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# Coverings with prescribed ramification and Virasoro Groups

By

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**Abstract**

In this paper we study coverings with prescribed ramification from the point of view of the Sato Grassmannian and of the algebro-geometric theory of solitons. We show that the moduli space of such coverings, which is a Hurwitz scheme, is a subscheme of the Grassmannian. We give its equations and show that there is a Virasoro group which uniformizes it. We also characterize when a curve is a covering in terms of bilinear identities.

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## 1. Introduction

A degree $n$ covering of the projective line, $Y \to \mathbb{P}^1$, is called *simple* if there is at least $n-1$ points of $Y$ over each point of $\mathbb{P}^1$. Hurwitz ([6]) proved that the set of such genus $g$ compact Riemann surfaces, $Y$, that are degree $n$ simple coverings of $\mathbb{P}^1$ is indeed a connected manifold. These manifolds are nowadays called *Hurwitz spaces*. It is remarkable that every genus $g$ Riemann surface may be representable by a degree $n$ simple covering of $\mathbb{P}^1$ (provided that $n \geq g+1$). Since then, many generalizations and applications have been given.

The algebro-geometric theory of Hurwitz spaces (e.g. its $\mathbb{Z}$-scheme structure) was developed by Fulton, who applied this fact to conclude the irreducibility of the moduli space of curves ([4]). Let us mention very briefly some current topics where the relevance of Hurwitz spaces have been unveiled.

In the case of simple coverings, the branch points of the covering (i.e. those points of $\mathbb{P}^1$ whose fiber has $n-1$ points) can serve as local coordinates of the Hurwtiz space. The study of isomonodromic deformations is essentially the study of families of coverings with respect to the variation of the branch points (see e.g. [9]).

On the other hand, if the branch points are fixed, then there exist finitely many coverings (of given genus and degree, note that these numbers are related by the Riemann-Hurwtiz formula). Thus, the explicit computation of these numbers is a hard and interesting problem in enumerative geometry which is deeply related to the intersection theory of the moduli space. Indeed, Gromov-Witten Theory is very well suited for carrying out such a study (e.g. [3, 15]).

These numbers show up in a bunch of different topics, let us mention a striking result by Kontsevich (conjectured by Witten, [18]); namely, that a suitable generating function builded from the Hurwitz numbers is a tau-function for the KdV hierarchy ([8]).

In this paper a different approach to Hurwitz spaces is offered. Recall that the algebraic theory of soliton equations (e.g. Sato infinite Grassmann manifold [17], Krichever map [10], etc.) has obtained several results in the interplay between the theory of algebraic curves (theta functions, jacobians, moduli spaces, etc.) and the theory of solitons (KP hierarchy, bosonization, representation theory, etc.). Motivated by this fruitful relation, we will apply these techniques to the case of Hurwtiz spaces.

We will consider coverings of curves $Y \to X$ with prescribed ramification data and, since we use the Sato Grassmannian, these data will be decorated with formal trivializations. To be more precise, let us fix a natural number $n$ and a set of $r$ partitions of it $E = \{\bar{e}_1, \dots, \bar{e}_r\}$, where $\bar{e}_i = \{e_1^{(i)}, \dots, e_{k_i}^{(i)}\}$. Let us denote by $V$ the $\mathbb{C}$-algebra $\mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_r))$ and by $W$ the $V$-algebra

$$W = \mathbb{C}((z_1^{1/e_1^{(1)}})) \times \cdots \times \mathbb{C}((z_1^{1/e_{k_1}^{(1)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_1^{(r)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_{k_r}^{(r)}}))$$

Then, the Hurwitz functor parametrizes data $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ where $\pi : Y \to X$ is a cover of curves, $\bar{x} = x_1 + \cdots + x_r \subset X$ and $\bar{y} = \sum_{i,j} y_j^{(i)} \subset Y$

are divisors such that $\pi^{-1}(x_i) = \sum_j e_j^{(i)} y_j^{(i)}$ and $t_{\bar{x}}$ and $t_{\bar{y}}$ are formal trivializations along $\bar{x}$ and $\bar{y}$ respectively (in particular, they induce isomorphisms $t_{\bar{x}} : (\widehat{\mathcal{O}}_{X,\bar{x}})_{(0)} \xrightarrow{\sim} V$ and $t_{\bar{y}} : (\widehat{\mathcal{O}}_{Y,\bar{y}})_{(0)} \xrightarrow{\sim} W$).

The corresponding functor will be called the Hurwitz functor and will be studied in §3. The Krichever map embeds it into the Sato Grassmannian and its image will be characterized. Then, we prove that there exists a subscheme, $\mathcal{H}_E^\infty[\bar{g}, g]$, of the Grassmannian of $W$ representing the Hurwitz functor, where $g$ and $\bar{g}$ are the genera of $X$ and $Y$ respectively. Since a covering of curves has a finite number of ramification points, this result allows us to characterize those curves $Y$ that are a covering of another curve $X$ with fully prescribed ramification profile, which is the main result of this section (see Theorem 3.4). Although this characterization is stated in terms of identities for the Baker-Akhiezer functions, it can easily be translated to a hierarchy for $\tau$-functions.

Study of the tangent spaces of these moduli spaces is carried out in §4. Apart from the importance of these results on their own (e.g. Theorem 4.1), they will also be used in the rest of the paper.

In §5, we introduce the group $G_V^W$ as a certain subgroup of the group of automorphisms of $W$ that induce an automorphism on $V$. We show that this formal group scheme acts canonically on $\mathrm{Gr}(W)$ leaving $\mathcal{H}_E^\infty[\bar{g}, g]$ stable. Further, we prove that this group "uniformizes" $\mathcal{H}_E^\infty[\bar{g}, g]$, more precisely, we prove that the previous action is locally transitive (Theorem 5.4). Proof of these facts requires the explicit computation of the tangent space to the Hurwitz space carried out in §4.

In the last section, to the previous data we add a pair $(L, \phi_{\bar{y}})$ consisting of a line bundle on $Y$ together with a formal trivialization of $L$ along $\bar{y}$. This functor is representable by a subscheme of the Grassmannian of $W$ which will be denoted by $\mathrm{Pic}_E^\infty[\bar{g}, g]$ (see Definition 6.1). Let $\Gamma_W$ denote the connected component of 1 in the scheme representing the functor of invertible elements of $W$ (see §2). Since $G_V^W$ acts canonically on $\Gamma_W$ we consider the semidirect product $G_V^W \ltimes \Gamma_W$. Then, we show that the group $G_V^W \ltimes \Gamma_W$ acts on $\mathrm{Pic}_E^\infty[\bar{g}, g]$ and that this action is locally transitive (Theorem 6.2). This roughly means that any deformation of a point of $\mathrm{Pic}_E^\infty[\bar{g}, g]$ can be obtained through the action of this group.

## 2. Preliminaries

We assume the base field to be the field of complex numbers. However, all our results hold for an algebraically closed field of characteristic zero.

This section recalls the definitions and generalizes some results of §§2–3 of [13] (see also [1]).

### Formal groups

Let $V$ be the trivial $\mathbb{C}$-algebra $\mathbb{C}((z_1)) \times \overset{r}{\cdots} \times \mathbb{C}((z_r))$. Let us fix an integer $n > 0$ and a set of $r$-tuple of partitions of $n$:

$$E = \{\bar{e}_1, \ldots, \bar{e}_r\}$$

that is, $\bar{e}_i = \{e_1^{(i)}, \dots, e_{k_i}^{(i)}\}$ and $n = e_1^{(i)} + \cdots + e_{k_i}^{(i)}$. Finally, let us denote $\bar{r} = k_1 + \cdots + k_r$.

