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1 Introduction

It is well-known that the Krichever map can be extended to the case when the geometric data are given by a finite covering of pointed Riemann surfaces and trivializations at the punctures. This has been studied in works authored by M. R. Adams and M. J. Bergvelt ([2]), M. J. Bergvelt and A. P. E. ten Kroode ([6]), and Y. Li and M. Mulase ([13, 14]). The soliton hierarchies appearing naturally in this problem are given by the flows defined by the Heisenberg algebras in $\widehat{\mathrm{sl}}(n, \mathbb{C})$ (the affine Kac–Moody algebra associated with the loop group of $\mathrm{Sl}(n, \mathbb{C})$).

The objective of this paper is to characterize the Hurwitz schemes parametrizing finite coverings of Riemann surfaces in terms of soliton equations satisfied by certain τ-functions.

For this goal, the first main step consists of generalizing the "formal geometry" developed by the authors in [17] to the case of the "formal spectral cover". Let V be a finite a separable $\mathbb{C}((z))$-algebra of dimension n and $V^+ \subset V$ a $\mathbb{C}[[z]]$-subalgebra of rank n over $\mathbb{C}[[z]]$. The properties of the infinite Grassmannian, $\mathrm{Gr}(V, V^+)$, and of the "formal spectral cover", $\operatorname{Spf} V^+$, are studied with the help of the $\mathbb{C}((z))$-algebra structure of V. This construction allows us to prove a new bilinear identity (Theorem 3.11) that depends on the algebra structure of V.

Let us consider the Hurwitz space $\mathcal{H}^\infty(e_1, \dots, e_r)$ (where $\{e_1, \dots, e_r\}$ is a partition of n, that is, $e_1 + \cdots + e_r = n$) parametrizing geometric data $(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}})$, where $Y \overset{\pi}{\to} X$ is a finite covering of degree n, $x \in X$, $\bar{y} = \pi^{-1}(x) = e_1 y_1 + \cdots + e_r y_r$ and t_x, $t_{\bar{y}}$ are formal parameters at x and $\bar{y}$ respectively. Theorem 4.5 shows that the Krichever functor defines an immersion

$$K \colon \mathcal{H}^\infty(e_1, \dots, e_r) \longrightarrow \mathrm{Gr}(V, V^+),$$

where $V = \mathbb{C}\big(\big(z^{1/e_1}\big)\big) \times \cdots \times \mathbb{C}\big(\big(z^{1/e_r}\big)\big)$. The main result of the paper is Theorem 4.7, which gives a characterization of the image of $\mathcal{H}^\infty(e_1, \dots, e_r)$ in $\mathrm{Gr}(V, V^+)$ uniquely in terms of the piece of data $(Y, \bar{y}, t_{\bar{y}})$ of the geometric data and the algebra structure of V. This characterization allows us to prove that the Hurwtiz space is a scheme (Theorem 4.9). Furthermore, the τ-functions of this space are explicitly characterized by a set of differential equations given in Theorems 4.13 and 4.15.

In the last section, we apply the above results to study the finite dimensional Hurwitz scheme $\mathcal{H}(g, 0; 1, \dots, 1)$, which parametrizes finite coverings of $\mathbb{P}^1$ with a fibre of type $(1, \dots, 1)$. It could be said that the paper provides an explicit method for constructing arbitrary finite coverings of Riemann surfaces from a local datum (the algebra structure of V) and a system of differential equations related to a soliton hierarchy.

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Another application of our results can be found in [11], where the equations defining the moduli space of curves with an automorphism of prime order as a subscheme of the Sato Grassmannian are given.

Throughout this paper we shall assume that the base field is $\mathbb{C}$, the field of complex numbers. Given a scheme X, its functor of points will be denoted by $\underline{X}$ and its structural sheaf by $\mathcal{O}(X)$.

2 Vector-valued infinite Grassmannians

Let V be a separable and finite $\mathbb{C}((z))$-algebra of dimension n and let $V^+ \subset V$ be a $\mathbb{C}[[z]]$-subalgebra of rank n over $\mathbb{C}[[z]]$. Let us denote by $\operatorname{Gr}(V)$ the infinite Grassmannian of (V, V^+) constructed in [3] (see also [20]–[22]). It is worth recalling that Sections 2 and 3 of [3] are concerned with the existence and basic properties of this Grassmannian. Let us summarize some of them.

The infinite Grassmannian of (V, V^+) is a $\mathbb{C}$-scheme whose set of rational points consists of

$$\text{subspaces } U \subset V \text{ such that } U \to V/V^+ \text{ has finite dimensional kernel and cokernel.}$$

The connected components of this scheme are indexed by the Poincaré–Euler characteristic of $U \to V/V^+$. The connected component of index m will be denoted by $\operatorname{Gr}^m(V)$. Recall that $\operatorname{Gr}(V)$ is equipped with the determinant bundle, whose dual has a canonical global section Ω_+.

Let us briefly recall the definition of Ω_+. On the connected component of index 0, $\operatorname{Gr}^0(V)$, it is the determinant of the natural map $\mathcal{L} \to V/V^+$ ($\mathcal{L}$ being the universal submodule). For an integer $m > 0$, set $v_m \in V^+$ such that $\dim_{\mathbb{C}} V^+/v_m V^+ = m$. Then, the section Ω_+ on $\operatorname{Gr}^m(V)$ for $m > 0$ (resp. $m < 0$), which will be denoted by Ω_+^m, is the determinant of the map $\mathcal{L} \to V/v_m V^+$ (resp. $\mathcal{L} \to V/v_{-m}^{-1} V^+$).

The fourth section of [3] is devoted to the study of some groups acting on the Grassmannian and uses the notions of the formal curve, its Jacobian, etc. We now shall employ the same techniques to generalize those notions to our present setting.

Example 2.1 The main examples of couples (V, V^+) of the above type are as follows:

1. $V = \mathbb{C}\big(\big(z^{1/e}\big)\big), V^+ = \mathbb{C}\big[\big[z^{1/e}\big]\big]$, where e is a positive integer.
2. $V = \mathbb{C}((z)) \otimes_{\mathbb{C}} A_0, V^+ = \mathbb{C}[[z]] \otimes_{\mathbb{C}} A_0$, where A_0 is a finite separable $\mathbb{C}$-algebra.
3. $V = \mathbb{C}\big(\big(z^{1/e_1}\big)\big) \times \cdots \times \mathbb{C}\big(\big(z^{1/e_r}\big)\big), V^+ = \mathbb{C}\big[\big[z^{1/e}\big]\big] \times \cdots \times \mathbb{C}\big[\big[z^{1/e_r}\big]\big]$, where $e_1, \dots, e_r$ are positive integers.

Definition 2.2 The *formal base curve associated with the couple* (V, V^+) is the formal scheme $\widehat{C} := \operatorname{Spf} \mathbb{C}[[z]]$. The *formal spectral cover associated with the couple* (V, V^+) is the formal scheme

$$\widehat{C}_V := \operatorname{Spf} V^+.$$

In the rest of this paper, it will be assumed that $\widehat{C}_V$ is a smooth curve. Let us observe that in general $\widehat{C}_V$ is not connected.

Let $V^+ = V_1^+ \times \cdots \times V_r^+$ be the the decomposition of V^+ as a product of local $\mathbb{C}[[z]]$-algebras. Thus, the smoothness of $\widehat{C}_V$ implies that there exist isomorphisms $V_i^+ \simeq \mathbb{C}[[z_i]]$ for all i. Further, note that the parameters z_i can be chosen such that

$$\begin{aligned} \mathbb{C}[[z]] &\hookrightarrow V_i^+ \simeq \mathbb{C}[[z_i]], \\ z &\longmapsto z_i^{e_i}, \end{aligned}$$

and such that one has isomorphisms $V_i \simeq \mathbb{C}\big[\big[z^{1/e_i}\big]\big]$. Summing up, the assumption of the smoothness of the formal spectral cover is equivalent to considering the third case of Example 2.1.

Below, we shall identify z^{1/e_i} with z_i.

From [3] we know that the restricted linear group $\operatorname{Gl}(V)$ of the couple (V, V^+), defined as a contravariant functor on the category of $\mathbb{C}$-schemes, acts on $\operatorname{Gr}(V)$. Moreover, if Det_V denotes the determinant bundle on $\operatorname{Gr}(V)$, then $g^* \operatorname{Det}_V \simeq \operatorname{Det}_V$ for every $g \in \operatorname{Gl}(V)$. Accordingly, the set of isomorphisms $g^* \operatorname{Det}_V \simeq \operatorname{Det}_V$

when g varies in $\mathrm{Gl}(V)$ defines a central extension of functors of groups over the category of $\mathbb{C}$-schemes (see [3, Theorem 4.3]):

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{\mathrm{Gl}(V)} \longrightarrow \mathrm{Gl}(V) \longrightarrow 0 \tag{2.1}$$

where $\mathbb{G}_m$ denotes the multiplicative group.

Let $\underline{V}^*$ be the contravariant functor over the category of $\mathbb{C}$-schemes with values in the category of abelian groups defined as follows:

$$\begin{aligned}\underline{V}^* :\ \{\text{category of } \mathbb{C}\text{-schemes}\} &\rightsquigarrow \{\text{category of groups}\}\\ S &\rightsquigarrow \underline{V}^*(S) := \big(V\widehat{\otimes}_{\mathbb{C}} H^0(S,\mathcal{O}_S)\big)^*\\ &\qquad = \big\{\text{ invertible elements of } V\widehat{\otimes}_{\mathbb{C}} H^0(S,\mathcal{O}_S)\big\}\end{aligned} \tag{2.2}$$

where $\widehat{\otimes}$ denotes the completion of the tensor product w.r.t. the linear topology defined by the subspaces $\{z^n V^+\}$.

In [3, Section 4] it was proved that the functor $\underline{V}_i^*$ (where $V_i \simeq \mathbb{C}((z_i))$) is representable by a formal group scheme Γ_i and that its connected component of the origin decomposes as $\Gamma_i^- \times \mathbb{G}_m \times \Gamma_i^+$. Analogously, one can prove that $\underline{V}^*$ is representable by a formal group scheme Γ_V and that its connected component of the origin, Γ_V^0, decomposes as

$$\Gamma_V^0 \ = \Gamma_V^- \times \mathbb{G}_m^r \times \Gamma_V^+,$$

where $\Gamma_V^- \simeq \Gamma_1^- \times \cdots \times \Gamma_r^-$ and $\Gamma_V^+ \simeq \Gamma_1^+ \times \cdots \times \Gamma_r^+$.

$\underline{\Gamma}_V^0$ being the functor of points of Γ_V^0, there is a bijection between the set of $\mathbb{C}$-scheme homomorphisms $\mathrm{Spec}(R) \to \Gamma_V^0$ and the set $\underline{\Gamma}_V^0(\mathrm{Spec}(R)) \subseteq \underline{V}^*(\mathrm{Spec}(R))$. Therefore, such a morphism is determined by a triple $(\gamma_-, \gamma_0, \gamma_+) \in \Gamma_V^- \times \mathbb{G}_m^r \times \Gamma_V^+$:

$$\begin{aligned}\gamma_- &= \Big(\gamma_-^{(1)},\dots,\gamma_-^{(r)}\Big), \quad \text{where} \quad \gamma_-^{(i)} = 1 + \sum_{j=-m}^{-1} a_j^{(i)} z_i^j,\ \ a_j^{(i)} \in \mathrm{Rad}(R),\\ \gamma_0 &= \Big(\gamma_0^{(1)},\dots,\gamma_0^{(r)}\Big), \quad \text{where} \quad \gamma_0^{(i)} \in R \ \text{ is invertible},\\ \gamma_+ &= \Big(\gamma_+^{(1)},\dots,\gamma_+^{(r)}\Big), \quad \text{where} \quad \gamma_+^{(i)} = 1 + \sum_{j\geq 1} a_j^{(i)} z_i^j,\ \ a_j^{(i)} \in R.\end{aligned} \tag{2.3}$$

Remark 2.3 For the case $n = r = 1$, this group was introduced in [7] and was applied to problems related to the tame symbol.

Remark 2.4 The central extension of $\mathrm{Gl}(V)$ of Equation (2.1) induces a central extension of the group scheme Γ_V:

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{\Gamma}_V \longrightarrow \Gamma_V \longrightarrow 0$$

whose restriction to $\Gamma_i = \Gamma_V^- \times \mathbb{G}_m \times \Gamma_V^+$ is the central extension constructed in [3]. Further, note that the central extension $\widetilde{\Gamma}_V$ gives rise to a pairing

$$\begin{aligned}\Gamma_V \times \Gamma_V \ &\longrightarrow \mathbb{G}_m,\\ (\gamma_1,\gamma_2) \ &\longmapsto \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_1^{-1}\tilde{\gamma}_2^{-1},\end{aligned}$$

where $\tilde{\gamma}_i \in \widetilde{\Gamma}_V$ is an element whose image is γ_i.

Remark 2.5 The algebraic version of the loop group of $\mathrm{Gl}(n,\mathbb{C})$ is the sub-functor of groups $\underline{\mathrm{LGl}}(n,\mathbb{C}((z))) \subset \mathrm{Gl}(V)$ defined by

$$\underline{\mathrm{LGl}}(n,\mathbb{C}((z)))(S) := \big\{\text{automorphisms of } V\widehat{\otimes}H^0(S,\mathcal{O}_S) \text{ as } \mathbb{C}((z))\widehat{\otimes}H^0(S,\mathcal{O}_S)\text{-module}\big\}$$

for a $\mathbb{C}$-scheme S. This functor is representable by a formal group $\mathbb{C}$-scheme which is denoted by $\mathrm{LGl}(n, \mathbb{C}((z)))$. It has some distinguished subgroups $\mathrm{LGl}(n)^-$, $\mathrm{Gl}(n, \mathbb{C})$ and $\mathrm{LGl}(n)^+$. The "big cell" of $\mathrm{LGl}(n, \mathbb{C}((z)))$ is defined by

$$\mathrm{LGl}(n)^0 := \mathrm{LGl}(n)^- \cdot \mathrm{Gl}(n, \mathbb{C}) \cdot \mathrm{LGl}(n)^+$$

and is an open subscheme of the connected component of the origin in $\mathrm{LGl}(n, \mathbb{C}((z)))$ ([4, Proposition 1.11], [20, Chap. 8.1.2], [9, Section 2]). The natural action of Γ_V on V (by homotheties) induces a natural action on the Grassmannian, from which one may deduce a natural immersion $\Gamma_V^0 \subset \mathrm{LGl}(n)^0$ such that $\Gamma_V^- \subset \mathrm{LGl}(n)^-$, $\mathbb{G}_m^r \subset \mathrm{Gl}(n, k)$ and $\Gamma_V^+ \subset \mathrm{LGl}(n)^+$. Therefore, the elements of Γ_V can be interpreted as matrices of size $n \times n$ with entries in $\mathbb{C}((z))$.

