \mathrm{Gr}_0(V)$$

Corollary 5.3. *The functor $\mathcal{P}_0$ is representable by a locally closed subscheme of the infinite Grassmannian* $\mathrm{Gr}(V)$*; namely, that whose (S-valued) points U satisfy:*

1. $\mathcal{O}_S \subset U$,
2. $U \cdot U \subseteq U$ *($\cdot$ being the product of $\mathcal{O}_S((z))$),*

3. *the map* $\mathcal{O}_S((z)) \to \mathcal{O}_S((z))$ *defined by* $z \mapsto -z$ *restricts to an isomorphism* $U \xrightarrow{\sim} U$.

Proof. Recall from [Al, MP] that the first two conditions are locally closed. The third condition is closed since it is where the identity and the involution of $\mathrm{Gr}(V)$ given by $z \mapsto -z$ coincide; recall also that $\mathrm{Gr}(V)$ is separated. □

We can now state a theorem characterising the points of $\mathrm{Gr}_0(V)$ in terms of bilinear identities; this is an analogous result to the characterization of $\mathrm{Gr}(V)$ of [DKJM, F, MP].

Theorem 5.4 (Bilinear Identities)**.**

$$U \in \mathrm{Gr}_0(V) \quad \Longleftrightarrow \quad \begin{cases} \mathrm{Res}_{z=0}\, \psi_U(z,t)\psi^*_U(z,t')\dfrac{dz}{z^2} = 0 \\ \mathrm{Res}_{z=0}\, \psi_U(-z,t)\psi^*_U(z,t')\dfrac{dz}{z^2} = 0 \end{cases} \quad \textit{for all } t, t'$$

Proof. Note that the third condition in the corollary is equivalent (for the rational points) to saying that: $\psi_U(z,t) = -\psi_U(-z,t)$ since for a point $U \in \mathrm{Gr}(V)$ one has that the Baker-Akhiezer function, $\psi_U(z,t)$, may be written as a series of the form $z \cdot \sum_{i>0} \psi_U^{(i)}(z) p_i(t)$ where $p_i(t)$ are universal polynomials (they do not depend on U) and $\{\psi_U^{(i)}(z)\}_{i>0}$ is a basis of U as a k-vector space (see [MP]).

Recalling the property $\mathrm{Res}_{z=0}\, \psi_U(z,t)\psi^*_{U'}(z,t')\frac{dz}{z^2} = 0$ if and only if $U = U'$, one concludes. □

This result holds when $char(k) \neq 2$; however, it can be translated in the language of differential equations when $char(k) = 0$. We arrive at two sets of differential equations in terms of τ-functions defining the set of rational points of $\mathrm{Gr}_0(V)$ and $\mathcal{P}_0$.

For a Young diagram λ, denote by χ_λ its associated Schur polynomial. If the diagram has only one row of length β, then denote the corresponding Schur polynomial by p_β. Let t be $(t_1, t_2, \dots)$ and $\widetilde{\partial}_t$ be $(\partial_{t_1}, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, \dots)$. Let $D_{\lambda,\alpha}$ be $\sum_\mu \chi(\widetilde{\partial}_t)$ where the sum is taken over the set of Young diagrams μ such that $\lambda - \mu$ is a horizontal α-strip.

Theorem 5.5 (P.D.E. for $\mathrm{Gr}_0(V)$)**.** *A function* $\tau(t)$ *is the* τ*-function of a rational point* $U \in \mathrm{Gr}_0(V)$ *if and only if it satisfies the following infinite set of differential equations (indexed by a pair of Young diagrams* λ_1, λ_2*):*

$$\Big(\sum p_{\beta_1}(-\widetilde{\partial}_t) D_{\lambda_1,\alpha_1}(\widetilde{\partial}_t) \cdot p_{\beta_2}(-\widetilde{\partial}_{t'}) D_{\lambda_2,\alpha_2}(\widetilde{\partial}_{t'})\Big)\,|_{t=t'=0} \tau_U(t) \cdot \tau_U(t') \;=\; 0$$

where the sum is taken over the 4-tuples $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ *of integers such that* $-\alpha_1 + \beta_1 - \alpha_2 + \beta_2 = 1$, $-\alpha_1 + \beta_1$ *is even.*