Associated with these data we consider the following $V$-algebras

$$W^i := \mathbb{C}((z_i^{1/e_1^{(i)}})) \times \cdots \times \mathbb{C}((z_i^{1/e_{k_i}^{(i)}}))$$

and

$$\begin{aligned} W &:= W^1 \times \cdots \times W^r = \\ &= \mathbb{C}((z_1^{1/e_1^{(1)}})) \times \cdots \times \mathbb{C}((z_1^{1/e_{k_1}^{(1)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_1^{(r)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_{k_r}^{(r)}})) \end{aligned}$$

Let $V_+$, $W_+^i$ and $W_+$ denote the subalgebras corresponding to the power series; that is,

$$\begin{aligned} V_+ &:= \mathbb{C}[[z_1]] \times \overset{r}{\cdots} \times \mathbb{C}[[z_r]] \\ W_+^i &:= \mathbb{C}[[z_i^{1/e_1^{(i)}}]] \times \cdots \times \mathbb{C}[[z_i^{1/e_{k_i}^{(i)}}]] \\ W_+ &:= W_+^1 \times \cdots \times W_+^r . \end{aligned}$$

One now introduces formal group schemes whose rational points are the groups of invertible elements of $V$ and $W$. Let $\Gamma_V$ and $\Gamma_W$ denote the connected components of the identity of these groups ([1]).

Let $R$ be a $\mathbb{C}$-algebra. Recall that the set of $R$-valued points of $\Gamma_V$ is the set of $r$-tuples $(\gamma_1, \dots, \gamma_r) \in V\widehat{\otimes}_{\mathbb{C}}R$ with $\gamma_i = \sum_j a_j^i z_i^j$ where $a_j^i \in \operatorname{Rad}(R)$ for $j < 0$ and $a_0^i$ are invertible. Here, $\widehat{\otimes}$ denotes the completion of the tensor product with respect to the $V_+$-topology. Similarly, the set of $R$-valued points of $\Gamma_W$ is the set of $r$-tuples $(\bar{\gamma}_1, \dots, \bar{\gamma}_r) \in W\widehat{\otimes}_{\mathbb{C}}R$ with

$$\bar{\gamma}_i = \big( \sum_j a_j^{(i,1)} z_i^{j/e_1^{(i)}}, \dots, \sum_j a_j^{(i,k_i)} z_i^{j/e_{k_i}^{(i)}} \big) \in W^i \widehat{\otimes}_{\mathbb{C}} R$$

where $a_j^{(i,h)} \in \operatorname{Rad}(R)$ for $j < 0$ and $a_0^{(i,h)}$ are invertible.

**Infinite Grassmannians**

It is well known that there is a $\mathbb{C}$-scheme $\operatorname{Gr}(W)$, which is called the infinite Grassmannian of the pair $(W, W_+)$, whose set of rational points is

$$\left\{ \begin{matrix} \text{subspaces } U \subset W \text{ such that } U \to W/W_+ \\ \text{has finite dimensional kernel and cokernel} \end{matrix} \right\} .$$

The connected components of the Grassmannian are indexed by the Euler–Poincaré characteristic of the complex. The connected component of index $m$ will be denoted by $\operatorname{Gr}^m(W)$.

The group $\Gamma_W$ acts by homotheties on $W$, and this action gives rise to a natural action on $\operatorname{Gr}(W)$

$$\Gamma_W \times \operatorname{Gr}(W) \longrightarrow \operatorname{Gr}(W) .$$

This infinite Grassmannian is equipped with the determinant bundle, $\mathrm{Det}_W$, which is the determinant of the complex of $\mathcal{O}_{\mathrm{Gr}(W)}$-modules

$$\mathcal{L} \oplus (W_+ \hat{\otimes}_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}(W)}) \longrightarrow W \hat{\otimes}_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}(W)} ,$$

where $\mathcal{L}$ is the universal submodule of $W \hat{\otimes}_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}(W)}$ over $\mathrm{Gr}(W)$ and the morphism is the natural projection. Furthermore, this bundle is preserved under the action of $\Gamma_W$.

**$\tau$-functions and Baker-Akhiezer functions**

The determinant of the morphism $\mathcal{L} \to W/W_+ \hat{\otimes}_{\mathbb{C}} \mathcal{O}_{\mathrm{Gr}(W)}$ gives rise to a canonical global section of the dual $\mathrm{Det}_W^*$ of the determinant bundle $\mathrm{Det}_W$

$$\Omega_+ \in H^0(\mathrm{Gr}^0(W), \mathrm{Det}_W^*) .$$

In order to extend this section to $\mathrm{Gr}(W)$ (in a non-trivial way), we fix elements $\{v_m \in W | m \in \mathbb{Z}\}$ such that: i) the multiplication by $v_m$ shifts the index by $m$; ii) $v_m \cdot v_{\bar{r}-r\cdot n-m} = 1$; and, iii) $v_0 = 1$. Given $U \in \mathrm{Gr}^m(W)$ it follows that for $v_m^{-1} U \in \mathrm{Gr}^0(W)$, and thus it makes sense to define $\Omega_+(U) := \Omega_+(v_m^{-1} U)$.

Now, the $\tau$-function and BA functions will be introduced following [13]. Recall that

$$W = \mathbb{C}((z_1^{1/e_1^{(1)}})) \times \cdots \times \mathbb{C}((z_1^{1/e_{k_1}^{(1)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_1^{(r)}})) \times \cdots \times \mathbb{C}((z_r^{1/e_{k_r}^{(r)}})) ,$$

and that $\Gamma_W$ parametrizes a certain subgroup of invertible elements of $W$. Let $t$ be the set of variables $(t^{(1,1)}, \dots, t^{(1,k_1)}, \dots, t^{(r,1)}, \dots, t^{(r,k_r)})$ where $t^{(a,b)} = (t_1^{(a,b)}, t_2^{(a,b)}, \dots)$. Consider the element of $\Gamma_W$ given by

$$g = \big(1 + \sum_{j<0} t_j^{(1,1)} z_1^{j/e_1^{(1)}}, \dots, 1 + \sum_{j<0} t_j^{(r,k_r)} z_r^{j/e_{k_r}^{(r)}}\big) \in \Gamma_W$$

Thus, the $\tau$-function of $U$, $\tau_U(t)$, is defined by

$$\tau_U(t) := \frac{\Omega_+(gU)}{g\delta_U} ,$$

where $\delta_U$ is a non-zero element in the fibre of $\mathrm{Det}_W^*$ over $U$.

Let $z_{\bullet}$ denote $(z_1^{1/e_1^{(1)}}, \dots, z_1^{1/e_{k_1}^{(1)}}, \dots, z_r^{1/e_1^{(r)}}, \dots, z_r^{1/e_{k_r}^{(r)}}) \in W$.

Let $(a,b)$ be a pair of natural numbers such that $a \in \{1, \dots, r\}$ and $b \in \{1, \dots, k_a\}$. Let $(c,d)$ be another pair satisfying the same constraints.

The $(a,b)$-th Baker-Akhiezer function of a point $U \in \mathrm{Gr}(W)$ is defined as the $W$-valued function whose $(c,d)$-th entry is given by

$$\psi_{a,b,U}^{(c,d)}(z_{\bullet}, t) := \exp\Big(-\sum_{i\geq 1} \frac{t_i^{(c,d)}}{z_c^{i/e_d^{(c)}}}\Big) \frac{\tau_{U_{a,b}^{c,d}}(t + [z_c^{1/e_d^{(c)}}])}{\tau_U(t)} , \tag{2.1}$$

where

- $[z_v] := (z_v, \frac{z_v^2}{2}, \frac{z_v^3}{3}, \dots)$,
- $t+[z_c^{1/e_d^{(c)}}] := (t^{(1,1)}, \dots, t^{(1,k_1)}, \dots, t^{(c,d)}+[z_c^{1/e_d^{(c)}}], \dots, t^{(r,1)}, \dots, t^{(r,k_r)})$,
- and $U_{a,b}^{c,d} := (1, \dots, z_a^{1/e_b^{(a)}}, \dots, (z_c^{1/e_d^{(c)}})^{-1}, \dots, 1) \cdot U$.