The Lie algebra of Γ_+^0 as a subalgebra of $\mathrm{Lie}\left(\mathrm{LGl}(n)^0\right)$ is precisely the principal Heisenberg algebra of type $\underline{n} = (e_1, \dots, e_r)$. Thus, as a subgroup of $\mathrm{LGl}(n)^0$, Γ_V^0 is a principal Heisenberg group of type $\underline{n} = (e_1, \dots, e_r)$ ([2, 6, 5]).

Definition 2.6 The *formal Jacobian of the formal spectral cover* $\widehat{C}_V$ is the formal group scheme Γ_V^- and will be denoted by $\mathcal{J}\big(\widehat{C}_V\big)$.

The above-defined Jacobian satisfies the functorial properties of the formal Jacobian of an integral formal curve of [3]. Note, moreover, that $\mathcal{J}\big(\widehat{C}_V\big)$ is the formal spectrum of the ring

$$\mathbb{C}\left\{\left\{x_1^{(1)}, x_2^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{x_1^{(r)}, x_2^{(r)}, \dots\right\}\right\}$$

where

$$\mathbb{C}\left\{\left\{x_1^{(i)}, x_2^{(i)}, \dots\right\}\right\} := \varprojlim_m \mathbb{C}\left[\left[x_1^{(i)}, \dots, x_m^{(i)}\right]\right].$$

By the very definition of the functor $\underline{V}^*$ (see Equation (2.2)) and the decomposition

$$\Gamma_V^- \simeq \Gamma_1^- \times \dots \times \Gamma_r^-,$$

one knows that a morphism $\widehat{C}_V \to \mathcal{J}\big(\widehat{C}_V\big)$ is defined by r series in one variable with coefficients in the ring

$$\mathcal{O}\big(\widehat{C}_V\big) = \mathbb{C}[[\bar{z}_1]] \times \dots \times \mathbb{C}[[\bar{z}_r]] \simeq V^+,$$

where we distinguish the variables $\bar{z}_i \in \mathcal{O}\big(\widehat{C}_V\big)$ and the variables $z_i \in \mathcal{J}\big(\widehat{C}_V\big)$.

We define the Abel morphism of degree one

$$\phi_1 \colon \widehat{C}_V \longrightarrow \mathcal{J}\big(\widehat{C}_V\big)$$

as the morphism corresponding to the series

$$\left(\left(1 - \frac{\bar{z}_1}{z_1}\right)^{-1}, \dots, \left(1 - \frac{\bar{z}_r}{z_r}\right)^{-1}\right).$$

Equivalently, this is the morphism induced by the ring homomorphism

$$\begin{aligned} \mathbb{C}\left\{\left\{x_1^{(1)}, x_2^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{x_1^{(r)}, x_2^{(r)}, \dots\right\}\right\} &\longrightarrow \mathbb{C}[[\bar{z}_1]] \times \dots \times \mathbb{C}[[\bar{z}_r]], \\ x_i^{(j)} &\longmapsto \bar{z}_j^{(i)}. \end{aligned}$$

One checks that the pair $\left(\mathcal{J}\big(\widehat{C}_V\big), \phi_1\right)$ verifies the Albanese property for $\widehat{C}_V$ (see [3, Section 4]).

Since $\mathbb{C}$ has characteristic zero, we can define the exponential map for the formal Jacobian as follows. Let $\widehat{\mathbb{A}}_\infty$ be the formal group scheme

$$\widehat{\mathbb{A}}_\infty := \varinjlim_n \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_n]]$$

endowed with the additive group law. Let us define $\mathbb{C}\{\{t_1, t_2, \dots\}\}$ as the ring $\mathcal{O}\big(\widehat{\mathbb{A}}_\infty\big) = \varprojlim_n \mathbb{C}[[t_1, \dots, t_n]]$.

Thus, the exponential map is the morphism

$$\begin{aligned} \widehat{\mathbb{A}}^r_\infty \,:=\, \widehat{\mathbb{A}}_\infty \times \overset{r}{\dots} \times \widehat{\mathbb{A}}_\infty &\overset{\exp}{\longrightarrow} \mathcal{J}\big(\widehat{C}_V\big), \\ \left(\left\{a_i^{(1)}\right\}_{i>0}, \dots, \left\{a_i^{(r)}\right\}_{i>0}\right) &\longmapsto \left(\exp\left(\sum_{i>0} \frac{a_i^{(1)}}{z_1^i}\right), \dots, \exp\left(\sum_{i>0} \frac{a_i^{(r)}}{z_r^i}\right)\right) \end{aligned}$$

induced by the following ring homomorphism

$$\begin{aligned} \mathbb{C}\left\{\left\{x_1^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{x_1^{(r)}, \dots\right\}\right\} &\longrightarrow \mathbb{C}\left\{\left\{t_1^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{t_1^{(r)}, \dots\right\}\right\}, \\ 1 \otimes \dots \otimes x_i^{(j)} \otimes \dots \otimes 1 &\longmapsto 1 \otimes \dots \otimes p_i(t^{(j)}) \otimes \dots \otimes 1, \end{aligned}$$

where $t^{(j)} = \left(t_1^{(j)}, t_2^{(j)}, \dots\right)$ and $p_i\big(t^{(j)}\big)$ is the i-th Schur polynomial on $t^{(j)}$; that is, the coefficient of z_j^{-i} in the series $\exp\left(\sum_{k>0} t_k^{(j)} z_j^{-k}\right)$. Obviously, the exponential map is an isomorphism of formal group schemes.

Therefore, henceforth we shall understand that the group $\mathcal{J}\big(\widehat{C}_V\big)$ is the formal spectrum of the ring

$$\mathbb{C}\left\{\left\{t_1^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{t_1^{(r)}, \dots\right\}\right\}$$

and its universal element will be

$$\prod_{i=1}^r \exp\left(\sum_{j\geq 1} \frac{t_j^{(i)}}{z_i^j}\right).$$

3 τ-functions and Baker–Akhiezer functions

Following [17, Section], this section generalizes the notions of τ-functions and Baker–Akhiezer functions and their properties to our situation ([8, 12]).

Let us consider the natural action:

$$\begin{aligned} \mu\colon \Gamma_V \times \mathrm{Gr}(V) &\longrightarrow \mathrm{Gr}(V), \\ (g, U) &\longmapsto g \cdot U \end{aligned}$$

given in [3] and let us define a Poincaré sheaf on $\Gamma_V \times \mathrm{Gr}(V)$ as

$$\mathcal{P} \,:=\, \mu^* \operatorname{Det}_V^* .$$

Then, for each rational point $U \in \mathrm{Gr}(V)$ one has an invertible sheaf on Γ_V given by

$$\widetilde{\mathcal{L}}_\tau(U) \,:=\, \mathcal{P}|_{\Gamma_V \times \{U\}},$$

and a natural homomorphism

$$H^0(\Gamma_V \times \mathrm{Gr}(V), \mathcal{P}) \,\longrightarrow\, H^0\big(\Gamma_V \times \{U\}, \widetilde{\mathcal{L}}_\tau(U)\big). \tag{3.1}$$

Definition 3.1 The *τ-section of the point U*, $\tilde{\tau}_U$, is the image of $\mu^*\Omega_+$ by the homomorphism (3.1). Here, Ω_+ is the canonical global section of Det_V^* defined in Section 2.

To generalize τ-functions, we must restrict our definition to the formal scheme $\mathcal{J}\big(\widehat{C}_V\big) = \Gamma_V^- \subset \Gamma_V$. Let us consider the invertible sheaf on $\mathcal{J}\big(\widehat{C}_V\big)$:

$$\mathcal{L}_\tau(U) \,:=\, \widetilde{\mathcal{L}}_\tau(U)|_{\mathcal{J}(\widehat{C}_V)\times\{U\}},$$

which is trivial. Indeed, a trivialization of $\mathcal{L}_\tau(U)$ can be given by the global section

$$\sigma_0(g) := g \cdot \delta_U, \quad g \in \mathcal{J}(\widehat{C}_V),$$

where δ_U is a non-zero element in the fibre of $\mathcal{L}_\tau(U)$ over the point $(1, U) \in \mathcal{J}(\widehat{C}_V) \times \{U\}$. The τ-function is defined as the trivialization of the restriction of $\tilde{\tau}_U$ to $\mathcal{J}(\widehat{C}_V)$.

Definition 3.2 The *τ-function of the point U*, τ_U, is the function

$$\tau_U(t) \in \mathcal{O}(\mathcal{J}(\widehat{C}_V)) = \mathbb{C}\left\{\left\{t_1^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{t_1^{(r)}, \dots\right\}\right\}$$

defined by

$$\tau_U(t) = \frac{\Omega_+(gU)}{\sigma_0(g)} \quad \text{for} \quad g = \prod_{i=1}^{r} \exp\left(\sum_{j\geq 1} \frac{t_j^{(i)}}{z_i^j}\right) \in \mathcal{J}(\widehat{C}_V).$$

Remark 3.3 Let V^- be the subspace $z_1^{-1}\mathbb{C}[z_1^{-1}] \times \dots \times z_r^{-1}\mathbb{C}[z_r^{-1}]$ and note that $V = V^- \oplus V^+$ and that $V^- \in \operatorname{Gr}(V)$. Let $X \subset \operatorname{Gr}(V)$ denote the orbit of $V^- \subset V$ under the action of Γ_V^+, which acts freely on $\operatorname{Gr}(V)$. Therefore, the bosonization isomorphism $B : \Omega(S) \xrightarrow{\sim} \mathcal{O}(\Gamma_V^+)$ is the composition of the restriction homomorphism

$$H^0\big(\operatorname{Gr}^0(V), \operatorname{Det}_V^*\big) \longrightarrow H^0\big(X, \operatorname{Det}_V^*|_X\big)$$

and the isomorphism $H^0\big(X, \operatorname{Det}_V^*|_X\big) \xrightarrow{\sim} \mathcal{O}(\Gamma_V^+)$ induced by Ω_+.

In order to write down expressions for the Baker–Akhiezer function analogous to the classical ones, let us observe that the composition of the Abel morphism with the exponential map is

$$\begin{array}{ccccc} \widehat{C}_V & \xrightarrow{\phi_1} & \mathcal{J}(\widehat{C}_V) & \xrightarrow{\exp^{-1}} & \widehat{\mathbb{A}}^r_\infty, \\ z_j & \longmapsto & \phi_1(z_j) & \longmapsto & [z_j], \end{array}$$

which maps z_j to the point of $\widehat{\mathbb{A}}^r_\infty$ with coordinates

$$[z_j] := \left((0, \dots), \dots, \left(z_j, \frac{z_j^2}{2}, \frac{z_j^3}{3}, \dots\right), \dots, (0, \dots)\right)$$

or, what amounts to the same, the map ϕ_1 is induced by the ring homomorphism

$$\begin{aligned} \mathbb{C}\left\{\left\{t_1^{(1)}, \dots\right\}\right\} \widehat{\otimes} \dots \widehat{\otimes} \mathbb{C}\left\{\left\{t_1^{(r)}, \dots\right\}\right\} &\longrightarrow \mathbb{C}[[z_1]] \times \dots \times \mathbb{C}[[z_r]], \\ t_i^{(j)} &\longmapsto \left(0, \dots, 0, \frac{z_j^i}{i}, 0 \dots, 0\right). \end{aligned}$$

It follows that there is a natural "addition" morphism

$$\begin{array}{ccc} \widehat{C}_V \times \mathcal{J}(\widehat{C}_V) & \xrightarrow{\beta} & \mathcal{J}(\widehat{C}_V), \\ (z_\bullet, t) & \longmapsto & t + [z_\bullet] \end{array} \tag{3.2}$$

where $z_\bullet = (z_1, \dots, z_r)$, $t = (t^{(1)}, \dots, t^{(r)})$ and $t + [z_\bullet]$ denotes the point of $\widehat{\mathbb{A}}^r_\infty$ with coordinates $\left(\dots, t_i^{(j)} + \frac{z_j^i}{i}, \dots\right)$.

Analogously to [3], one defines the Baker–Akhiezer function of a point $U \in \operatorname{Gr}(V)$ (see also [19]). Recall that $V = \prod V_i$. Following [16], let ϵ_{ui} be -1 for $u > i$ and 1 for $u \leq i$, then

Definition 3.4 The *u-th Baker–Akhiezer function of a point $U \in \mathrm{Gr}(V)$* is the V-valued function

$$\psi_{u,U}(z_{\bullet}, t) \;:=\; \left(\epsilon_{ui}\exp\left(-\sum_{i\geq 1}\frac{t_i^{(1)}}{z_1^i}\right)\frac{\tau_{U_{u1}}(t+[z_1])}{\tau_U(t)}, \dots, \epsilon_{ur}\exp\left(-\sum_{i\geq 1}\frac{t_i^{(r)}}{z_r^i}\right)\frac{\tau_{U_{ur}}(t+[z_r])}{\tau_U(t)}\right)$$

where $1 \leq u \leq r$, $U_{uv} := \big(1,\dots,z_u,\dots,z_v^{-1},\dots,1\big)\cdot U$ and $t+[z_v] := \big(t^{(1)},\dots,t^{(v)}+[z_v],\dots,t^{(r)}\big)$.