Proof. Recall from [MP] how the KP hierarchy is deduced from the Residue Bilinear Identity. Apply the same procedure to the identities in the preceding Theorem, and add and subtract both identities. □

We finish with the partial differential equations for τ-functions that characterize $\mathcal{P}_0$ as a subscheme of $\check{\mathbb{P}}\Omega$.

Theorem 5.6 (P.D.E. for $\mathcal{P}_0$). *A function* $\tau(t)$ *is the* τ*-function of a point* $U \in \mathcal{P}_0$ *if and only if it satisfies the following infinite set of differential equations:*

1. *the p.d.e. of Theorem 5.5,*
2. $$P(\lambda_1, \lambda_2, \lambda_3)|_{\substack{t=0\\ t'=0\\ t''=0}} \left(\tau_U(t) \cdot \tau_U(t') \cdot \tau_U(t'')\right) = 0$$
   for every three Young diagrams $\lambda_1, \lambda_2, \lambda_3$, *where* $P(\lambda_1, \lambda_2, \lambda_3)$ *is the differential operator defined by:*
   $$\sum p_{\beta_1}(\widetilde{\partial}_t) D_{\lambda_1,\alpha_1}(-\widetilde{\partial}_t) \cdot p_{\beta_2}(\widetilde{\partial}_{t'}) D_{\lambda_2,\alpha_2}(-\widetilde{\partial}_{t'}) \cdot p_{\beta_3}(-\widetilde{\partial}_{t''}) D_{\lambda_3,\alpha_3}(\widetilde{\partial}_{t''})$$
   where the sum is taken over the 6-tuples $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ *of integers such that* $-\alpha_1 + \beta_1 - \alpha_2 + \beta_2 - \alpha_3 + \beta_3 = 2$, $-\alpha_1 + \beta_1$ *is even,*
3. *the p.d.e.'s:*
   $$\left(\sum_{\alpha \geq 0} p_\alpha(-\widetilde{\partial}_t) D_{\lambda,\alpha}(\widetilde{\partial}_t)\right)|_{t=0} \tau_U(t) = 0 \quad \textit{for all Young diagram } \lambda$$

Proof. By Theorems 5.2 and 5.5, it is enough to recall from [MP] the p.d.e. defining $\mathcal{M}_\infty$ in the infinite Grassmannian. □

Note that a theta function of the Prym variety associated to a 4-uple (C, p, z, σ) (where σ is the given by $z \mapsto -z$) satisfy these differential equations which are not a consequence of the BKP hierarchy.

References

[Al] A Álvarez-Vázquez, *Estructuras aritméticas de las curvas algebraicas*, Ph.D. Thesis, Salamanca, June 1996

[AM] M Atiyah, I G MacDonald *Introduction to commutative algebra*, Addison Wesley Publ. Company, London (1969)

[AMP] A Álvarez Vázquez; J M Muñoz Porras and F J Plaza Martín, *The algebraic formalism of soliton equations over arbitrary base fields*, alg-geom/9606009 (To appear in "Proceedings of Workshop on Abelian Varieties and Theta Functions", edited by Aportaciones Matemáticas de la Sociedad Matemática Mexicana)

[B] D Borthwick, *The Pfaffian line bundle*, Comm. Math. Physics **149** (1992), pp. 463–493

[Bo] N Bourbaki,*Commutative Algebra*, Hermann (1972)

[BS] A A Beilinson and V V Schechtman, *Determinant Bundles and Virasoro Algebras* Commun. Math. Phys. **118** (1988), pp. 651-701