The main property of these Baker-Akhiezer functions is that they can be understood as generating functions for $U$ as a subspace of $W$, as we recall below. This eventually implies an analogue of the Bilinear Identity.

**Theorem 2.1** ([13]). *Let $U \in \mathrm{Gr}^m(W)$. Then the Baker-Akhiezer functions have the following expansion:*

$$\begin{aligned}\psi_{a,b,U}(z_\bullet, t) &= v_m^{-1} \cdot (1, \dots, z_a^{1/e_b^{(a)}}, \dots, 1) \cdot \\ &\cdot \sum_{i>0} \left( \psi_{a,b,U}^{(1,1)}(z_1^{1/e_1^{(1)}}), \dots, \psi_{a,b,U}^{(r,k_r)}(z_r^{1/e_{k_r}^{(r)}}) \right) p_{a,b,i,U}(t)\end{aligned},$$

*where*

$$\left\{ \left( \psi_{a,b,U}^{(1,1)}(z_1^{1/e_1^{(1)}}), \dots, \psi_{a,b,U}^{(r,k_r)}(z_r^{1/e_{k_r}^{(r)}}) \right) \mid a \in \{1, \dots, r\} \ , \ b \in \{1, \dots, k_a\} \right\}$$

*is a basis of $U$ and $p_{a,b,i,U}(t)$ are functions in $t$.*

**Bilinear Identity**

Note that $W$ is endowed with the following natural pairing

$$\begin{aligned} T_2 \colon W \times W &\longrightarrow \mathbb{C} \\ (w_1, w_2) &\longmapsto \sum_{i=1}^{r} \mathrm{Res}_{z_i=0} \mathrm{Tr}^i (w_1^{(i)} w_2^{(i)}) dz_i \end{aligned}$$

where $\mathrm{Tr}^i : W^i \to \mathbb{C}((z_i))$ is the trace map of $W^i$ as a $\mathbb{C}((z_i))$-algebra, and $w_j = (w_j^{(1)}, \dots, w_j^{(r)})$ w.r.t. the decomposition $W = W^1 \times \dots \times W^r$.

From the separability of $W^i$ as $\mathbb{C}((z_i))$-algebra, it follows that $T_2$ is a non-degenerate bilinear pairing. Furthermore, it induces an involution of the Grassmannian

$$\begin{aligned} \mathrm{Gr}(W) &\longrightarrow \mathrm{Gr}(W) \\ U &\longmapsto U^\perp \end{aligned},$$

where $U^\perp$ is the orthogonal of $U$ w.r.t. $T_2$. This involution sends the connected component of index $m$ to that of index $\bar{r} - r \cdot n - m$.

Finally, the $(a, b)$-th adjoint Baker-Akhiezer functions of $U$ are defined by

$$\psi_{a,b,U}^*(z_\bullet, t) := \psi_{a,b,U^\perp}(z_\bullet, -t)$$

**Theorem 2.2** (Bilinear Identity). *Let $U, U' \in \mathrm{Gr}^m(W)$ be two rational points lying on the same connected component. Then, $U = U'$ if and only if the following condition holds*

$$T_2\Big(\frac{1}{z.}\psi_U(z., t), \frac{1}{z.}\psi_{U'}^*(z., t')\Big) = 0$$

**The Krichever morphism**

Let $\mathcal{M}^\infty(\bar{r})$ be the moduli functor parametrizing the classes of sets of data $(Y, \bar{y}, t_{\bar{y}})$ of geometrically reduced proper curves with $\bar{r}$ pairwise distinct marked smooth points $\bar{y} := y_1 + \cdots + y_{\bar{r}}$ and formal parameters $t_{\bar{y}} := \{t_1, \dots, t_{\bar{r}}\}$ at these points such that each irreducible component of $Y$ contains at least one of the marked points.

The Krichever morphism for $\mathcal{M}^\infty(\bar{r})$ is the morphism of functors

$$\mathrm{Kr}: \ \mathcal{M}^\infty(\bar{r}) \longrightarrow \underline{\mathrm{Gr}}(W)$$

that sends $(Y, \bar{y}, t_{\bar{y}})$ to the following submodule of $W \widehat{\otimes}_{\mathbb{C}} \mathcal{O}_S$

$$t_{\bar{y}}\Big(\varinjlim_m (p_Y)_* \mathcal{O}_Y(m\bar{y})\Big) \subset W\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S \tag{2.2}$$

Applying Theorem 4.3 of [13], we have the following

**Theorem 2.3.** *The Krichever morphism (*3.1*) identifies $\mathcal{M}^\infty(\bar{r})(S)$ with the set of submodules $U \in \mathrm{Gr}(W)(S)$ such that $U \cdot U \subseteq U$ and $\mathcal{O}_S \subseteq U$. In particular, this functor is representable by a closed subscheme of the infinite Grassmannian, which will be also denoted by $\mathcal{M}^\infty(\bar{r})$.*

**Remark 1.** Note that the Krichever morphisms maps $(Y, \bar{y}, t_{\bar{y}})$ to the connected component of the infinite Grassmannian of index equal to the Euler-Poincaré characteristic of $\mathcal{O}_Y$. This follows from the exactness of the exact sequence

$$\begin{array}{ccccccccc} 0 \to H^0(Y, \mathcal{O}_Y) \to & H^0(Y - \bar{y}, \mathcal{O}_Y) & \to & (\widehat{\mathcal{O}}_{\bar{y}})_{(0)} \,/\, \widehat{\mathcal{O}}_{\bar{y}} & \to H^1(Y, \mathcal{O}_Y) \to 0 \\ & \downarrow\wr & & \downarrow\wr & \\ & \mathrm{Kr}(Y, \bar{y}, t_{\bar{y}}) & & W/W_+ & \end{array}$$

## 3. Coverings with prescribed ramification

**Hurwitz schemes**

Let $\pi : Y \to X$ be a finite morphism between proper curves over $\mathbb{C}$. Let us assume $Y$ and $X$ to be reduced. Let us fix a set of pairwise distinct smooth points in $X$, $x = \{x_1, \dots, x_r\}$ and let $y := \pi^{-1}(x_1) + \cdots + \pi^{-1}(x_r)$.

Let us define $A := H^0(X - x, \mathcal{O}_X)$, $B := H^0(Y - y, \mathcal{O}_Y)$, $\Sigma_X$ (resp. $\Sigma_Y$) to be the total quotient ring of $A$ (resp. $B$). Let $\mathrm{Tr}_{\Sigma_X}^{\Sigma_Y}$ denote the trace of $\Sigma_Y$ as a finite $\Sigma_X$-algebra.

*The triple $(Y, X, x)$ is said to have the property $(*)$ if $\mathrm{Tr}_{\Sigma_X}^{\Sigma_Y}(B) \subseteq A$.*

It is worth pointing out that every covering $\pi : Y \to X$ has the property $(*)$ whenever $X$ is smooth or $\pi$ is flat.