Let $\widehat{C}_V^N$ be the formal scheme

$$\widehat{C}_V^N := \mathrm{Spf}\left(\left(V_1^+\right)^{\widehat{\otimes} N}\right)\times\cdots\times\mathrm{Spf}\left(\left(V_r^+\right)^{\widehat{\otimes} N}\right),$$

which is the formal spectrum of the $\mathbb{C}$-algebra

$$\mathcal{O}\big(\widehat{C}_V^N\big) = \big(V_1^+\big)^{\widehat{\otimes} N}\widehat{\otimes}\dots\widehat{\otimes}\big(V_r^+\big)^{\widehat{\otimes} N}.$$

Accordingly, if we denote $(V_i^+)^{\widehat{\otimes} N} = \mathbb{C}[[z_i]]^{\widehat{\otimes} N}$ by $\mathbb{C}\left[\left[x_1^{(i)},\dots,x_N^{(i)}\right]\right]$, we have

$$\mathcal{O}\big(\widehat{C}_V^N\big) \;=\; \mathbb{C}\left[\left[x_1^{(1)},\dots,x_N^{(1)},\dots,x_1^{(r)},\dots,x_N^{(r)}\right]\right]. \tag{3.3}$$

Bearing in mind the correspondence between morphisms $\widehat{C}_V^N \to \mathcal{J}\big(\widehat{C}_V\big)$ and r-tuples of series $(s_1(z_1),\dots, s_r(z_r)) \in \Gamma_V^-\big(\widehat{C}_V^N\big)$ as in Formula (2.3), we define the Abel morphism of degree N as the morphism

$$\phi_N\colon \widehat{C}_V^N \longrightarrow \mathcal{J}\big(\widehat{C}_V\big),$$

corresponding to

$$\left(\prod_{k=1}^N\left(1-\frac{x_k^{(1)}}{z_1}\right)^{-1}, \dots, \prod_{k=1}^N\left(1-\frac{x_k^{(r)}}{z_r}\right)^{-1}\right). \tag{3.4}$$

A closed point of $\mathrm{Gr}(V)$ is a point whose residue field is an extension of $\mathbb{C}$. Accordingly, when dealing with closed points we must consider non-finite extensions of $\mathbb{C}$ since local rings of $\mathrm{Gr}(V)$ do not need to be finitely generated. However, our arguments only make use of the fact that $\operatorname{char}\mathbb{C} = 0$, so they remain valid for any such extension. Therefore, and for the sake of simplicity, we write $\mathbb{C}$.

Lemma 3.5 *Let $U \in \mathrm{Gr}^0(V)$ be a closed point. Let $N > 0$ be an integer number such that $V/\big(V^+ + z_{\bullet}^N U\big) = (0)$. Let*

$$\left\{f_i := \left(f_i^{(1)}(z_1),\dots,f_i^{(r)}(z_r)\right) \mid 1 \leq i \leq N\cdot r\right\}$$

be a basis of $V^+ \cap z_{\bullet}^N U$ as $\mathbb{C}$-vector space such that $f_i^{(j)} \in V_j^+$.

It then holds:

$$\phi_N^*\tau_U \;=\; c\cdot\prod_{\substack{1\leq k<l\leq N\\ 1\leq i\leq r}}\left(x_l^{(i)}-x_k^{(i)}\right)^{-1}\begin{vmatrix} f_1^{(1)}\left(x_1^{(1)}\right) & \dots & f_1^{(r)}\left(x_1^{(r)}\right) & \dots & f_1^{(1)}\left(x_N^{(1)}\right) & \dots & f_1^{(r)}\left(x_N^{(r)}\right)\\ \vdots & & \vdots & & \vdots & & \vdots\\ f_{Nr}^{(1)}\left(x_1^{(1)}\right) & \dots & f_{Nr}^{(r)}\left(x_1^{(r)}\right) & \dots & f_{Nr}^{(1)}\left(x_N^{(1)}\right) & \dots & f_{Nr}^{(r)}\left(x_N^{(r)}\right)\end{vmatrix}$$

as functions of $\mathcal{O}\big(\widehat{C}_V^N\big)$ where c belongs to $\mathbb{C}-\{0\}$.

In the case $V = \mathbb{C}((z))$, this result is deeply connected with Fay's trisecant formula ([10]) and Sato's theory of infinite Grassmann manifolds ([22]).

P r o o f. The proof consists of repeating the arguments of the proof of Lemma 4.6 of [17] but taking into account the decomposition (3.3).

Let A denote the $\mathbb{C}$-algebra $\mathcal{O}\big(\widehat{C}_V^N\big)$ and let g be the element (3.4). Then, $\phi_N^*\tau_U(x)$ is the determinant of the natural map

$$gU \longrightarrow V/V^+ \tag{3.5}$$

(henceforth, we understand that the subspaces $U, V, \dots$ have been tensorialized by A).

Let α_N be the following homomorphism of A-modules:

$$\begin{gathered}\alpha_N \colon V^+ \longrightarrow A^{\oplus N\cdot r} = A^{\oplus r} \times \cdots \times A^{\oplus r},\\ \left(f^{(1)}(z_1), \dots, f^{(r)}(z_r)\right) \longmapsto \left(f^{(1)}\left(x_1^{(1)}\right), \dots, f^{(r)}\left(x_1^{(r)}\right), \dots, f^{(1)}\left(x_N^{(1)}\right), \dots, f^{(r)}\left(x_N^{(r)}\right)\right)\end{gathered}$$

whose kernel is generated by

$$\bar{g} := \left(\prod_{k=1}^{N}\left(z_1 - x_k^{(1)}\right), \dots, \prod_{k=1}^{N}\left(z_r - x_k^{(r)}\right)\right) \in V^+.$$

Consider the following exact sequence of complexes of A-modules (written vertically):

$$\begin{tikzcd}
0 \arrow[r] & \bar{g}\cdot V^+ \arrow[r] \arrow[d, "\beta"'] & V^+ \arrow[r] \arrow[d, "{(\pi,\alpha_N)}"'] & V^+/\bar{g}\cdot V^+ \arrow[r] \arrow[d, "\bar{\alpha}_N"'] & 0\\
0 \arrow[r] & V/z_{\cdot}^N U \arrow[r] & (V/z_{\cdot}^N U) \oplus A^{\oplus N\cdot r} \arrow[r] & A^{\oplus N\cdot r} \arrow[r] & 0
\end{tikzcd}$$

Observe that the complex on the l.h.s. is quasi-isomorphic to the complex (3.5) and that, therefore, $\phi_N^*\tau_U(x)$ equals the determinant of β. Furthermore, it turns out that

$$|\bar{\alpha}_N| = \prod_{\substack{1\le k<l\le N\\ 1\le i\le r}}\left(x_l^{(i)} - x_k^{(i)}\right) \in A.$$

It remains for us to compute the determinant of the complex in the middle. To this end, we consider another exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[r] & V^+\cap z_{\cdot}^N U \arrow[r] \arrow[d, "\alpha_N^U"'] & V^+ \arrow[r] \arrow[d, "{(\alpha_N,\pi)}"'] & V^+/V^+\cap z_{\cdot}^N U \arrow[r] \arrow[d, "\gamma"', "\wr"] & 0\\
0 \arrow[r] & A^{\oplus N\cdot r} \arrow[r] & A^{\oplus N\cdot r}\oplus V/z_{\cdot}^N U \arrow[r] & V/z_{\cdot}^N U \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

The hypothesis implies that the morphism of the complex on the r.h.s. is an isomorphism and that its determinant belongs to $\mathbb{C}^*$. In order to compute $\left|\alpha_N^U\right|$, we consider a basis of the A-module $V^+\cap z_{\cdot}^N U$. We may choose a basis

$$\left\{f_i := \left(f_i^{(1)}(z_1), \dots, f_i^{(r)}(z_r)\right) \mid 1 \le i \le N\cdot r\right\},$$

as in the statement. Thus, the determinant $\left|\alpha_N^U\right|$ is given by

$$\begin{vmatrix} f_1^{(1)}\left(x_1^{(1)}\right) & \dots & f_1^{(r)}\left(x_1^{(r)}\right) & \dots & f_1^{(1)}\left(x_N^{(1)}\right) & \dots & f_1^{(r)}\left(x_N^{(r)}\right)\\ \vdots & & \vdots & & \vdots & & \vdots\\ f_{Nr}^{(1)}\left(x_1^{(1)}\right) & \dots & f_{Nr}^{(r)}\left(x_1^{(r)}\right) & \dots & f_{Nr}^{(1)}\left(x_N^{(1)}\right) & \dots & f_{Nr}^{(r)}\left(x_N^{(r)}\right)\end{vmatrix},$$

and the conclusion follows from the multiplicative behavior of determinants. □

Theorem 3.6 *Let $U \in \mathrm{Gr}^0(V)$. Then*

$$\psi_{u,U}(z_{\bullet}, t) \,=\, (1, \dots, z_u, \dots, 1) \cdot \sum_{i>0} \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) p_{ui,U}(t),$$

where

$$\left\{ \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) \mid i > 0\,,\, 1 \leq u \leq r \right\}$$

is a basis of U, and the $p_{ui,U}(t)$'s are functions in t.

P r o o f. Observe that the canonical morphism $\varinjlim_N \widehat{C}_V^N \to \Gamma_V^-$ is a quotient by a permutation group and that $\widehat{C}_V = \coprod \operatorname{Spf} V_i^+$. Therefore, the first step consists of computing $\psi_{u,U}|_{\operatorname{Spf} V_i^+ \times \widehat{C}_V^N}$ for all $N >> 0$. Note that the canonical morphism

$$\phi_{i,N} \colon \operatorname{Spf} V_i^+ \times \widehat{C}_V^N \xrightarrow{\phi_1 \times \phi_N} \Gamma_V^- \times \Gamma_V^- \longrightarrow \Gamma_V^-$$

is given by the series

$$\left(1, \dots, \left(1 - \frac{\bar{z}_i}{z_i}\right)^{-1}, \dots, 1\right) \cdot g \in V^+ \widehat{\otimes} \mathcal{O}\big(\widehat{C}_V^N\big),$$

where $V_i^+ \simeq \mathbb{C}[[\bar{z}_i]]$ and g is the element (3.4).

Accordingly, the restriction of the Baker–Akhiezer function of U to the product $\operatorname{Spf} V_i^+ \times \widehat{C}_V^N$ is

$$\epsilon_{ui} \cdot g^{(i)}(z_i)^{-1} \cdot \frac{\phi_{i,N}^* \tau_{U_{ui}}}{\phi_N^* \tau_U}.$$

In order to compute $\phi_{i,N}^* \tau_{U_{ui}}$, which coincides with the determinant of

$$z_{\bullet}^N U_u \longrightarrow V/\bar{g}\bar{g}_i V^+$$

($\bar{g}_i$ being $(1, \dots, z_i - \bar{z}_i, \dots, 1)$ and $U_u := (1, \dots, z_u, \dots, 1)U$), we proceed as in the proof of Lemma 3.5. We replace α_N by

$$\begin{aligned} &\alpha_{i,N}\left(f^{(1)}(z_1), \dots, f^{(r)}(z_r)\right) \\ &\quad := \left(f^{(1)}\left(x_1^{(1)}\right), \dots, f^{(r)}\left(x_1^{(r)}\right), \dots, f^{(1)}\left(x_N^{(1)}\right), \dots, f^{(r)}\left(x_N^{(r)}\right), f^{(i)}(\bar{z}_i)\right) \end{aligned}$$

which takes values in $A^{\oplus N \cdot r+1}$. We thus have that

$$|\bar{\alpha}_{i,N}| \,=\, |\bar{\alpha}_N| \cdot \prod_{k=1}^N \left(\bar{z}_i - x_k^{(i)}\right).$$

The analogous of diagram (3.6) reads as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^+ \cap z_{\bullet}^N U_u & \longrightarrow & V^+ & \longrightarrow & V^+/V^+ \cap z_{\bullet}^N U_u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{i,N}^U} & & \downarrow{\scriptstyle (\alpha_{i,N}, \pi)} & & \downarrow{\scriptstyle \bar{\gamma}}\,\wr & & \\ 0 & \longrightarrow & A^{\oplus N \cdot r+1} & \longrightarrow & A^{\oplus N \cdot r+1} \oplus V/z_{\bullet}^N U_u & \longrightarrow & V/z_{\bullet}^N U_u & \longrightarrow & 0 \end{array}$$

It will suffice to calculate the determinant of the following restriction of $\alpha_{i,N}$:

$$\alpha_{i,N}^U \,:\, V^+ \cap z_{\bullet}^N (1, \dots, z_u, \dots, 1)U \,\longrightarrow\, A^{\oplus N \cdot r+1}. \tag{3.7}$$

Consider elements $g_k = \left(g_k^{(1)}, \dots, g_k^{(r)}\right) \in U$ such that $\{g_1, \dots, g_{Nr}\}$ is a basis of $z_{\bullet}^{-N} V^+ \cap U$, $\{g_1, \dots, g_{(N+1)r}\}$ is a basis of $z_{\bullet}^{-N-1} V^+ \cap U$, $\{g_1, \dots, g_{Nr}, g_{Nr+u}\}$ is a basis of $z_{\bullet}^{-N}(1, \dots, z_u^{-1}, \dots, 1)V^+ \cap U$ and the coefficient of z_u^{-1} in $f_{Nr+u}^{(u)}$ is 1. Let $f_k := (-1)^k z_{\bullet}^N g_k$. Then, $\{(1, \dots, z_u, \dots, 1) f_k \mid k = 1, \dots, Nr, Nr+u\}$ is a basis of $V^+ \cap z_{\bullet}^N (1, \dots, z_u, \dots, 1)U$. It is now easy to check that $\frac{|\bar{\gamma}|}{|\gamma|} = (-1)^{u-1}$.