[C] C Contou-Carrére, *Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré*, C. R. Acad. Sci. Paris, série I **318** (1994), pp. 743-746

[DKJM] E Date; M Kashiwara; M Jimbo and T Miwa, *Transformation groups for soliton equations*, Proceedings of RIMS Symposium, Kyoto 1981

[DKJM2] E Date; M Kashiwara; M Jimbo and T Miwa, *Quasi-periodic solutions of the orthogonal KP equations — Transformation groups for soliton equations V*, Publ. RIMS, Kyoto Univ. **18** (1982), pp.1111-1119

[EGA] A Grothendieck and J A Dieudonné, *Eléments de géométrie algébrique* **I**, Springer–Verlag (1971)

[F] J D Fay, *Bilinear Identities for Theta Functions*, Preprint

[H] R Hirota, *Solitons solutions to the BKP equations. I. The pfaffian technique*, Journal of the Physical Society of Japan **58**, 7, (1989), pp. 2285-2296

[K] I M Krichever, *Methods of algebraic geometry in the theory of non-linear equations*, Russian Math. Surveys **32**:6, (1977), pp. 185-213

[KSU] T Katsura, Y Shimizu and K Ueno, *Formal Groups and Conformal Field Theory over* $\mathbb{Z}$, Advanced Studies in Pure Mathematics **19** (1989), Integrable systems in quantum field theory and statistical mechanics, pp. 347-366

[KM] F Knudsen and D Mumford, *The projectivity of the moduli space of stable curves I: preliminaries on det and div*, Math. Scand.**39** (1976), pp. 19–55

[LM] Y Li and M Mulase, *Category of morphisms of algebraic curves and a characterization of Prym varieties*, Max-Planck Preprint 1992 and alg-geom/9203002

[M1] M Mulase, *Cohomological structure in soliton equations and jacobian varieties*, J. Differetnial Geom. **19** (1984), pp. 403-430

[M2] M Mulase, *A correspondence between an Infinite Grassmannian and arbitrary vector bundles on algebraic curves*, Proc. Symp. Puer Math. **49** Series A (1989), pp. 39-50

[Mu] D Mumford, *Prym varieties*, Contributions to Analyis (1974), pp. 325-350

[MP] J M Muñoz Porras and F J Plaza Martín, *Equations of the moduli space of pointed curves in the infinite Grassmannian* (to appear)

[N] Y Namikawa, *A Conformal Field Theory on Riemann Surfaces Realized as Quantized Moduli Theory of Riemann Surfaces*, Proceedings of Symposia in Pure Mathematics **49** (1989), Part I, pp. 413-443

[O] Y Ohta, *Paffian solutions for the Veselov-Novikov equations*, Journal of the Physical Society of Japan **61**, 11, (1992), pp. 3928-3933

[P] F J Plaza Martín, *Picard group and automorphism group of* $\mathrm{Gr}(k((z)))$ (to appear)

[PS] A Pressley and G Segal, *Loop Groups*, Oxford University Press

[S] T Shiota, *Characterization of Jacobian varieties in terms of soliton equations*, Inventiones Mathematicae **83** (1986), pp. 333–382

[S2] T Shiota, *Prym varieties and soliton equations*, Infinite-dimensional Lie algebras and groups, Adv. Ser. Math. Phys., 7, (Luminy-Marseille, 1988), pp. 407–448, World Sci. Publishing

[SW] G Segal and G Wilson, *Loop groups and equations of KdV type*, Publ. Math. I.H.E.S. **61** (1985), pp. 3–64

[SS] M Sato and Y Sato, *Soliton equations as dynamical systems on infinite Grassmann manifold*, Lecture Notes in Num. Appl. Anal. **5** (1982), pp. 259–271

Departamento de Matemática Pura y Aplicada, Universidad de Salamanca, Plaza de la Merced 1-4, 37008 Salamanca. Spain.

E-mail address: fplaza@gugu.usal.es