Let us fix a set of numerical data as in the previous section

$$E = \{\bar{e}_1, \dots, \bar{e}_r\}$$

with $\bar{e}_i = \{e_1^{(i)}, \dots, e_{k_i}^{(i)}\}$ with $n = e_1^{(i)} + \cdots + e_{k_i}^{(i)} > 0$. Let $V$ and $W$ be the $\mathbb{C}((z))$-algebras defined by the data $E$ as in §2. For a $\mathbb{C}$-scheme $S$, we write $\widehat{V}_S := V\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S$ and $\widehat{W}_S := W\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S$.

**Definition 3.1.** The Hurwitz functor $\overline{\underline{\mathcal{H}}}_E^\infty$ of pointed coverings of curves of degree $n$ with fibres of type $E$ and formal parameters along the fibers is the contravariant functor on the category of $\mathbb{C}$-schemes

$$\overline{\underline{\mathcal{H}}}_E^\infty : \begin{Bmatrix} \text{category of} \\ \mathbb{C}\text{-schemes} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{category of} \\ \text{sets} \end{Bmatrix}$$
$$S \mapsto \overline{\underline{\mathcal{H}}}_E^\infty(S) := \{(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})\}$$

where

1. $p_Y : Y \to S$ and $p_X : X \to S$ are proper and flat morphisms whose fibres are geometrically reduced curves.

2. $\pi : Y \to X$ is a finite morphism of $S$-schemes of degree $n$ such that its fibres over closed points $s \in S$ have the property $(*)$.

3. $\bar{x} = \{x_1, \dots, x_r\}$ is a set of disjoint smooth sections of $p_X$ such that the Cartier divisors $x_i(s)$ for $i = 1, \dots, r$ are smooth points of $X_s := p_X^{-1}(s)$ for all closed points $s \in S$.

4. $\bar{y} = \{\bar{y}_1, \dots, \bar{y}_r\}$ and, for each $i$, $\bar{y}_i = \{y_1^{(i)}, \dots, y_{k_i}^{(i)}\}$ is a set of disjoint smooth sections of $p_Y$ such that the Cartier divisor $\pi^{-1}(x_i(S))$ is $e_1^{(i)} y_1^{(i)}(S) + \cdots + e_{k_i}^{(i)} y_{k_i}^{(i)}(S)$.

5. For all closed point $s \in S$ and each irreducible component of the fibre $X_s$, there is at least one point $x_i(s)$ lying on that component.

6. For all closed point $s \in S$ and each irreducible component of the fibre $Y_s$, there is at least one point $y_j^{(i)}(s)$ lying on that component.

7. $t_{\bar{x}}$ is a formal parameter along $\bar{x}(S)$, $t_{\bar{x}} \colon \widehat{\mathcal{O}}_{X,\bar{x}(S)} \xrightarrow{\sim} \widehat{V}_S$, such that it induces

$$t_{x_i} := (t_{\bar{x}})_{x_i} : \widehat{\mathcal{O}}_{X,x_i(S)} \xrightarrow{\sim} \mathcal{O}_S[[z_i]]$$

for all $i$.

8. $t_{\bar{y}} = \{t_{\bar{y}_1}, \dots, t_{\bar{y}_r}\}$ are formal parameters along $\bar{y}_1(S), \dots, \bar{y}_r(S)$ such that

$$\pi^*(t_{x_i})_{y_j^{(i)}(S)} = t_{y_j^{(i)}}^{e_j^{(i)}}$$

9. $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ and $(Y', X', \pi', \bar{x}', \bar{y}', t_{\bar{x}'}, t_{\bar{y}'})$ are said to be equivalent when there is a commutative diagram of $S$-schemes

$$\begin{array}{ccc} Y & \xrightarrow{\sim} & Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow{\sim} & X' \end{array}$$

compatible with all the data.

Observe that the map $\pi : Y \to X$ of a point $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ in $\overline{\underline{\mathcal{H}}}_E^\infty$ is surjective.

The forgetful map which sends $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ to $(Y, \bar{y}, t_{\bar{y}})$ defines a map

$$\overline{\underline{\mathcal{H}}}_E^\infty \longrightarrow \mathcal{M}^\infty(\bar{r}),$$

whose composition with the Krichever morphism defines the Krichever morphism for the Hurwitz functor. Explicitly, it is the morphism of functors

$$\mathrm{Kr}\colon \overline{\underline{\mathcal{H}}}_E^\infty \longrightarrow \underline{\mathrm{Gr}}(W) \tag{3.1}$$

that sends $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \in \overline{\underline{\mathcal{H}}}_E^\infty(S)$ to the submodule of $W\widehat{\otimes}_{\mathbb{C}}\mathcal{O}_S$ given by (2.2).

Let $\mathrm{Tr} : W \to V$ denote the trace map of $W$ as a $V$-algebra. It is then clear that $\mathrm{Tr} = \oplus_{i=1}^r \mathrm{Tr}^i$, where $\mathrm{Tr}^i : W^i \to \mathbb{C}((z_i))$ is the trace map of the $\mathbb{C}((z_i))$-algebra $W^i$. Furthermore, one has a commutative diagram

$$\begin{tikzcd} H^0(Y-y,\mathcal{O}_Y) \arrow[r, hook, "t_y"] \arrow[d, "\mathrm{Tr}^{\Sigma_Y}_{\Sigma_X}"'] & \widehat{W}_K := W\widehat{\otimes}_{\mathbb{C}}K = (W^1\times\cdots\times W^r)\widehat{\otimes}_{\mathbb{C}}K \arrow[d, "\mathrm{Tr}"] \\ H^0(X-x,\mathcal{O}_X) \arrow[r, hook, "t_x"] & \widehat{V}_K := V\widehat{\otimes}_{\mathbb{C}}K = K((z_1))\times\cdots\times K((z_r)) \end{tikzcd}$$

for each geometric point $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ in $\overline{\underline{\mathcal{H}}}_E^\infty(K)$ ($K$ being an extension of $\mathbb{C}$).

**Proposition 3.1.** *Let $\mathcal{Y} = (Y, X, \pi, x, y, t_x, t_y)$ be an $S$-valued point of $\overline{\underline{\mathcal{H}}}_E^\infty$. It then holds that*

$$\mathrm{Kr}(X, x, t_x) \;=\; \mathrm{Kr}(\mathcal{Y}) \cap \widehat{V}_S \;=\; \mathrm{Tr}(\mathrm{Kr}(\mathcal{Y})) \;\in\; \mathrm{Gr}(V)(S)$$

*and also that, in particular, the Krichever map (*3.1*) is injective.*

**Theorem 3.1.** *Let $U \in \mathcal{M}^\infty(\bar{r}) \subset \mathrm{Gr}(W)$ be an $S$-valued point. Then, the following conditions are equivalent:*

1. $U \in \overline{\underline{\mathcal{H}}}_E^\infty(S)$,
2. $\mathrm{Tr}(U) \subseteq U$.

*In particular, the functor $\overline{\underline{\mathcal{H}}}_E^\infty$ is representable by a closed subscheme $\overline{\mathcal{H}}_E^\infty$ of* $\mathrm{Gr}(W)$.

**Definition 3.2.** The Hurwitz functor $\underline{\mathcal{H}}_E^\infty$ of pointed coverings of smooth curves of degree $n$ with fibres of type $E$ and formal parameters along the fibers is the subfunctor of $\overline{\underline{\mathcal{H}}}_E^\infty$ consisting of data $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ where the fibres of $Y \to S$ are nonsingular curves.