The determinant of the morphism (3.7) is

$$\begin{vmatrix} & & & \bar{z}_i^{\delta_{ui}} f_1^{(i)}(\bar{z}_i) \\ M_1 & \dots & M_N & \vdots \\ & & & \bar{z}_i^{\delta_{ui}} f_{Nr}^{(i)}(\bar{z}_i) \\ & & & \bar{z}_i^{\delta_{ui}} f_{Nr+u}^{(i)}(\bar{z}_i) \end{vmatrix}$$

(observe that $\bar{z}_i^{\delta_{ui}}$ equals $\bar{z}_u$ if $i = u$ and 1 otherwise), where M_j is the $(Nr+1) \times r$–matrix

$$M_j := \begin{pmatrix} f_1^{(1)}\left(x_j^{(1)}\right) & \dots & x_j^{(u)} f_1^{(u)}\left(x_j^{(u)}\right) & \dots & f_1^{(r)}\left(x_j^{(r)}\right) \\ \vdots & & \vdots & & \vdots \\ f_{Nr}^{(1)}\left(x_j^{(1)}\right) & \dots & x_j^{(u)} f_{Nr}^{(u)}\left(x_j^{(u)}\right) & \dots & f_{Nr}^{(r)}\left(x_j^{(r)}\right) \\ f_{Nr+u}^{(1)}\left(x_j^{(1)}\right) & \dots & x_j^{(u)} f_{Nr+u}^{(u)}\left(x_j^{(u)}\right) & \dots & f_{Nr+u}^{(r)}\left(x_j^{(r)}\right) \end{pmatrix}.$$

Substituting, we have that

$$\begin{aligned} \frac{\phi_{i,N}^* \tau_{U_{ui}}}{\phi_N^* \tau_U} &= \frac{\epsilon_{ui} (-1)^{u-1} \bar{z}_i^{\delta_{ui}}}{\prod_{j=1}^N \left(\bar{z}_i - x_j^{(i)}\right)} \left((-1)^{Nr+1} f_{Nr+u}^{(i)}(\bar{z}_i) \prod_{j=1}^N x_j^{(u)} + \sum_{j=1}^{Nr} (-1)^j f_j^{(i)}(\bar{z}_i) \bar{p}_{uj,U}(x) \right) \\ &= \frac{\epsilon_{ui} \bar{z}_i^{\delta_{ui}} \bar{z}_i^{-N}}{\prod_{j=1}^N \left(1 - \frac{x_j^{(i)}}{\bar{z}_i}\right)} \left((-1)^{Nr+u} f_{Nr+u}^{(i)}(\bar{z}_i) \prod_{j=1}^N x_j^{(u)} + \sum_{j=1}^{Nr} f_j^{(i)}(\bar{z}_i) (-1)^{j+u-1} \bar{p}_{uj,U}(x) \right) \\ &= \frac{\epsilon_{ui} \bar{z}_i^{\delta_{ui}}}{\prod_{j=1}^N \left(1 - \frac{x_j^{(i)}}{\bar{z}_i}\right)} \left(g_{Nr+u}^{(i)}(\bar{z}_i) \prod_{j=1}^N x_j^{(u)} + \sum_{j=1}^{Nr} g_j^{(i)}(\bar{z}_i) p_{uj,U}(x) \right), \end{aligned}$$

where the $p_{uj,U}(x)$'s are certain polynomials independent of i.

The statement now follows from the definitions of $\psi_{u,U}$ and g and of the variables t, which are a basis of the ring of symmetric functions on the variables x. The explicit expression is

$$\exp\left(\sum_{j \geq 1} t_j^{(i)} \bar{z}_i^j \right) = \prod_{j \geq 1} \left(1 - \frac{x_j^{(i)}}{\bar{z}_i} \right)^{-1}.$$

□

Recall that in order to define Ω_+^m and Ω_+^{-m} ($m > 0$) we need to choose an element v_m with $\dim_{\mathbb{C}} V^+/v_m V^+ = m$. Let us set v_m as follows:

- for $m \leq \frac{1}{2}(r-n)$, let q, p, s, t be integer numbers defined by $-m = q \cdot (n-r) + p$, $0 \leq p < n-r$, $p = s \cdot r + t$, $0 \leq t < r$. Therefore, we set:

$$v_m := \left(z^{-1} \cdot z_{\bullet}\right)^q z_1^{s+1} \dots z_t^{s+1} z_{t+1}^s \dots z_r^s;$$

- for $m > \frac{1}{2}(r-n)$, we set:

$$v_m := \left(z^{-1} \cdot z_{\bullet}\right) \cdot v_{r-n-m}^{-1}.$$

Theorem 3.7 *Let $U \in \mathrm{Gr}^m(V)$. It holds:*

$$\psi_{u,U}(z_\bullet, t) \;=\; v_m^{-1} \cdot (1, \dots, z_u, \dots, 1) \cdot \sum_{i>0} \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) p_{ui,U}(t),$$

where

$$\left\{ \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) \mid i > 0,\; 1 \le u \le r \right\}$$

is a basis of U and the $p_{ui,U}(t)$'s are functions in t. In particular, an element of V lies in U if and only if it can be expressed as a linear combination of $\psi_{1,U}(z_\bullet, t), \dots, \psi_{r,U}(z_\bullet, t)$ for certain values of the parameter t.

P r o o f. The commutative diagram

$$\begin{tikzcd} v_m^{-1}U \arrow[r] \arrow[d, "\simeq"'] & V/V^+ \arrow[d, "\simeq"'] \\ U \arrow[r] & V/v_mV^+ \end{tikzcd}$$

shows that $\tau_{v_m^{-1}U}(g) = \tau_U(g)$. We thus have

$$\begin{aligned} \psi_{u,U}(g,z) &= \psi_{u,v_m^{-1}U}(g,z) \\ &= (1, \dots, z_u, \dots, 1) \cdot \sum_{i>0} \left(\psi_{u,v_m^{-1}U}^{(i,1)}(z_1), \dots, \psi_{u,v_m^{-1}U}^{(i,r)}(z_r)\right) p_{ui,v_m^{-1}U}(t) \\ &= v_m^{-1}(1, \dots, z_u, \dots, 1) \cdot \sum_{i>0} \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) p_{ui,U}(t), \end{aligned}$$

where $\psi_{u,U}^{(i,j)}(z_j) := v_m \cdot \psi_{u,v_m^{-1}U}^{(i,j)}(z_j)$. In particular, observe that

$$\left\{ \left(\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)\right) \mid i > 0,\; 1 \le u \le r \right\}$$

is a basis of U. □

The above results allow us to prove a generalization of the bilinear identities of the KP-hierarchy.

Since V is a finite separable $\mathbb{C}((z))$-algebra, it carries the metric of the trace $\mathrm{Tr} : V \times V \to \mathbb{C}((z))$ is non-degenerate. Therefore, V can be endowed with the non-degenerate pairing

$$\begin{aligned} T_2\colon V \times V &\longrightarrow \mathbb{C}, \\ (a,b) &\longmapsto \mathrm{Res}_{z=0}(\mathrm{Tr}(a,b))dz. \end{aligned}$$

Lemma 3.8 *The pairing T_2 gives rise to an isomorphism of $\mathbb{C}$-schemes:*

$$\begin{aligned} R\colon \mathrm{Gr}(V) &\longrightarrow \mathrm{Gr}(V), \\ U &\longmapsto U^\perp, \end{aligned}$$

where $U^\perp$ denotes the orthogonal of U w.r.t. T_2.

P r o o f. The proof easily reduces to the $r = 1$ case; that is, $V = \mathbb{C}\big(\big(z^{1/e}\big)\big)$. In this situation, the metric given by the trace is

$$\mathrm{Tr}(z^{i/e}, z^{j/e}) = \begin{cases} e & \text{if} \quad i+j=0, \\ ez & \text{if} \quad i+j=e, \qquad \text{for} \quad 0 \le i,j < e. \\ 0 & \text{otherwise}, \end{cases}$$

Taking into account that V is a $\mathbb{C}((z))$-algebra, one has that

$$T_2\big(z^{i/e}, z^{j/e}\big) \;=\; \begin{cases} e & \text{if} \quad i+j=-e, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for} \quad i,j\in\mathbb{Z}.$$

A straightforward calculation shows that $U^{\perp}$ belongs to $\mathrm{Gr}(V)$ for any $U\in\mathrm{Gr}(V)$ ([17, Section 5]). □

It is worth pointing out the following identities:

$$R(\mathrm{Gr}^m(V)) = \mathrm{Gr}^{r-n-m}(V),$$
$$R^*\,\mathrm{Det}_V \simeq \mathrm{Det}_V,$$

and $(g\cdot U)^{\perp} = g^{-1}\cdot U^{\perp}$ for $U\in\mathrm{Gr}(V)$ and $g\in\mathcal{J}\big(\widehat{C}_V\big)$. Further, for $m\neq\frac12(r-n)$ and $U\in\mathrm{Gr}^m(V)$, it holds:

$$R^*\Omega_+^m = \Omega_+^{r-n-m},$$
$$\tau_{U^\perp}(g) = \tau_U(g^{-1}).$$

Remark 3.9 The latter two identities also hold for $m=\frac12(r-n)$ whenever one can take v_m such that $v_m^2 = z^{-1}\cdot z_{\bullet}$ (e.g. when e_i is odd for all i). Since this is not possible in general, we shall omit this case. However, although with different explicit expressions, our techniques can be applied to it.

Definition 3.10 The *u-th adjoint Baker–Akhiezer function of a point* $U\in\mathrm{Gr}(V)$ is defined by

$$\psi^*_{u,U}(z_\bullet,t) \;:=\; \psi_{u,U^\perp}(z_\bullet,-t).$$

Note that one has the following identity:

$$\psi^{*(j)}_{u,U}(z_j,t) \;=\; \epsilon_{uj}\exp\left(\sum_{i\geq1}\frac{t_i^{(j)}}{z_j^i}\right)\frac{\tau_{U_ju}(t-[z_j])}{\tau_U(t)}.$$

Theorem 3.11 (Bilinear Identity) *Let $U,U'\in\mathrm{Gr}^m(V)$ $\big(m\neq\frac12(r-n)\big)$ be two rational points of the same index.*

Then, $U=U'$ if and only if the following conditions hold:

$$T_2\left(\frac{z_\bullet\cdot\psi_{u,U}(z_\bullet,t)}{(1,\dots,z_u,\dots,1)},\frac{\psi^*_{v,U'}(z_\bullet,t')}{z(1,\dots,z_v,\dots,1)}\right) \;=\; 0, \quad 1\leq u,v\leq r.$$

P r o o f. Since T_2 is non-degenerate we know that a vector $w\in V$ lies in $U'\in\mathrm{Gr}^m(V)$ if and only if

$$T_2\left(w,\frac{v_{r-n-m}\psi^*_{v,U'}(z_\bullet,t')}{(1,\dots,z_v,\dots,1)}\right) = 0, \quad 1\leq v\leq r.$$

Recalling that $v_mv_{r-n-m}=z^{-1}z_\bullet$ and the properties of the trace, the conclusion follows from Theorem 3.7. □

Corollary 3.12 *Let U,U' be two rational points of $U,U'\in\mathrm{Gr}^m(V)$ $\big(m\neq\frac12(r-n)\big)$. Then, $U=U'$ if and only if*

$$\sum_{i=1}^r\mathrm{Res}_{z=0}\left(\sum_{j=1}^{e_i}\left(\xi_i^jz^{1/e_i}\right)^{1-\delta_{iu}-\delta_{iv}}\psi^{(i)}_{u,U}\left(\xi_i^jz^{1/e_i},t\right)\psi^{*(i)}_{v,U'}\left(\xi_i^jz^{1/e_i},t'\right)\right)\frac{dz}{z} \;=\; 0$$

for all $1\leq u,v\leq r$ (ξ_i is a primitive e_i-th root of 1 in $\mathbb{C}$).

Proof. It suffices to make explicit the condition of the previous theorem. The very definition of the metric T_2 yields

$$\sum_{i=1}^{r} \operatorname{Res}_{z=0}\left(\operatorname{Tr}\left(z_i^{1-\delta_{iu}-\delta_{iv}}\psi_{u,U}^{(i)}(z_i,t)\psi_{v,U'}^{*(i)}(z_i,t')\right)\right)\frac{dz}{z},$$

and the claim follows since the trace map of V as a $\mathbb{C}((z))$-algebra is given by

$$\begin{aligned}\operatorname{Tr}\colon V = V_1\times\cdots\times V_r &\longrightarrow \mathbb{C}((z))\\ (f_1(z_1),\dots,f_r(z_r)) &\longmapsto \sum_{j=1}^{e_1} f_1\left(\xi_1^j z^{1/e_1}\right)+\cdots+\sum_{j=1}^{e_r} f_r\left(\xi_r^j z^{1/e_r}\right).\end{aligned}$$

□

This set of equations is equivalent to a set of differential equations for the τ-functions.

Definition 3.13 Let $\underline{n}$ denote the partition of n given by $\{e_1,\dots,e_r\}$. The $\underline{n}$*-KP hierarchy* is the following set of equations

$$\sum_{i=1}^{r} \operatorname{Res}_{z=0}\left(\sum_{j=1}^{e_i}\left(\xi_i^j z^{1/e_i}\right)^{1-\delta_{iu}-\delta_{iv}}\psi_{u,U}^{(i)}\left(\xi_i^j z^{1/e_i},t\right)\psi_{v,U}^{*(i)}\left(\xi_i^j z^{1/e_i},t'\right)\right)\frac{dz}{z}=0$$

for all $1\le u,v\le r$ (ξ_i is a primitive e_i-th root of 1 in $\mathbb{C}$).

It would be interesting to compare the other hierarchies with those given in [6], which are expressed in terms of pseudodifferential operators and representation theory of infinite dimensional Lie algebras.