**Theorem 3.2.** *The functor $\underline{\mathcal{H}}_E^\infty$ is representable by a subscheme of* $\mathrm{Gr}(W)$*, which will be denoted by $\mathcal{H}_E^\infty$.*

*Proof.* Note that if a family $Y \to X$ has the property $(*)$ and the fibres of $Y$ are nonsingular curves, then the fibres of $X$ are also nonsingular curves. Now let $\mathcal{Y} \to \overline{\mathcal{H}}_E^\infty$ be the universal family given by the representability of the functor $\overline{\underline{\mathcal{H}}}_E^\infty$. Then the desired subscheme, $\mathcal{H}_E^\infty$, consists precisely of the points $s \in \overline{\mathcal{H}}_E^\infty$ such that $\mathcal{Y}_s$ is smooth. Since the set $\{s \in S \mid \mathcal{Y}_s \text{ is smooth}\}$ is open in $S$, the result follows. $\square$

**Moduli and characterization of coverings**

The study of the functor $\overline{\underline{\mathcal{H}}}_E^\infty$ has been carried out exhaustively in [13] for the case $r=1$. However, since an arbitrary covering may have more than one ramification point, this case does not allow us to characterize general coverings.

Since the number of ramification points of a covering is finite, the previous study of $\overline{\underline{\mathcal{H}}}_E^\infty$ does make it possible to find characterizations for *all* coverings.

Let $\pi : Y \to X$ be a finite covering of degree $n$ of smooth integral curves. Let $\bar{g}$ and $g$ be the genus of $Y$ and $X$ respectively. Then, the Hurwitz formula reads

$$1-\bar{g} \;=\; n(1-g) - \frac{1}{2}\sum_{y\in Y}(e_y - 1)\,,$$

where $e_y$ is the ramification index of the point $y \in Y$.

Let $\{x_1,\dots,x_r\} \subset X$ be the branch locus and let us denote

$$\begin{aligned} \bar{y} \;&=\; \pi^{-1}(x_1)+\cdots+\pi^{-1}(x_r) \\ \pi^{-1}(x_i) \;&=\; e_1^{(i)}y_1^{(i)}+\cdots+e_{k_i}^{(i)}y_{k_i}^{(i)} \end{aligned}$$

Considering $E=\{\bar{e}_1,\dots,\bar{e}_r\}$, $\bar{e}_i=\{e_1^{(i)},\dots,e_{k_i}^{(i)}\}$ and $\bar{r}=\sum_{i=1}^r k_i$, one has that

$$\sum_{y\in Y}(e_y-1) \;=\; \sum_{i=1}^{r}\sum_{j=1}^{k_i}(e_j^{(i)}-1) \;=\; rn-\bar{r}$$

Thus the Hurwitz formula can be rewritten as

$$1-\bar{g} \;=\; n(1-g)-\frac{1}{2}(rn-\bar{r})$$

**Definition 3.3.** For integers $i,j$, we define the following subschemes of $\mathcal{H}_E^\infty$

$$\begin{aligned} \mathcal{H}_E^\infty[j] \;&:=\; \{U\in\mathcal{H}_E^\infty\cap \mathrm{Gr}^{1-j}(W) \text{ such that } U \text{ is an integral domain}\} \\ \mathcal{H}_E^\infty[j,i] \;&:=\; \{U\in\mathcal{H}_E^\infty[j] \text{ such that } \operatorname{Tr}(U)\in\mathrm{Gr}^{1-i}(V)\} \end{aligned}$$

Note that since $\operatorname{Tr}(U)\subseteq U$ for any $U\in\mathcal{H}_E^\infty$, the condition that $U$ is an integral domain implies that $\operatorname{Tr}(U)$ is integral too.

From the representability of $\mathcal{H}_E^\infty$ and the Hurwitz formula for coverings, one has the following

**Theorem 3.3.** *Let $(E,n,\bar{r})$ be a set of numerical data as in §2, and let $g,\bar{g}$ be two non-negative integer numbers satisfying*

$$\bar{g}-1 \;=\; n(g-1)+\frac{1}{2}(rn-\bar{r})$$

*Accordingly, the subscheme $\mathcal{H}_E^\infty[\bar{g},g]\subset \mathrm{Gr}^{1-\bar{g}}(W)$ is the moduli scheme parametrizing geometrical data $(Y,X,\pi,\bar{x},\bar{y},t_{\bar{x}},t_{\bar{y}})\in\mathcal{H}_E^\infty$, where $Y$ has genus $\bar{g}$, $X$ has genus $g$, the covering $\pi:Y\to X$ is non-ramified outside $\bar{y}$, and the ramification index at the point $y_j^{(i)}\in Y$ is $e_j^{(i)}$.*

**Remark 2.** Let us observe that there exists a natural forgetful morphism

$$\begin{aligned}\Phi\colon \mathcal{H}_E^\infty[\bar{g},g] &\longrightarrow \mathcal{M}_g^\infty(r)\\ (Y,X,\pi,\bar{x},\bar{y},t_{\bar{x}},t_{\bar{y}}) &\longmapsto (X,\bar{x},t_{\bar{x}})\end{aligned}$$

Given $(X,\bar{x},t_{\bar{x}})\in\mathcal{M}_g^\infty(r)$ with $X$ smooth and integral, the fiber of this point will be denoted by $\mathcal{H}_E^\infty(X,\bar{x},t_{\bar{x}})$. Recall that there is a finite number of coverings $\pi: Y\to X$ with $Y$ and $X$ smooth, $X$ integral, $\pi$ is dominant on each component of $Y$, non-ramified outside $\bar{x}$, and with ramification indices $\bar{e}_i$ at $x_i$ (see [3, 15]). Then, we may conclude that $\mathcal{H}_E^\infty(X,\bar{x},t_{\bar{x}})$ is a finite set.

Let $E$ be a set of partitions as above, $Y$ be a proper smooth curve and let $\bar{y}$ be $\{y_1^{(1)},\dots,y_{k_1}^{(1)},\dots,y_1^{(r)},\dots,y_{k_r}^{(r)}\}$. Then, we say that $(Y,\bar{y})$ is a covering with ramification profile $E$ if and only if there is another curve $X$ and a map $\pi: Y\to X$ such that the ramification divisor of $\pi$ is equal to $\sum_{i=1}^r\sum_{j=1}^{k_i} e_j^{(i)}y_j^{(i)}$.

**Theorem 3.4.** *Fix $m$ a non-positive integer $m$, and let $\{u_{1,1},\dots,u_{r,k_r}\}$ be integers defined by $v_m = z_1^{u_{1,1}/e_1^{(1)}}\dots z_r^{u_{r,k_r}/e_{k_r}^{(r)}}$ where $v_m\in W$ was chosen to define $\tau$-functions. Let $\xi_e$ be a primitive $e$-th root of unity and $\delta$ denote the Kronecker symbol.*

*Let $(Y,\bar{y},t_{\bar{y}})\in\mathcal{M}^\infty(\bar{r})\subset \mathrm{Gr}^m(W)$ be a closed point such that $Y$ is smooth.*

*Then, $(Y,\bar{y},t_{\bar{y}})$ is a covering of another curve with ramification profile $E$ if and only if the following "bilinear identities" are satisfied; that is, the form*

$$\Big(\sum_{l=1}^{r}\sum_{j=1}^{k_l}\sum_{i=1}^{e_j^{(l)}}\frac{\psi^{(l,j)}_{a,b,B}(\xi^i_{e_j^{(l)}}z_l^{1/e_j^{(l)}},t)}{(\xi^i_{e_j^{(l)}}z_l^{1/e_j^{(l)}})^{\delta_{al}\delta_{bj}-u_{l,j}}}\Big)\Big(\sum_{l=1}^{r}\sum_{j=1}^{k_l}\sum_{i=1}^{e_j^{(l)}}\frac{\psi^{(l,j)}_{c,d,B}(\xi^i_{e_j^{(l)}}z_l^{1/e_j^{(l)}},t)}{(\xi^i_{e_j^{(l)}}z_l^{1/e_j^{(l)}})^{u_{l,j}-1+\delta_{cl}\delta_{dj}}}\Big)\frac{dz}{z}$$

*has residue zero at $z=0$ for all $a,b,c,d$.*

*Proof.* The claim is equivalent to characterizing $\overline{\mathcal{H}}_E^\infty$ as a subscheme of $\mathcal{M}^\infty(\bar{r})$. The idea is standard and consists of translating the second condition of the previous statement in terms of Baker-Akhiezer functions by means of Theorem 2.1. □

**Remark 3.** Using formula 2.1, the above set of equations can be rewritten as a hierarchy of differential equations for $\tau$ functions. Furthermore, since we are considering points in $\mathcal{M}^\infty(\bar{r})$ it would be possible to express those equations as differential equations for theta functions ([12]).

**Remark 4.** For the case $r=1$ and $\bar{e}_1=\{1,\overset{r}{\dots},1\}$ the previous hierarchy coincides with the $r$-component KP hierarchy. This links our work with the theory of pseudodifferential operators and representation theory (see [7]).

## 4. Tangent space to the Hurwitz scheme

This section is devoted to an explicit computation of the tangent space of the Hurwitz schemes constructed in the previous section. To achieve this goal, we begin by recalling the computation of the tangent spaces to the infinite Grassmannian and to the moduli space of pointed curves.

**Proposition 4.1.** *Let $U$ be a rational point of* $\mathrm{Gr}(W)$. *There is a canonical isomorphism*

$$T_U \,\mathrm{Gr}(W) \xrightarrow{\sim} \mathrm{Hom}(U, W/U)$$

*Proof.* Let $A \sim W_+$ and $U \in \mathrm{Gr}(W)$ such that $U \oplus A \simeq W$. Let $F_A$ be the open subscheme of $\mathrm{Gr}(W)$ parametrizing those subspaces $U' \in \mathrm{Gr}(W)$ such that $U' \oplus A \simeq W$. Then, it is well known that $F_A$ is isomorphic to the affine space $\mathrm{Hom}(U, A)$ and that the embedding

$$\mathrm{Hom}(U, A) \overset{\sim}{\to} F_A \hookrightarrow \mathrm{Gr}(W)$$

maps $f : U \to A$ to its graph $\Gamma_f := \{u + f(u) | u \in U\}$.

Since $U \in F_A$ (it corresponds to the zero map) and $F_A$ is open, we obtain an isomorphism of vector spaces

$$\mathrm{Hom}(U, A) \overset{\sim}{\to} T_0 \,\mathrm{Hom}(U, A) \overset{\sim}{\to} T_U \,\mathrm{Gr}(W) \overset{\sim}{\to} \mathrm{Gr}(W)(k[\epsilon]/\epsilon^2) \times_{\mathrm{Gr}(W)(k)} \{U\}$$

which maps $f \in \mathrm{Hom}(U, A)$ to the $(k[\epsilon]/\epsilon^2)$-valued point of $\mathrm{Gr}(W)$ given by $\{u + \epsilon f(u) | u \in U\}$.

Composing the inverse of this map with the isomorphism

$$\begin{aligned} \mathrm{Hom}(U, A) &\xrightarrow{\sim} \mathrm{Hom}(U, W/U) \\ f &\longmapsto \quad U \overset{\mathrm{Id}+f}{\to} U \oplus A \overset{\sim}{\to} W \to W/U \end{aligned}$$

we obtain the desired isomorphism (observe that it does not depend on the choice of $A$). $\square$

**Proposition 4.2.** *Let $U$ be a rational point of $\mathcal{M}^\infty(\bar{r})$. The isomorphism of Proposition* 4.1 *induces a canonical identification*

$$T_U \mathcal{M}^\infty(\bar{r}) \simeq \mathrm{Der}(U, W/U)$$

*(where* Der *means derivations trivial over* $\mathbb{C}$*).*

*Furthermore, if $U$ is associated to the geometrical data $(C, \bar{p}, \bar{z})$ under the Krichever map, then $W/U \simeq H^0(C - \bar{p}, \omega_C)^*$.*

*Proof.* For $U \in \mathcal{M}^\infty(\bar{r})$, one has that

$$T_U \mathcal{M}^\infty(\bar{r}) = \{\bar{U} \in T_U \,\mathrm{Gr}(W) \text{ such that } \bar{U} \cdot \bar{U} \subseteq \bar{U} \text{ and } k[\epsilon] \subset \bar{U}\}$$

From Proposition 4.1, there is a map $f \in \mathrm{Hom}(U, A)$ such that $\bar{U} = \{u + \epsilon f(u) | u \in U\}$.

The condition $\bar{U} \cdot \bar{U} \subseteq \bar{U}$ means that for $u, u' \in U$ there exists $u'' \in U$ satisfying $(u + \epsilon f(u)) \cdot (u' + \epsilon f(u')) = u'' + \epsilon f(u'')$; that is

$$f(u \cdot u') \;=\; u f(u') + f(u) u' . \tag{4.1}$$

The second condition, $k[\epsilon] \subset \bar{U}$, implies that there exists $u_0 \in U$ such that $u_0 + \epsilon f(u_0) = 1$; or, in other words

$$f(1) \;=\; 0 . \tag{4.2}$$

It is now easy to check that the image of $f \in \operatorname{Hom}(U, A)$ in $\operatorname{Hom}(U, W/U)$ gives rise to a derivation $D_f \in \operatorname{Der}(U, W/U)$ (note that $W/U$ is a $U$-module). The first part follows from a straightforward check.

Consider the following exact sequence of $\mathcal{O}_C$-modules

$$0 \to \mathcal{O}_C(-m\bar{p}) \to \mathcal{O}_C(n\bar{p}) \to \mathcal{O}_C(n\bar{p})/\mathcal{O}_C(-m\bar{p}) \to 0$$

Taking inverse limit in $m$ and direct limit in $n$ in the induced long exact sequence of cohomology groups, one obtains

$$\begin{tikzcd} 0 \arrow[r] & H^0(C - \bar{p}, \mathcal{O}_C) \arrow[r] \arrow[d, "\wr"] & W \simeq (\widehat{\mathcal{O}}_{\bar{p}})_{(0)} \arrow[r] \arrow[d, "\wr"] & \varprojlim_m H^1(C, \mathcal{O}_C(-m\bar{p})) \arrow[r] & 0 \\ & U & W & & \end{tikzcd}$$

and the second claim follows. $\square$

Note that a similar result holds for points of $\mathcal{M}^\infty(r) \subset \operatorname{Gr}(V)$.

Let us denote

$$\operatorname{Der}(U, W/U)^{\operatorname{Tr}} := \left\{ D \in \operatorname{Der}(U, W/U) \text{ such that } \begin{tikzcd} U \arrow[r, "D"] \arrow[d, "\operatorname{Tr}"'] & W/U \arrow[d, "\operatorname{Tr}^1"] \\ \operatorname{Tr} U \arrow[r, "D"] & V/\operatorname{Tr} U \end{tikzcd} \right\} \tag{4.3}$$

where $\operatorname{Tr}^1$ is the map induced by the trace (since $\operatorname{Tr} U \subseteq U$ it makes sense to consider the restriction of $D$ to $\operatorname{Tr} U$).