3.1 Subschemes of the Grassmannian

For each subset $i_{\cdot} = \{i_1,\dots,i_s\}$ of $\{1,\dots,r\}$, let us denote by $V_{i_\cdot}$ the vector space $\prod_{j\in i_\cdot} V_j$, $(V_{i_\cdot})^+ = \prod_{j\in i_\cdot} V_j^+$, by $V^{i_\cdot}$ the vector space $\prod_{j\notin i_\cdot} V_j$ and by $(V^{i_\cdot})^+ = \prod_{j\notin i_\cdot} V_j^+$. One can consider the following morphism

$$\begin{aligned}\operatorname{Gr}(V_{i_\cdot})\times\operatorname{Gr}(V^{i_\cdot}) &\xrightarrow{\gamma_{i_\cdot}} \operatorname{Gr}(V),\\ (W,W') &\longmapsto W\times W'\subset V.\end{aligned}\tag{3.8}$$

Definition 3.14 A subspace U is said tobe *decomposable* if it lies in the image of $\gamma_{i_\cdot}$ for some $i_\cdot$. That is, there exists a subset $i_\cdot$ and subspaces $W\in\operatorname{Gr}(V_{i_\cdot})$, $W'\in\operatorname{Gr}(V^{i_\cdot})$ such that $U=W\times W'$.

The *decomposable Grassmannian* of V is the subscheme of $\operatorname{Gr}(V)$ whose points are the decomposable subspaces, that is,

$$\operatorname{Gr}^{\text{dec}}(V) = \bigcup_{i_\cdot\subsetneq\{1,\dots,r\}} \operatorname{Im}\gamma_{i_\cdot}$$

($\operatorname{Im}\gamma_{i_\cdot}$ being the scheme-theoretic image of $\gamma_{i_\cdot}$).

Proposition 3.15 *The morphism* (3.8) *is a closed immersion for any* $i_\cdot$*. In particular,* $\operatorname{Gr}^{dec}(V)$ *is a closed subscheme of* $\operatorname{Gr}(V)$.

Proof. The map is clearly injective. If U denotes an S-valued point of $\operatorname{Gr}(V)(S)$, one has to show that the subset of S of those $s\in S$ such that U_s decomposes as a product of subspaces is a closed subscheme of S.

Let $p_{i_\cdot}$ (resp. $p^{i_\cdot}$) denote the projection $V\to V_{i_\cdot}$ (resp. $V\to V^{i_\cdot}$). Then, there is a natural injective morphism

$$U\longrightarrow p_{i_\cdot}(U)\times p^{i_\cdot}(U)\subseteq V_{i_\cdot}\times V^{i_\cdot}=V.$$

The desired subset consists exactly of those points where $p_{i_\cdot}(U)\times p^{i_\cdot}(U)\subseteq U$, which is a closed subscheme. □

One can compute explicitly the equations of all these closed subschemes of $\operatorname{Gr}(V)$:

Theorem 3.16 *Let U be a closed point of* $\mathrm{Gr}^m(V)$ $\big(m \neq \frac{1}{2}(r-n)\big)$. *It holds:*

1. $U \in \operatorname{Im} \gamma_{i.}$ *if and only if its Baker–Akhiezer function satisfies*

$$\operatorname{Res}_{z=0}\left(\sum_{i\in i.} \operatorname{Tr}\left(z_i^{1-\delta_{iu}-\delta_{iv}} \psi^{(i)}_{u,U}(z_i,t)\psi^{*(i)}_{v,U}(z_i,t')\right)\right)\frac{dz}{z} = 0$$

   for all $1 \le u, v \le r$.

2. $U \in \mathrm{Gr}^{dec}(V)$ *if and only if its Baker–Akhiezer function satisfies*

$$\prod_{i.\subsetneq\{1,\dots,r\}} \operatorname{Res}_{z=0}\left(\sum_{i\in i.} \operatorname{Tr}\left(z_i^{1-\delta_{iu}-\delta_{iv}} \psi^{(i)}_{u,U}(z_i,t)\psi^{*(i)}_{v,U}(z_i,t')\right)\right)\frac{dz}{z} \;=\; 0$$

   for all $1 \le u, v \le r$.

P r o o f. This follows from Theorem 3.11 and the Bilinear Identity for the KP hierachy. □

4 Algebro-geometric points of $\mathrm{Gr}(V)$

The goal of this section consists of defining a subscheme of $\mathrm{Gr}(V)$ representing the Hurwitz functor of pointed coverings with formal parameters at the marked points. Similar tasks have been carried out in the literature (see [2, 13, 14, 17]) where the set of points of $\mathrm{Gr}(V)$ defined by certain geometric data has been characterized.

In fact, we restrict ourselves to the following type of coverings. Let $\pi : Y \to X$ be a finite morphism between proper curves over $\mathbb{C}$ where Y is reduced and X integral. Let $x \in X$ be a smooth point. Define $A := H^0(X - x, \mathcal{O}_X)$, $B := H^0(Y - \pi^{-1}(x), \mathcal{O}_Y)$, Σ_X to be the function field of A and Σ_Y to be the total quotient ring of B. Let Tr denote the trace of Σ_Y as a finite Σ_X-algebra.

The triple (Y, X, x) is said to have the *property* $(*)$ when $\mathrm{Tr}(B) \subseteq A$.

Let us observe that every covering $\pi : Y \to X$ with either X smooth or π flat has the property $(*)$.

From now on, we set the numerical invariants n and $\underline{n} = \{e_1, \dots, e_r\}$ (with $e_1 + \dots + e_r = n$), which define the $\mathbb{C}((z))$-algebra V.

Definition 4.1 The *Hurwitz functor* $\underline{\mathcal{H}}^\infty$ of pointed coverings of curves of degree n with a fibre of type $(e_1, \dots, e_r)$ and a formal parameter along the fibre is the contravariant functor on the category of $\mathbb{C}$-schemes

$$\underline{\mathcal{H}}^\infty\colon\ \{\text{category of}\mathbb{C}-\text{schemes}\} \rightsquigarrow \{\text{category of sets}\}\,,$$

that associates with a $\mathbb{C}$-scheme S the set of equivalence classes of data $\big\{Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big\}$, where:

1. $p_Y : Y \to S$ is a proper and flat morphism whose fibres are geometrically reduced curves.
2. $p_X : X \to S$ is a proper and flat morphism whose fibres are geometrically integral curves.
3. $\pi : Y \to X$ is a finite morphism of S-schemes of degree n such that its fibres over closed points $s \in S$ have the property $(*)$.
4. $x : S \to X$ is a section of p_X such that the divisor $x(s)$ is a smooth point of $X_s := p_X^{-1}(s)$ for all closed points $s \in S$.
5. $\bar{y} = \{y_1, \dots, y_r\}$ is a set of disjoint smooth sections of p_Y such that the Cartier divisor $\pi^{-1}(x(S))$ is $e_1 y_1(S) + \dots + e_r y_r(S)$.
6. For all closed point $s \in S$ and each irreducible component of the fibre Y_s, there is at least one point $y_j(s)$ lying on that component.
7. t_x is a formal parameter along $x(S)$:

$$t_x\colon \widehat{\mathcal{O}}_{X,x(S)} \xrightarrow{\sim} \mathcal{O}_S[[z]].$$

8. $t_{\bar{y}} = \big\{t_{y_1}, \dots, t_{y_r}\big\}$ are formal parameters along $y_1(S), \dots, y_r(S)$ such that $\pi^*(t_x)_{y_j(S)} = t_{y_j}^{e_j}$.

9. $\{Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\}$ and $\{Y', X', \pi', x', \bar{y}', t_{x'}, t_{\bar{y}'}\}$ are said to be equivalent when there is a commutative diagram of S-schemes

$$\begin{tikzcd} Y \arrow[r, "\sim"] \arrow[d, "\pi"'] & Y' \arrow[d, "\pi'"] \\ X \arrow[r, "\sim"] & X' \end{tikzcd}$$

compatible with all the data.

Now, a Krichever morphism can be defined for this functor as the morphism of functors

$$K\colon \underline{\mathcal{H}}^{\infty}(e_1, \dots, e_r) \longrightarrow \underline{\mathrm{Gr}}(V) \tag{4.1}$$

given by

$$K\big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big) \;=\; t_{\bar{y}}\left(\varinjlim_i\, (p_Y)_* \mathcal{O}_Y(i\pi^{-1}(x))\right) \;\subset\; V\hat{\otimes}_{\mathbb{C}}\mathcal{O}_S$$

where $\mathcal{O}_Y\big(i\pi^{-1}(x)\big)$ is the sheaf associated with the Cartier divisor $i\pi^{-1}(x)$ and $t_{\bar{y}}$ is understood to be the isomorphism induced by

$$\widehat{\mathcal{O}}_{Y,y_1(S)} \times \cdots \times \widehat{\mathcal{O}}_{Y,y_r(S)} \;\simeq\; \mathcal{O}_S[[z_1]] \times \cdots \times \mathcal{O}_S[[z_r]] \;\simeq\; V^+\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S.$$

Note that for a closed point $\big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big) \in \underline{\mathcal{H}}^{\infty}(e_1, \dots, e_r)$ these definitions yield

$$K\big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big) = t_{\bar{y}}\big(H^0(Y - \pi^{-1}(x), \mathcal{O}_Y)\big) \;\subset\; V.$$

Let $\mathcal{M}^{\infty}(r)$ be the moduli scheme representing the classes of sets of data $(Y; y_1, \dots, y_r; t_1, \dots, t_r)$ of geometrically reduced curves with r marked pairwise distinct smooth points $\{y_1, \dots, y_r\}$ and formal parameters $\{t_1, \dots, t_r\}$ at these points and such that any irreducible component contains at least one of the marked points. Following the arguments of [17] for $\mathcal{M}^{\infty}(1)$, we can prove that the Krichever morphism induces a closed immersion

$$\begin{aligned} \mathcal{M}^{\infty}(r) &\overset{K}{\hookrightarrow} \mathrm{Gr}(V), \\ (Y; y_1, \dots, y_r; t_1, \dots, t_r) &\longmapsto t_{\bar{y}}\big(H^0(Y - \{y_1, \dots, y_r\}, \mathcal{O}_Y)\big) \subset V, \end{aligned}$$

whose image is characterized by the following

Theorem 4.2 *A point $U \in \mathrm{Gr}(V)(S)$ lies in $K(\mathcal{M}^{\infty}(r))$ if and only if $U \cdot U \subseteq U$ and $\mathcal{O}_S \subseteq U$, where $\cdot$ denotes the product of V.*

Proof. The direct proof is trivial. Let us prove the converse. Consider the filtration of $V = V_1 \times \cdots \times V_r = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_r))$ defined by

$$\dots \subset V(m-1) \subset V(m) \subset V(m+1) \subset \dots$$

where $V(m) := z^{-m}V^+$.

Accordingly, any point $U \in \mathrm{Gr}(V)(S)$ ($S = \mathrm{Spec}(R)$ being a $\mathbb{C}$-scheme) carries a natural filtration $\{U(m) := U \cap V(m)\}_{m \geq 0}$. Let us denote by $\mathcal{U}$ the corresponding graded R-module. If U satisfies $U \cdot U \subseteq U$ and $R \subseteq U$, then $\mathcal{U}$ is also a graded R-algebra.

It is easy to check that $Y = \mathrm{Proj}\,\mathcal{U}$ is an algebraic curve over $\mathrm{Spec}(R)$. Observe that the filtrations induced by $z_1, \dots, z_r$ give rise to pairwise disjoint sections of Y (smooth and of degree 1). The other geometric data are constructed using the same arguments as in the proof of Theorem 6.4 of [17]. □

Let us denote the trace map of the separable $\mathbb{C}((z))$-algebra V by

$$\mathrm{Tr}\colon V \longrightarrow \mathbb{C}((z)),$$

which is a $\mathbb{C}((z))$-linear map. For a point $U \in \mathrm{Gr}(V)$, let us denote by $\mathrm{Tr}(U) \subseteq \mathbb{C}((z))$ the image of U under the trace map.

Note that for a point $\mathcal{Y} := \big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big)$ of $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$, we have a commutative diagram

$$\begin{tikzcd} B \arrow[r, hook] & V \\ A \arrow[u, hook] \arrow[r, hook] & \mathbb{C}((z)) \arrow[u, hook] \end{tikzcd}$$

where $B := K(\mathcal{Y}) = t_{\bar{y}}\big(H^0(Y - \pi^{-1}(x), \mathcal{O}_Y)\big)$ and $A := t_x\big(H^0(X - x, \mathcal{O}_X)\big)$. Thus, the trace of V as a $\mathbb{C}((z))$-algebra restricted to B coincides with the trace of Σ_Y as a Σ_X-algebra restricted to B. Therefore, both trace maps will be denoted by Tr.

Lemma 4.3 *Let $\mathcal{B}$ be a S-valued point of $\mathrm{Gr}(V)$ such that $\mathrm{Tr}(\mathcal{B}) \subseteq \mathcal{B}$. Then*

$$\mathrm{Tr}(\mathcal{B}) \in \mathrm{Gr}(\mathbb{C}((z)))(S)$$

and $\mathrm{Tr}(\mathcal{B}) = \mathcal{B} \cap \mathcal{O}_S((z))$.

Proof. By the local nature of the hypotheses, we may assume that S is affine, $S = \operatorname{Spec} R$. Since $\mathcal{B} \in \mathrm{Gr}(V)$, there exists m such that $\widehat{V}_S/\big(\mathcal{B} + z^m \cdot \widehat{V}_S^+\big) = (0)$ and that $\mathcal{B} \cap z^m \cdot \widehat{V}_S^+$ is a free R-module of finite rank.

Observe that $\mathrm{Tr}(\mathcal{B})$ is quasicoherent and that $\mathrm{Tr}(\mathcal{B})_s \subset \widehat{V}_s$ for all closed point $s \in S$. So, in order to show that $\mathrm{Tr}(\mathcal{B}) \in \mathrm{Gr}(\mathbb{C}((z)))$ it suffices to check that $R((z))/(\mathrm{Tr}(\mathcal{B}) + z^m \cdot R[[z]]) = (0)$ and that $\mathrm{Tr}(\mathcal{B}) \cap z^m \cdot R[[z]]$ is free of finite rank (see [3]).