**Theorem 4.1.** *Let $U$ be a rational point of $\overline{\mathcal{H}}_E^\infty$. The embedding $\overline{\mathcal{H}}_E^\infty \hookrightarrow \mathcal{M}^\infty(\bar{r})$ yields an identification*

$$T_U \overline{\mathcal{H}}_E^\infty \;\simeq\; \operatorname{Der}(U, W/U)^{\operatorname{Tr}}$$

*Moreover, if $U$ corresponds to the geometrical data $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ in $\mathcal{H}_E^\infty$, then $\operatorname{Tr}^1$ is the map*

$$\operatorname{Tr}^1 \colon H^0(Y - y, \omega_Y)^* \;\longrightarrow\; H^0(X - x, \omega_X)^*$$

*canonically induced by the trace $\pi_* \mathcal{O}_Y \to \mathcal{O}_X$.*

*Proof.* Let us keep the notations of the proofs of the two previous propositions. Then, let $\bar{U} \in \operatorname{Gr}(W)$ be a $k[\epsilon]/\epsilon^2$-valued point lying in $T_U\mathcal{M}^\infty(\bar{r})$. Let $f \in \operatorname{Hom}(U, A)$ correspond to $\bar{U}$.

Thus, the condition $\operatorname{Tr}(\bar{U}) \subseteq \bar{U}$ (Theorem 3.1) is equivalent to saying that for each $u \in U$ there exists $u' \in U$ satisfying $\operatorname{Tr}(u + \epsilon f(u)) = u' + \epsilon f(u')$. Since $\operatorname{Tr}(u) \in U$, this condition is

$$\operatorname{Tr}^1(f(u)) \;=\; f(\operatorname{Tr}(u)) \qquad \forall u \in U \tag{4.4}$$

and the first part of the claim follows easily.

Let us prove the second part. Note that the trace is a sheaf homomorphism

$$\operatorname{Tr}\colon \pi_*\mathcal{O}_Y \;\longrightarrow\; \mathcal{O}_X$$

This map induces

$$H^1(X, (\pi_*\mathcal{O}_Y)(-n\bar{x})) \;\longrightarrow\; H^1(X, \mathcal{O}_X(-n\bar{x}))$$

Bearing in mind the adjunction formula, that $\pi$ is affine, Serre duality, and taking limits, one obtains the desired map

$$\operatorname{Tr}^1\colon H^0(Y - \bar{y}, \omega_Y)^* \;\longrightarrow\; H^0(X - \bar{x}, \omega_X)^*\,.$$

Using arguments similar to those of the proof of Proposition 4.2, it is not difficult to check that this map is compatible with the isomorphisms $H^0(Y - \bar{y}, \omega_Y)^* \simeq W/U$ and $H^0(X - \bar{x}, \omega_X)^* \simeq V/\operatorname{Tr}(U)$. □

**Theorem 4.2.** *Let $(E, n, r, \bar{g}, g)$ as in Theorem 3.3. Let $U \in \mathcal{H}_E^\infty[\bar{g}, g]$ be a rational point. Then, there is a canonical injection*

$$T_U\mathcal{H}_E^\infty[\bar{g}, g] \;\hookrightarrow\; T_{\operatorname{Tr} U}\mathcal{M}^\infty(r)\,.$$

*Proof.* By Proposition 4.2 and Theorem 4.1, the claim is equivalent to the injectivity of the restriction map

$$\operatorname{Der}(U, W/U)^{\operatorname{Tr}} \;\longrightarrow\; \operatorname{Der}(\operatorname{Tr} U, V/\operatorname{Tr} U)\,.$$

Let $D \in \operatorname{Der}(U, W/U)^{\operatorname{Tr}}$ be in the kernel of the above map; that is, $D|_{\operatorname{Tr}(U)} = 0$. Let $u$ be any element in $U$ and let us see that $Du = 0$. Since $A := \operatorname{Tr}(U) \to U$ is an integral morphism, there is a monic minimal polynomial $p(x) = \sum_i a_i x^i \in A[x]$ such that $p(u) = 0$. Therefore, the following identity holds true

$$0 \;=\; Dp(u) \;=\; \sum_i (Da_i)u^i + p'(u)Du \;=\; p'(u)Du\,.$$

($p'(x)$ denoting the derivative w.r.t. $x$).

Since $p(x)$ is separable and $\pi$ is unramified over $X - \bar{x} = \operatorname{Spec} A$, then $\frac{1}{p'(u)} \in A[u] \subseteq U$. Therefore, one has $Du = 0$. □

## 5. The multicomponent Virasoro group

The algebraic Virasoro group, defined as $G := \underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}((z))$, was introduced and studied in [14]. In this section it will be generalized for certain $\mathbb{C}((z))$-algebras. Note that $V$ carries the linear topology given by $\{z^n V_+\}_{n\in\mathbb{Z}}$.

It is convenient to recall the following result: *let $R$ be a $\mathbb{C}$-algebra and let $f(z) \in R((z))$. Then $f(z)$ is invertible if and only if there exists $n \in \mathbb{Z}$ such that $a_i \in \mathrm{Rad}(R)$ for $i < n$ and $a_n$ is invertible.*

**Definition 5.1.** The functor of automorphisms of $V$ is the functor defined from the category of $\mathbb{C}$-schemes to the category of groups defined as follows

$$S \rightsquigarrow \underline{\mathrm{Aut}}_{\mathbb{C}}(V)(S) := \mathrm{Aut}_{H^0(S,\mathcal{O}_S)}\, V\widehat{\otimes}_{\mathbb{C}} H^0(S,\mathcal{O}_S)$$

where $\mathrm{Aut}_R$ means continuous automorphisms of $R$-algebras.

**Lemma 5.1.** *Let $S$ be a $\mathbb{C}$-scheme and let $\phi \in \underline{\mathrm{Aut}}_{\mathbb{C}}(V)(S)$. For any point $s \in S$, let $p_{\phi(s)}$ be the permutation defined by $\phi(s)$ on the set $\mathrm{Spec}(V_s)$ (note that $V_s := V\widehat{\otimes}_{\mathbb{C}} k(s)$ consists of $r$ points).*

*Then, the map*

$$\begin{aligned} S &\longrightarrow \mathcal{S}_r \\ s &\longmapsto p_{\phi(s)} \end{aligned}$$

*($\mathcal{S}_r$ being the symmetric group of $r$ letters) is locally constant.*

*Proof.* Let us assume that $S = \mathrm{Spec}(R)$ is an irreducible $\mathbb{C}$-scheme. Let $\phi \in \underline{\mathrm{Aut}}_{\mathbb{C}}(V)(S)$. Since $V_s := V\widehat{\otimes}_{\mathbb{C}} k(s) \simeq \prod^r k(s)((z))$ is a product of fields, it follows that $\phi(s) \in \underline{\mathrm{Aut}}_{\mathbb{C}}(V)(k(s))$ acts by permutation on $\mathrm{Spec}(V_R)$; that is, on the set of ideals $I_1 := ((z_1, 0, \dots, 0))$, $\dots$, $I_r := ((0, \dots, 0, z_r))$. In other words, there exists $p_{\phi(s)} \in \mathcal{S}_r$ such that

$$\phi(s)(I_i) \;=\; I_{p_{\phi(s)}(i)}$$

Let us write $\phi(0, \dots, z_i, \dots, 0) = (\phi_1^{(i)}, \dots, \phi_r^{(i)}) \in R((z_1)) \times \cdots \times R((z_r))$. Let $s_0$ denote the point associated with the minimal prime ideal of $R$. The very definition of the permutation $p_{\phi(s_0)}$ states that

$$\phi_j^{(i)}(s_0) \in k(s_0)((z_j)) \text{ is } \begin{cases} \text{invertible if } j = p_{\phi(s_0)}(i) \\ \text{zero if } j \neq p_{\phi(s_0)}(i) \end{cases}$$

and, therefore, $s_0 \in (\phi_j^{(i)})_0$ for $j \neq p_{\phi(s_0)}(i)$. Since $s_0$ is minimal, $S$ is irreducible, and $(\phi_j^{(i)})_0$ is closed, it follows that the closure of $s_0$, $S$, is contained in $(\phi_j^{(i)})_0$ for $j \neq p_{\phi(s_0)}(i)$.