For the first claim, note that the trace gives rise to a surjection

$$\widehat{V}_S/\big(\mathcal{B} + z^m \cdot \widehat{V}_S^+\big) \xrightarrow{\frac{1}{n}\mathrm{Tr}} R((z))/(\mathrm{Tr}(\mathcal{B}) + z^m \cdot R[[z]]).$$

The second claim follows from the fact that the composition

$$\mathrm{Tr}(\mathcal{B}) \cap z^m \cdot R[[z]] \hookrightarrow \mathcal{B} \cap z^m \cdot \widehat{V}_S^+ \xrightarrow{\frac{1}{n}\mathrm{Tr}} \mathrm{Tr}(\mathcal{B}) \cap z^m \cdot R[[z]]$$

is the identity map because $\mathrm{Tr}\left(z^m \cdot \widehat{V}_S^+\right) = z^m\,\mathrm{Tr}\left(\widehat{V}_S^+\right) = z^m \cdot R[[z]]$.

The second part of the statement follows easily from the $R((z))$-linearity of Tr and from the hypothesis $\mathrm{Tr}(\mathcal{B}) \subseteq \mathcal{B}$. $\square$

Lemma 4.4 *Let $\mathcal{Y} := \big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big)$ be an S-valued point of $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$. It holds:*

$$K(X, x, t_x) = K(\mathcal{Y}) \cap \mathcal{O}_S((z)) = \mathrm{Tr}(K(\mathcal{Y})).$$

Proof. As in the previous lemma we may assume that S is affine, $S = \operatorname{Spec} R$. For the point $\mathcal{Y}$, define $\mathcal{B} := K(\mathcal{Y}) = t_{\bar{y}}\big(H^0(Y - \pi^{-1}(x), \mathcal{O}_Y)\big)$ and $\mathcal{A} := K(X, x, t_x) = t_x\big(H^0(X - x, \mathcal{O}_X)\big)$.

From the commutative diagram

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook] & \widehat{V}_S \\ \mathcal{A} \arrow[u, hook] \arrow[r, hook] & R((z)) \arrow[u, hook] \end{tikzcd}$$

one has that $\mathcal{A} \subseteq \mathcal{B} \cap R((z))$. The inclusion $\mathcal{B} \cap R((z)) \subseteq \mathcal{B}$ implies that $\mathrm{Tr}(\mathcal{B} \cap R((z))) \subseteq \mathrm{Tr}(\mathcal{B})$. Bearing in mind that Tr is $R((z))$-linear, one concludes that $\mathcal{B} \cap R((z)) \subseteq \mathrm{Tr}(\mathcal{B})$. Summing up, we have proved that

$$\mathcal{A} \subseteq \mathcal{B} \cap R((z)) \subseteq \mathrm{Tr}(\mathcal{B}).$$

Since $\mathcal{B}$ is a finite $\mathcal{A}$-module, so is $\mathrm{Tr}(\mathcal{B})$. Bearing in mind the compatibility of the trace w.r.t. base changes, the above inclusion implies that $\mathcal{A}_s \subseteq \mathrm{Tr}(\mathcal{B})_s$ for all closed points $s \in S$. Now, recalling that the data $\mathcal{Y}$ satisfies the property $(*)$ at closed points, one has that $\mathrm{Tr}(\mathcal{B})_s \subseteq \mathcal{A}_s$ for all s and that, therefore,

$$\mathcal{A} = \mathcal{B} \cap R((z)) = \mathrm{Tr}(\mathcal{B}).$$

□

Theorem 4.5 *The Krichever morphism* (4.1) *is injective.*

P r o o f. We shall keep the same notations as in the previous lemma. It suffices to show that the geometric data $\mathcal{Y} := \big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big)$ can be recovered from the point $\mathcal{B} = K(\mathcal{Y}) \in \mathrm{Gr}(V)$. Observe that $\mathcal{B}$ determines the data $\big(Y, \bar{y}, t_{\bar{y}}\big)$ uniquely.

Lemma 4.4 shows that

$$\mathcal{A} = \mathrm{Tr}(\mathcal{B}) = \mathcal{B} \cap \mathcal{O}_S((z)) \in \mathrm{Gr}(\mathbb{C}((z)))(S),$$

and note that this is an $\mathcal{O}_S$-subalgebra of $\mathcal{O}_S((z))$ because $\mathcal{B}$ is an $\mathcal{O}_S$-subalgebra of $V\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S$.

Theorem 4.2 shows that $\mathcal{A} = \mathrm{Tr}(\mathcal{B})$ corresponds to the point (X, x, t_x) of $\mathcal{M}^\infty(1)(S)$. Since the inclusion $\mathrm{Tr}(\mathcal{B}) \subseteq \mathcal{B}$ is compatible with the filtrations induced by those of $\mathbb{C}((z))$ and V, respectively, it gives rise to a morphism $\pi : Y \to X$, which is finite and satisfies $\pi^{-1}(x) = \bar{y}$ and $(\pi^* t_x)_{y_j} = t_{y_j}^{e_j}$. □

All the above results allow us to give characterizations of the functor $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$ as a subset of $\mathcal{M}^\infty(r) \subset \underline{\mathrm{Gr}}(V)$.

Corollary 4.6 *Let $\mathcal{B}$ be an S-valued point of $\underline{\mathcal{M}}^\infty(r)$. The point $\mathcal{B}$ belongs to $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)(S)$ if and only if*

$$\mathcal{B} \cap \mathcal{O}_S((z)) \in \underline{\mathrm{Gr}}(\mathbb{C}((z)))(S).$$

P r o o f. The first part of the proof follows from Lemma 4.4. The converse is a consequence of the proof of Theorem 4.5. □

Theorem 4.7 *Let $\mathcal{B} \in \mathcal{M}^\infty(r) \subset \mathrm{Gr}(V)$ be an S-valued point. The following conditions are equivalent:*

1. $\mathcal{B} \in \underline{\mathcal{H}}^\infty(e_1, \dots, e_r)(S)$.
2. $\mathrm{Tr}(\mathcal{B}) \subseteq \mathcal{B}$.

P r o o f. (1) ⇒ (2) is a consequence of Lemma 4.4.
(2) ⇒ (1) follows from Lemma 4.3 and Corollary 4.6. □

Remark 4.8 Our approach to the Hurwitz functor is closely related to those given in [2, 6, 13]. Let us review their approaches in a very concise way. Those authors study the moduli space of coverings carrying a line bundle on the curve upstairs; more precisely, the geometric data $\big(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}, L, \phi\big)$, where the first data are as in our approach and the pair (L, ϕ) consists of a line bundle on Y endowed with an isomorphism $\widehat{L}_{\bar{y}} \simeq \widehat{\mathcal{O}}_{\bar{y}}$. To such data one associates a point in the infinite grassmannian as follows:

$$H^0(Y - \bar{y}, L) = H^0(X - x, \pi_* L) \in \mathrm{Gr}\left(\mathbb{C}((z))^{\oplus n}\right).$$

In order to recover the geometric data from such a point one considers the action of the Heisenberg group on $\mathrm{Gr}\left(\mathbb{C}((z))^{\oplus n}\right)$. Then, roughly said, one shows that $Y - \bar{y}$ is the spectrum of the stabilizer and that $X - x$ is the spectrum of the intersection of the stabilizer with $\mathbb{C}((z))$ (as diagonal matrices). Another important result is that since we are given points of $\mathrm{Gr}\left(\mathbb{C}((z))^{\oplus n}\right)$ these points are solutions of the multicomponent KP hierarchy corresponding to the flows of the Heisenberg group.

Let us now compare these approaches with the present work. In our setting, the role of the Heisenberg group of a partition $\underline{n}$ is encoded in the $\mathbb{C}((z))$-algebra structure of V, since we are taking $V = \mathbb{C}\big(\big(z^{1/e_1}\big)\big) \times \cdots \times \mathbb{C}\big(\big(z^{1/e_r}\big)\big)$ and, in this way, the Bilinear Identity of Theorem 3.11 is related to $\underline{n}$-KP-hierarchy (see [6]).

However, the multicomponent KP-hierarchy alone does not allow one to characterize the image of the Krichever map since this problem requires more hierarchies. While Adams–Bergvelt and Li–Mulase characterize the above geometric data (with line bundles) in terms of stabilizers, finite-dimensional orbits, and Schur pairs, we characterize coverings (i.e. with no line bundle or, equivalently, $(L, \phi) = (\mathcal{O}_Y, t_{\bar{y}})$) in terms of the trace map (Theorem 4.7). Such a characterization, together with results of [17], allow us to derive the hierarchies that solve the characterization problem.

Theorem 4.9 *The functor $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$ is representable by a closed subscheme $\mathcal{H}^\infty(e_1, \dots, e_r)$ of $\mathrm{Gr}(V)$.*

P r o o f. Recall that Theorem 4.5 states that $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$ is a subfunctor of $\underline{\mathcal{M}}^\infty(r)$. Our task consists in proving that it is a closed subfunctor, since $\mathcal{M}^\infty(r)$ is closed in $\mathrm{Gr}(V)$ ([17]).

Theorem 4.7 reduces the proof to checking that for $\mathcal{B} \in \mathcal{M}^\infty(r)(S)$ the condition $\mathrm{Tr}(\mathcal{B}) \subseteq \mathcal{B}$ is fulfilled on a closed subscheme of S. Recall that such a condition is closed because $\mathcal{B} \in \mathrm{Gr}(V)(S)$ and $\mathrm{Tr}(\mathcal{B})$ is quasi-coherent. $\square$

Remark 4.10 Note that the set of points of $\mathcal{H}^\infty(e_1, \dots, e_r)$ corresponding to coverings where the curve upstairs is not connected is given by the intersection $\mathcal{H}^\infty(e_1, \dots, e_r) \cap \mathrm{Gr}^{\mathrm{dec}}(V)$ (see Definition 3.14).

Theorem 4.11 *Let $B \in \mathcal{M}^\infty(r) \subset \mathrm{Gr}^m(V)$ $\big(m \neq \frac{1}{2}(r-n)\big)$ be a closed point. Let $u_1, \dots, u_r$ be integer numbers defined by $v_m = z_1^{u_1} \dots z_r^{u_r}$.*

Then, $B \in \mathcal{H}^\infty(e_1, \dots, e_r)$ if and only if the following "bilinear identity"

$$\mathrm{Res}_{z=0}\left(\left(\sum_{j=1}^{r}\sum_{i=1}^{e_j}\frac{\psi_{u,B}^{(j)}\big(\xi_j^i z^{1/e_j}, t\big)}{\big(\xi_j^i z^{1/e_j}\big)^{\delta_{uj}-u_j}}\right)\cdot\left(\sum_{j=1}^{r}\sum_{i=1}^{e_j}\frac{\psi_{v,B}^{*(j)}\big(\xi_j^i z^{1/e_j}, s\big)}{\big(\xi_j^i z^{1/e_j}\big)^{u_j-1+\delta_{vj}}}\right)\right)\frac{dz}{z} = 0$$

is satisfied for all $1 \le u, v \le r$.

P r o o f. Let us observe that from the bilinear identity given in Theorem 3.11, the condition $\mathrm{Tr}(B) \subseteq B$ is equivalent to the condition

$$T_2\left(\mathrm{Tr}\left(\frac{v_m \psi_{u,B}(z_\bullet, t)}{(1, \dots, z_u, \dots, 1)}\right), \frac{v_{r-n-m}\psi^*_{v,B}(z_\bullet, s)}{(1, \dots, z_v, \dots, 1)}\right) = 0$$

for all $1 \le u, v \le r$. Recalling the definition of T_2 and the explicit expression of the trace of V as a $\mathbb{C}((z))$-algebra, one concludes the statement. $\square$

Remark 4.12 Let us observe that the bilinear identities do not characterize the points of $\mathcal{H}^\infty(e_1, \dots, e_r)$ in $\mathrm{Gr}^m(V)$; in fact, a point of $\mathcal{H}^\infty(e_1, \dots, e_r)$ is characterized by these bilinear equations and the equations characterizing $\mathcal{M}^\infty(r) \subset \mathrm{Gr}^m(V)$ (see [17]), which are not a hierarchy of soliton equations. This is clarified in Theorem 4.15.

Theorem 4.13 *A closed point $B \in \mathcal{M}^\infty(r)$ $\Big(B \notin \mathrm{Gr}^{\frac{1}{2}(r-n)}(V)\Big)$ is a point of $\mathcal{H}^\infty(e_1, \dots, e_r)$ if and only if its τ-function fulfills the following set of equations:*

$$\sum_{\substack{1\le j\le r\\ 1\le k\le r}}\Bigg(\epsilon_{uj}\cdot\epsilon_{vk}\sum_{\substack{1\le i\le e_j\\ 1\le l\le e_k}}\sum \xi_j^{i(u_j-\delta_{uj}-\alpha_1+\beta_1)} D_{\lambda_j,\alpha_1}\big(-\widetilde{\partial}_{t^{(j)}}\big)p_{\beta_1}\big(\widetilde{\partial}_{t^{(j)}}\big)\check{D}^j_{\lambda_\bullet}\big(-\widetilde{\partial}_t\big)$$
$$\cdot\,\xi_k^{l(1-u_k-\delta_{kv}-\alpha_2+\beta_2)} D_{\mu_k,\alpha_2}\big(\widetilde{\partial}_{s^{(k)}}\big)p_{\beta_2}\big(-\widetilde{\partial}_{s^{(k)}}\big)\check{D}^k_{\mu_\bullet}\big(\widetilde{\partial}_s\big)\Bigg)\tau_{B_{uj}}(t)\cdot\tau_{B_{kv}}(s) = 0$$

for all Young diagrams $\lambda_1, \mu_1, \dots, \lambda_r, \mu_r$ *and* $1 \leq u, v \leq r$. *Here,* $D_{\lambda,\alpha}$ *and* $\check{D}^j_{\lambda}$ *are differential operators whose explicit expression will be given in the proof. The third sum runs over the set of* 4*-tuples* $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ *of nonnegative integers such that* $\frac{u_j - \delta_{uj} - \alpha_1 + \beta_1}{e_j} + \frac{1 - u_k - \delta_{kv} - \alpha_2 + \beta_2}{e_k} = 0$, *and* u_i *are given by* $v_m = z_1^{u_1} \dots z_r^{u_r}$.

P r o o f. The proof is similar to that of Theorem 5.4 in [17] and is based on that given by Fay in [10]. To begin with, let us state some results.

First, let χ_λ be the Schur polynomial corresponding to a Young diagram λ, let t (resp. s) be the set of variables $(t_1, t_2, \dots)$ (resp. $(s_1, s_2, \dots)$), and let $\widetilde{\partial}_t$ denote $\big(\frac{\partial}{\partial t_1}, \frac{\partial}{2\partial t_2}, \dots\big)$. From the fact that $\{\chi_\lambda(\widetilde{\partial}_t)\}_\lambda$ is the dual basis of $\{\chi_\mu(t)\}_\mu$, we have that a function $f(t) \in \mathbb{C}\{\{t\}\}$ admits an expansion of the type

$$f(t) \;=\; \left(\sum_\lambda \chi_\lambda(t)\chi_\lambda\big(\widetilde{\partial}_s\big) f(s)\right)\Bigg|_{s=0},$$

where λ runs over the set of Young diagrams.

Second, recall Pieri's formula ([15, Formula I.5.16])

$$\chi_\lambda(t) p_m(t) = \sum_{\mu - \lambda = (m)} \chi_\mu(t),$$

where the condition $\mu - \lambda = (m)$ means that the skew diagram $\mu - \lambda$ is a horizontal m-strip and $p_m(t)$ is defined by the identity $p_m(t) = \chi_{(m)}(t)$ or, equivalently, by

$$\sum_{k\geq 0} p_k(t) z^k \;=\; \exp\left(\sum_{k\geq 1} t_k z^k\right). \tag{4.2}$$

Third, the following computation

$$\begin{aligned}
\chi_\lambda\big(\widetilde{\partial}_t\big)\big(p_m(t) f(t)\big)\big|_{t=0} &= \chi_\lambda\big(\widetilde{\partial}_t\big)\left(\sum_\mu p_m(t)\chi_\mu(t)\chi_\mu\big(\widetilde{\partial}_s\big) f(s)\big|_{s=0}\right)\Bigg|_{t=0} \\
&= \chi_\lambda\big(\widetilde{\partial}_t\big)\left(\sum_\mu \sum_{\gamma - \mu = (m)} \chi_\gamma(t)\chi_\mu\big(\widetilde{\partial}_s\big) f(s)\big|_{s=0}\right)\Bigg|_{t=0} \\
&= \sum_{\lambda - \mu = (m)} \chi_\mu\big(\widetilde{\partial}_s\big) f(s)\big|_{s=0}
\end{aligned}$$

will be useful, and let us define the differential operator $D_{\lambda,m}\big(\widetilde{\partial}_t\big)$ by

$$D_{\lambda,m}\big(\widetilde{\partial}_t\big) f(t) := \sum_{\lambda - \mu = (m)} \chi_\mu\big(\widetilde{\partial}_t\big) f(t)\big|_{t=0}.$$

Finally, observe that Definition 3.4 allows us to write down the following explicit expression for $\psi^{(j)}_{u,B}(z_j, t)$:

$$\begin{aligned}
\psi^{(j)}_{u,B}(z_j, t) &= \epsilon_{uj} \exp\left(-\sum_{i\geq 1} \frac{t^{(j)}_i}{z^i_j}\right) \frac{\tau_{B_{uj}}(t + [z_j])}{\tau_B(t)} \\
&= \epsilon_{uj} \left(\sum_{i\geq 0} \frac{p_i\big(-t^{(j)}\big)}{z^i_j}\right) \frac{\big(\sum_{i\geq 0} p_i\big(\widetilde{\partial}_{t^{(j)}}\big) z^i_j\big)\tau_{B_{uj}}(t)}{\tau_B(t)}
\end{aligned} \tag{4.3}$$

since $\exp\Big(\sum_{i\geq 1} z^i_j \widetilde{\partial}_{t^{(j)}_i}\Big)\, \tau_{B_{uj}}(t) = \tau_{B_{uj}}(t + [z_j])$. Similarly, one has

$$\psi^{*(k)}_{v,B}(z_k, t) \;=\; \epsilon_{vk} \left(\sum_{i\geq 0} \frac{p_i\big(t^{(k)}\big)}{z^i_k}\right) \frac{\big(\sum_{i\geq 0} p_i\big(-\widetilde{\partial}_{t^{(k)}}\big) z^i_k\big)\tau_{B_{kv}}(t)}{\tau_B(t)}.$$

We are now ready to prove the statement. The bilinear identity of Theorem 4.11 claims that the coefficient of z^{-1} of a certain function vanishes. Note that the coefficient of z_j^m in $\psi^{(j)}_{u,B}\big(\xi_j^i z^{1/e_j}, t\big)$ is equal to ξ_j^{im} times the coefficient of z_j^m in $\psi^{(j)}_{u,B}(z_j, t)$. From Formulae (4.2) and (4.3), the residue condition reads

$$\sum_{\substack{1\leq j\leq r\\1\leq k\leq r}} \epsilon_{uj}\cdot\epsilon_{vk} \sum_{\substack{1\leq i\leq e_j\\1\leq l\leq e_k}} \sum \left(\frac{p_{\alpha_1}\big(-t^{(j)}\big)p_{\beta_1}\big(\widetilde{\partial}_{t^{(j)}}\big)}{\xi_j^{i(\delta_{uj}-u_j+\alpha_1-\beta_1)}}\tau_{B_{uj}}(t)\cdot \frac{p_{\alpha_2}\big(s^{(k)}\big)p_{\beta_2}\big(-\widetilde{\partial}_{s^{(k)}}\big)}{\xi_k^{l(u_k-1+\delta_{kv}+\alpha_2-\beta_2)}}\tau_{B_{kv}}(s)\right) = 0,$$

where the third sum runs over the set of 4-tuples $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ of nonnegative integers such that

$$\frac{u_j-\delta_{uj}-\alpha_1+\beta_1}{e_j}+\frac{1-u_k-\delta_{kv}-\alpha_2+\beta_2}{e_k}=0.$$

This is a function, F, on $2r$ sets of variables; namely, $t = \big(t^{(1)}, \dots, t^{(r)}\big)$ and $s = \big(s^{(1)}, \dots, s^{(r)}\big)$. Its vanishing is equivalent to the vanishing of

$$\left(\prod_{\substack{1\leq a\leq r\\1\leq b\leq r}} \chi_{\lambda_a}\big(-\widetilde{\partial}_{t^{(a)}}\big)\chi_{\mu_b}\big(\widetilde{\partial}_{s^{(b)}}\big)\right) F\Big|_{t=s=0} = 0$$

for all Young diagrams $\lambda_1, \mu_1, \dots, \lambda_r, \mu_r$.

Using the facts discussed at the beginning of the proof, we arrive at the following identity

$$\sum_{\substack{1\leq j\leq r\\1\leq k\leq r}} \epsilon_{uj}\cdot\epsilon_{vk} \sum_{\substack{1\leq i\leq e_j\\1\leq l\leq e_k}} \sum \Big(\xi_j^{i(u_j-\delta_{uj}-\alpha_1+\beta_1)} D_{\lambda_j,\alpha_1}\big(-\widetilde{\partial}_{t^{(j)}}\big)p_{\beta_1}\big(\widetilde{\partial}_{t^{(j)}}\big)\check{D}^j_{\lambda_\bullet}\big(-\widetilde{\partial}_t\big)\tau_{B_{uj}}(t)$$
$$\cdot\, \xi_k^{l(1-u_k-\delta_{kv}-\alpha_2+\beta_2)} D_{\mu_k,\alpha_2}\big(\widetilde{\partial}_{s^{(k)}}\big)p_{\beta_2}\big(-\widetilde{\partial}_{s^{(k)}}\big)\check{D}^k_{\mu_\bullet}(\widetilde{\partial}_s)\tau_{B_{kv}}(s)\Big) = 0,$$

where $\check{D}^j_{\lambda_\bullet}\big(\widetilde{\partial}_t\big)$ is the operator $\prod_{a\neq j}\chi_{\lambda_a}\big(\widetilde{\partial}_{t^{(a)}}\big)\big|_{t^{(a)}=0}$. □

Remark 4.14 The technique used in the above proof allows one to translate some of our previous results, such as Theorem 3.16, into a set a differential equations. Further, if we succeed in expressing τ-functions in terms of theta functions, then we would obtain a characterization of these coverings in terms of differential equations for theta functions.

Theorem 4.15 *Let U be a closed point of $\mathrm{Gr}^m(V)$ $\big(m\neq \frac{1}{2}(r-n)\big)$ and let $\{u_1, \dots, u_r\}$ be integer numbers defined by $v_m = z_1^{u_1}\cdot\ldots\cdot z_r^{u_r}$.*

Then, U is a point of $\mathcal{H}^\infty(e_1, \dots, e_r)$ if and only if its τ-functions satisfy:

- *the equations of Theorem 4.13;*
- *the equation*

$$\sum_{j=1}^r \epsilon_{ui}\epsilon_{vj}\,\epsilon_{wj} \sum \Big(D_{\lambda_j,\alpha_1}\big(-\widetilde{\partial}_{t^{(j)}}\big)p_{\beta_1}\big(\widetilde{\partial}_{t^{(j)}}\big)\check{D}^j_{\lambda_\bullet}\big(-\widetilde{\partial}_t\big)\cdot D_{\mu_j,\alpha_2}\big(-\widetilde{\partial}_{t'^{(j)}}\big)p_{\beta_2}\big(\widetilde{\partial}_{t'^{(j)}}\big)\check{D}^j_{\mu_\bullet}\big(-\widetilde{\partial}_{t'}\big)$$
$$\cdot\, D_{\nu_j,\alpha_3}\big(\widetilde{\partial}_{t''^{(j)}}\big)p_{\beta_3}\big(-\widetilde{\partial}_{t''^{(j)}}\big)\check{D}^j_{\nu_\bullet}\big(\widetilde{\partial}_{t''}\big)\Big)\tau_{U_{uj}}(t)\tau_{U_{vj}}(t')\tau_{U_{jw}}(t'') \;=\; 0$$

  for all $1\leq u, v, w\leq r$, all Young diagrams λ, μ, ν and all t, t', t'' (the inner sum runs over the 6-tuples $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ such that $-\alpha_1+\beta_1-\alpha_2+\beta_2-\alpha_3+\beta_3 = \delta_{uj}+\delta_{vj}+\delta_{wj}-u_j$) ;
- *the equation*

$$\sum_{j=1}^r \epsilon_{uj}\sum\Big(D_{\lambda_j,\alpha}\big(\widetilde{\partial}_{t^{(j)}}\big)p_\beta\big(-\widetilde{\partial}_{t^{(j)}}\big)\check{D}^j_{\lambda_\bullet}\big(\widetilde{\partial}_t\big)\Big)\tau_{U_{ju}}(t) \;=\; 0$$

for all $1 \le u \le r$*, all Young diagrams* λ *and all* t *(the inner sum runs over the pairs* (α, β) *such that* $-\alpha + \beta = u_j + \delta_{uj} - 1$*).*

Proof. It will suffice to check that the second and third sets of differential equations are the equations characterizing $\mathcal{M}^\infty(r)$ as a subscheme of $\mathrm{Gr}(V)$. From Theorem 4.2, we know that $\mathcal{M}^\infty(r)$ consists of those $U \in \mathrm{Gr}(V)$ such that $U \cdot U \subseteq U$ and $\mathbb{C} \subset U$.

Theorem 3.7 implies that these two conditions are equivalent to

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left(\frac{v_m \psi_{u,U}(z_\centerdot, t) \psi_{v,U}(z_\centerdot, t') \psi^*_{w,U}(z_\centerdot, t'')}{(1, \dots, z_u, \dots, z_v, \dots, z_w, \dots, 1)} \right) \frac{dz}{z} = 0$$

and

$$\mathrm{Res}_{z=0} \mathrm{Tr} \left(\frac{v_{r-n-m} \psi^*_{u,U}(z_\centerdot, t)}{(1, \dots, z_u, \dots, 1)} \right) dz = 0,$$

respectively. Proceeding as in the previous proof, one concludes the proof. □

5 Curves with prescribed involutive series

Theorems 4.7, 4.11, and 4.13 completely characterize when an algebraic curve is a finite covering of another curve with a prescribed ramification profile over a point. We shall now apply these results to characterize the existence of algebraic 1-dimensional series on a curve with prescribed numerical invariants. However, an explicit solution of this problem would require us to compute the Baker–Akhiezer function of $\mathrm{Tr}(B)$ as a point of $\mathrm{Gr}(\mathbb{C}((z)))$ for $B \in \mathcal{H}^\infty(e_1, \dots, e_r)$.

Let Y be a smooth algebraic curve of genus g over $\mathbb{C}$. An involutive algebraic series of genus g_0 and degree n over Y is the algebraic series γ^1_n

$$\gamma^1_n = \left\{ \pi^{-1}(x) \,|\, x \in X \right\} \subset S^n Y$$

or, equivalently,

$$\gamma^1_n \equiv \Gamma_\pi \hookrightarrow X \times Y$$

(Γ_π being the graph of the morphism π) defined by a finite morphism

$$\pi \colon Y \longrightarrow X,$$

where X is a smooth algebraic curve of genus g_0.

If X is the projective line ($g_0 = 0$), then the algebraic series defined by π is a linear series, g^1_n, of degree n.

For instance, recall that a curve with a linear series g^1_2 is a hyperelliptic curve; a curve of genus $g > 3$ with a linear series g^1_3 is a trigonal curve; a curve with an algebraic series γ^1_2 of genus $g_0 > 0$ is called a g_0-hyperelliptic curve ([1]).

The simplest case is the moduli space of curves of genus g with a linear series g^1_n (this problem is trivial for big enough n). Let us denote by $\underline{\mathcal{H}}^\infty(g, 0; e_1, \dots, e_r)$ the subfunctor of $\underline{\mathcal{H}}^\infty(e_1, \dots, e_r)$ consisting of coverings of the type

$$\pi \colon Y \longrightarrow \mathbb{P}_1$$

where Y has arithmetic genus g, $x \in \mathbb{P}_1$ and $\pi^{-1}(x) = e_1 y_1 + \dots + e_r y_r$ (with $e_1 + \dots + e_r = n$).

In other words, the set $\underline{\mathcal{H}}^\infty(g, 0; e_1, \dots, e_r)(\mathbb{C})$ is the set of curves of genus g with a linear series g^1_n and a divisor $D \in g^1_n$ of the type $D = e_1 y_1 + \dots + e_r y_r$.

Theorem 5.1 *The functor* $\underline{\mathcal{H}}^\infty(g, 0; e_1, \dots, e_r)$ *is representable by a closed subscheme,* $\mathcal{H}^\infty(g, 0; e_1, \dots, e_r)$*, of* $\mathcal{H}^\infty(e_1, \dots, e_r)$*.*

Proof. The condition that the fibres of the family of curves $Y \to S$ have arithmetic genus g means that $\underline{\mathcal{H}}^\infty(g, 0; e_1, \dots, e_r)$ lies inside the connected component $\mathrm{Gr}^{1-g}(V)$, which is a closed subscheme of $\mathrm{Gr}(V)$.

The second condition, namely that $X_s = \mathbb{P}_1$ for all $s \in S$, is equivalent to saying that $\mathrm{Tr}(K(Y))$ lies in the connected component of index 1, $\mathrm{Gr}^1(\mathbb{C}((z)))$. This is also a closed condition. □

In particular, if we set $e_1 = \cdots = e_r = 1$ and $r = n$, then the moduli space $\mathcal{H}^\infty(g, 0; 1, \ldots, 1)$ represents all curves of genus g with a linear series g^1_n and parameters at the marked points.

Let $\underline{\mathcal{H}}(g, 0; 1, \ldots, 1)$ be the Hurwitz functor classifying the set of data $(Y, y_1, \ldots, y_n)$ of coverings $Y \to \mathbb{P}_1$ with a distinguished fibre of pairwise different points $\{y_1, \ldots, y_n\}$.

Note that there is a canonical morphism

$$\Phi \colon \underline{\mathcal{H}}^\infty(g, 0; 1, \ldots, 1) \longrightarrow \underline{\mathcal{H}}(g, 0; 1, \ldots, 1) \tag{5.1}$$

that forgets the formal parameters.

Theorem 5.2 *The functor $\underline{\mathcal{H}}(g, 0; 1, \ldots, 1)$ is representable by a closed subscheme, $\mathcal{H}(g, 0; 1, \ldots, 1)$, of $\mathcal{H}^\infty(g, 0; 1, \ldots, 1)$.*

P r o o f. Let us define a morphism

$$\sigma \colon \underline{\mathcal{H}}(g, 0; 1, \ldots, 1) \longrightarrow \underline{\mathcal{H}}^\infty(g, 0; 1, \ldots, 1)$$

as follows: $\sigma(Y, y_1, \ldots, y_n)$ is the unique $\mathcal{Y} \in \underline{\mathcal{H}}^\infty(g, 0; 1, \ldots, 1)$ on the fibre $\Phi^{-1}(Y, y_1, \ldots, y_n)$ such that $K(\mathcal{Y}) \cap \mathbb{C}((z)) = \mathbb{C}\big[z^{-1}\big]$.

Geometrically, this construction corresponds to choosing the set of data $\big(Y, \mathbb{P}_1, \pi, x, \bar{y}, t_x, t_{\bar{y}}\big)$ (with $\bar{y} = \{y_1 \ldots, y_r\}$) such that $(\mathbb{P}_1, x, t_x)$ satisfies

$$t_x\big(H^0(\mathbb{P}_1 - x, \mathcal{O}_{\mathbb{P}_1})\big) = \mathbb{C}\big[z^{-1}\big] \in \mathrm{Gr}(\mathbb{C}((z))).$$

Since σ is injective, it suffices to show that $\underline{\mathcal{H}}(g, 0; 1, \ldots, 1)$ is a closed subfunctor of $\underline{\mathcal{H}}^\infty(g, 0; 1, \ldots, 1)$ (via the morphism σ). Since an S-valued point, $\mathcal{Y} \in \underline{\mathcal{H}}^\infty(g, 0; 1, \ldots, 1)$ belongs to $\underline{\mathcal{H}}(g, 0; 1, \ldots, 1)$ if and only if $\mathrm{Tr}(K(\mathcal{Y})) \subseteq \mathcal{O}_S\big[z^{-1}\big]$ and since this is a closed condition $\big($because $\mathcal{O}_S\big[z^{-1}\big]$ is a point of $\mathrm{Gr}(\mathbb{C}((z)))\big)$, the theorem is proved. □

Remark 5.3 Note that the morphism σ defines a canonical section of Φ (see (5.1)). Further, since V is the $\mathbb{C}((z))$-algebra $\mathbb{C}((z)) \times \cdots \times \mathbb{C}((z))$, the group $G := \underline{\mathrm{Aut}}(\mathbb{C}((z)))$ (see [18] for its definition and properties) acts on $\mathrm{Gr}(\mathbb{C}((z)))$ and on $\mathrm{Gr}(V)$. If G^+ is the subgroup of G representing "coordinate changes" of the curve downstairs ([18]), then Φ is a G^+-principal bundle, that is, $\mathcal{H}(g, 0; 1, \ldots, 1) = \mathcal{H}^\infty(g, 0; 1, \ldots, 1)/G^+$.

Theorem 5.4 *The equations defining the subscheme $\mathcal{H}(g, 0; 1, \ldots, 1)$ as a subscheme of $\mathcal{H}^\infty(g, 0; 1, \ldots, 1)$ (via the section σ) are as follows:*

$$\mathrm{Res}_{z=0}\left(z^{-i} \cdot \mathrm{Tr}\left(\frac{v_{1-g}\psi_{u,B}(z_{\bullet}, t)}{(1, \ldots, z_u, \ldots, 1)}\right)\right) dz = 0, \quad i \geq 2, \quad 1 \leq u \leq r.$$

P r o o f. Note that a point $B \in \mathcal{H}^\infty(g, 0; 1, \ldots, 1)$ lies in $\mathcal{H}(g, 0; 1, \ldots, 1)$ if and only if $\mathrm{Tr}\, B = \mathbb{C}\big[z^{-1}\big]$. We can write this condition in terms of Baker–Akhiezer functions.

The equality is equivalent to saying that $z^{-i} \in (\mathrm{Tr}\, B)^\perp$ for all $i \geq 2$. And the claim follows. □

In order to study the moduli space of curves of genus g with an involutive algebraic series γ^1_n of genus g_0 and degree n, one has to consider the subfunctor $\underline{\mathcal{H}}^\infty(g, g_0; 1, \ldots, 1)$ of $\underline{\mathcal{H}}^\infty(1, \ldots, 1)$ consisting of those coverings $Y \to X$ where Y has arithmetic genus g and X has arithmetic genus g_0. Analogously to Theorem 5.1, one proves that this subfunctor is representable by a closed subscheme of $\mathcal{H}^\infty(1, \ldots, 1)$, which will be denoted by $\mathcal{H}^\infty(g, g_0; 1, \ldots, 1)$, since we know that the following two conditions

- $B \in \mathrm{Gr}^{1-g}(V)$,
- $\mathrm{Tr}(B) \in \mathrm{Gr}^{1-g_0}(\mathbb{C}((z)))$

are closed (B belongs to $\mathcal{H}^\infty(1, \ldots, 1)$).

If we assume that $B \in \mathrm{Gr}^{1-g}(V)$, the second condition can be translated, in certain particular cases, into a set of differential equations. We shall study this problem elsewhere and shall obtain, for those cases, explicit characterizations (for instance, characterizations of g_0-hyperelliptic curves).

Remark 5.5 One can consider the Hurwitz functor parametrizing families of n-sheeted coverings $Y \to X$ of smooth connected curves with Y of genus g and X of genus g_0. Let us denote it by $\underline{\mathcal{H}}_n(g, g_0)$. Let $\mathcal{W}$ denote the open subscheme of $\mathcal{H}^\infty(g, g_0; 1, \dots, 1)$, whose points corresponds to geometric data where both curves are smooth and connected. We therefore have a canonical forgetful morphism

$$\Psi \colon \mathcal{W} \longrightarrow \underline{\mathcal{H}}_n(g, g_0).$$

Note that the fibre of Ψ of the covering $Y \to X$ is isomorphic to the complementary in X of the set of ramification points.

6 Final remarks

It was pointed out by Li and Mulase [14] that a Zariski open subset of the moduli space of Higgs pairs over a curve can be embedded into a quotient Grassmannian and that the restriction of the n-component KP-flow is precisely the Hamiltonian flow of the Hitchin system.

In our setting, a local analogue for the Hitchin system appears naturally when we attempt to solve the following question: how do the algebro-geometric objects introduced here (τ-functions, Baker functions, etc.) depend on the structure of $\mathbb{C}((z))$-algebra of V? Let us discuss this question briefly.

For a monic polynomial of degree n, $P(T) \in \mathbb{C}[[z]][T]$, with coefficients in $\mathbb{C}[[z]]$, let us define the finite $\mathbb{C}((z))$-algebra

$$V_P := \mathbb{C}((z))/(P(T))$$

and a rank n free $\mathbb{C}[[z]]$-module $V_P^+ := \mathbb{C}[[z]]/(P(T))$. Now, $\widehat{C}_P := \operatorname{Spf} V_P^+$ is called the formal spectral cover of polynomial P.

Let us denote by $\mathbb{A}^n_\infty$ the infinite dimensional affine space representing $\mathbb{C}[[z]] \times \overset{n}{\dots} \times \mathbb{C}[[z]]$ (see [17, Section 3.A]). For each point $s = (s_1, \dots, s_n) \in \mathbb{A}^n_\infty$, let us denote by P_s the polynomial $T^n - s_1 T^{n-1} + \dots + (-1)^n s_n$ and, for the sake of simplicity, let us write $V_s = V_{P_s}$, $V_s^+ = V_{P_s}^+$ and $\widehat{C}_s = \widehat{C}_{P_s}$.

One can define a family of infinite Grassmannians parametrized by the space $\mathbb{A}^n_\infty$:

$$\mathcal{G}r \xrightarrow{\ \pi\ } \mathbb{A}^n_\infty \tag{6.1}$$

such that the fibre of s, $\pi^{-1}(s)$, is the infinite Grassmannian of the couple $\left(V_s, V_s^+\right)$.

There is an open dense subscheme $U \subset \mathbb{A}^n_\infty$ such that for each $s \in U$ the formal spectral cover $\widehat{C}_s$ is smooth. The fibre of π at $s \in U$ corresponds to the Grassmannian of $\left(V_s, V_s^+\right)$ where V_s is a separable $\mathbb{C}((z))$-algebra and it has been studied in Sections 1–4 of this paper.

Note that there is a natural representation of $\operatorname{End}_{\mathbb{C}[[z]]} V^+$ in the Lie algebra of vector fields over $\operatorname{Gr}(V)$:

$$\operatorname{End}_{\mathbb{C}[[z]]} V^+ \hookrightarrow \operatorname{End}_{\mathbb{C}((z))} V \xrightarrow{\ \Psi\ } T\operatorname{Gr}(V),$$

where the fibre of Ψ at the point $L \in \operatorname{Gr}(V)$ is given by

$$\begin{aligned} \Psi_L \colon \operatorname{End}_{\mathbb{C}((z))} V &\longrightarrow T_L \operatorname{Gr}(V) = \operatorname{Hom}(L, V/L), \\ \varphi &\longmapsto \Psi_L(\varphi) := \left(L \hookrightarrow V \xrightarrow{\varphi} V \to V/L\right). \end{aligned}$$

Using this representation, a notion of a local Higgs pair can be introduced and the corresponding moduli space can be studied. The local analogue of the Hitchin fibration is then related to the fibration (6.1). Further, the different $\underline{n}$-KP hierarchies can be interpreted as the flows defining the fibres of the local Hitchin map. We hope to study all these aspects elsewhere.

Remark 6.1 Let us denote by V_{s_1} and V_{s_2} two different $\mathbb{C}((z))$-algebra structures on V and let us assume that these structures are determined by two partitions of n, $\underline{n}_1 = \left(e_1^1, \dots, e_{r_1}^1\right)$ and $\underline{n}_2 = \left(e_1^2, \dots, e_{r_2}^2\right)$. In [2, Section 9], the following question is stated: given a point U coming from two geometric data

$$\left(Y^i, X^i, \pi^i, x^i, \bar{y}^i, t_{x^i}, t_{\bar{y}^i}\right) \in \mathcal{H}^\infty\left(e_1^1, \dots, e_{r_i}^1\right) \subset \operatorname{Gr}(V), \quad i = 1, 2,$$

is there any relation between the curves Y^1 and Y^2? The answer to this question is that in general Y^1 and Y^2 do not have to be isomorphic (even if both curves are irreducible).

The following example, related to the constructions of Section 5, is instructive for understanding the above question. Let $\pi^i\colon Y^i \to \mathbb{P}^1$ $(i = 1, 2)$ be two different finite coverings of degree n such that

$$\pi^1_*\mathcal{O}_{Y^1} \simeq \pi^2_*\mathcal{O}_{Y^2}.$$

Obviously, we can find non-isomorphic smooth curves Y^1 and Y^2 fulfilling this condition, since the set of rank n locally free sheaves on $\mathbb{P}^1$ is discrete (they are of the form $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$) while the set of degree n coverings is not. Choosing a point $x \in \mathbb{P}^1$ and a formal trivialization at that point, t_x, one observes that $\big(Y^1, \mathbb{P}^1, x, t_x\big)$ and $\big(Y^2, \mathbb{P}^1, x, t_x\big)$ define the same point of $\mathrm{Gr}(V)$.

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