Thus, it turns out that $\phi_j^{(i)}(s) \neq 0$ for $j = p_{\phi(s_0)}(i)$ and for all $s \in S$; equivalently, $\phi_j^{(i)} \in R((z_j))$ is invertible for $j = p_{\phi(s_0)}(i)$, and therefore, $p_{\phi(s_0)} = p_{\phi(s)}$ for all $s \in S$. The statement follows. $\square$

As a consequence of the previous lemma, we have a well defined map of group functors

$$\underline{\mathrm{Aut}}_{\mathbb{C}}(V) \xrightarrow{p} \mathcal{S}_r$$

given by $p(\phi) := p_\phi$. Here, the scheme structure of the finite group $\mathcal{S}_r$ is considered to be that of the $\mathbb{C}$-scheme $\operatorname{Spec}\big(\prod^{r!}\mathbb{C}\big)$.

**Theorem 5.1.** *Let $V$ be the $\mathbb{C}$-algebra $\mathbb{C}((z_1))\times\cdots\times\mathbb{C}((z_r))$ and $V_+$ the subalgebra $\mathbb{C}[[z_1]]\times\cdots\times\mathbb{C}[[z_r]]$.*

*The canonical exact sequence of group functors*

$$0 \to G\times\overset{r}{\dots}\times G \xrightarrow{i} \underline{\mathrm{Aut}}_{\mathbb{C}}(V) \xrightarrow{p} \mathcal{S}_r \to 0$$

*splits.*

*In particular, $\underline{\mathrm{Aut}}_{\mathbb{C}}(V)$ is representable by a formal $\mathbb{C}$-group scheme which will be denoted by $G_V$.*

*Proof.* The map $i$ is the canonical inclusion

$$\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}((z_1))\times\cdots\times\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}((z_r)) \hookrightarrow \underline{\mathrm{Aut}}_{\mathbb{C}}(V)$$

Let us note that one can associate with every permutation $\sigma\in\mathcal{S}_r$ the automorphism of $V$ defined by $z_i\mapsto z_{\sigma^{-1}(i)}$. □

Let $G_V^0$ be the connected component of the identity of $G_V$. Accordingly, $G_V^0$ acts on $\mathrm{Gr}(V)$ and this action yields an action on the subscheme $\mathcal{M}^\infty(r)\subset\mathrm{Gr}(V)$.

Let us now consider the groups $G_V$ and $G_W$ corresponding to the $\mathbb{C}$-algebras $V$ and $W$ respectively.

The $V$-algebra structure of $W$ allows us to define a subgroup $G_V^W\subseteq G_W^0$ as follows

$$G_V^W := \left\{ \begin{array}{c} \begin{array}{ccc} W & \xrightarrow[\sim]{\bar g} & W \\ \uparrow & & \uparrow \\ V & \xrightarrow[\sim]{g} & V \end{array} \quad \text{where } \bar g\in G_W^0 \text{ and } g\in G_V^0 \end{array}\right\} \tag{5.1}$$

The element of $G_V^W$ given by such a diagram will be denoted by $\bar g$.

**Proposition 5.1.** *There exists an exact sequence of formal $\mathbb{C}$-group schemes*

$$0\to\mu_E\to G_V^W \xrightarrow{\pi} G_V^0\to 0$$

*where $\pi(\bar g)=g$ and*

$$\mu_E := (\mu_{e_1^{(1)}}\times\cdots\times\mu_{e_{k_1}^{(1)}})\times\cdots\times(\mu_{e_1^{(r)}}\times\cdots\times\mu_{e_{k_r}^{(r)}})$$

*($\mu_{e_i^{(j)}}\subset\mathbb{C}^*$ being the group of $e_i^{(j)}$-th roots of unity).*

*Proof.* From the previous result it can be assumed that $r = 1$ and $k_1 = 1$; that is, $V = \mathbb{C}((z))$ and $W = \mathbb{C}((z^{1/e}))$. Then, one has that $G_V \simeq G$ and $G_W \simeq G$.

Note that an element $\bar{g} \in G^0_W$ belongs to $G^W_V$ if and only if $\bar{g}(z^{1/e})^e \in V$. And the result follows. $\square$

**Corollary 5.1.** *The canonical restriction map $G^W_V \to G^0_V$ yields an isomorphism of Lie algebras*

$$\operatorname{Lie} G^W_V \xrightarrow{\sim} \operatorname{Lie} G^0_V$$

**Lemma 5.2.** *Let $R$ be a $\mathbb{C}$-algebra and $f(z^{1/e}) \in R((z^{1/e}))$.*

*If $f(z^{1/e})^e \in R((z))$ and $f(z^{1/e})$ is invertible, then there exist $i$ such that $z^{i/e} f(z^{1/e}) \in R((z))$.*

*Proof.* We may assume that $f(z^{1/e}) = \sum_i a_i z^{i/e}$ where $a_i \in \operatorname{Rad}(R)$ for $i < 0$ and $a_0$ is invertible. Let $I \subseteq \operatorname{Rad}(R)$ be the ideal generated by $\{a_i | i < 0\}$ (recall that this set is finite) and let $n \geq 0$ be the smallest integer such that $I^{n+1}$ vanishes. Let us proceed by induction on $n$.

Case $n = 1$; that is $a_i \cdot a_j = 0$ for all negative integers $i, j$. Let $i_0$ be the smallest index for which $a_{i_0} \neq 0$. The hypothesis implies that

$$f(z^{1/e})^e \;=\; e \cdot a_0^{e-1} \cdot a_{i_0} \cdot z^{i_0/e} + (\text{higher order terms}) \;\in\; R((z))$$

and, therefore, $e | i_0$. To conclude it suffices to consider $(a_0 + a_{i_0} z^{i_0/e})^{-1} \cdot f(z^{1/e})$ and iterate this argument.

General case. Let $\bar{f}_n(z^{1/e})$ be the class of $f(z^{1/e})$ in $R/I^n((z^{1/e}))$. From the induction hypothesis it follows that $z^{i/e} \bar{f}_n(z^{1/e}) \in R/I^n((z))$. Let $f_n(z) \in R((z))$ be a preimage of $\bar{f}_n$. Consider now the element $\frac{f(z^{1/e})}{f_n(z)} \in R((z^{1/e}))$. From the $n = 1$ case, it follows that $f(z^{1/e})(f_n(z))^{-1} \in R((z))$, and the claim follows. $\square$

**Lemma 5.3.** *Let $R$ be a $\mathbb{C}$ algebra, let $\operatorname{Tr} : \widehat{W}_R \to \widehat{V}_R$ be $\widehat{V}_R$ be the trace map and let $\bar{g}$ be an element of $G^0_W$.*

*Then, $\bar{g} \in G^W_V$ if and only if $\operatorname{Tr} \circ \bar{g} = \bar{g} \circ \operatorname{Tr}$.*

*Proof.* Note that it suffices to prove the claim for the following case

$$V = R((z)) \hookrightarrow W = R((z^{1/e}))\,.$$

Let us show that

$$\operatorname{Tr}(\bar{g} w) \;=\; \pi(\bar{g}) \operatorname{Tr}(w) \;=\; g(\operatorname{Tr}(w)) \qquad \forall w \in \widehat{W}_R\,. \tag{5.2}$$

An element $\bar{g} \in G^0_W(R)$ is of the form

$$\bar{g}(z^{1/e}) \;=\; z^{1/e} \cdot (\sum_i a_i z^{i/e}) \;\in\; R((z^{1/e}))\,,$$

where $a_i$ is nilpotent for $i<0$ and $a_0$ invertible. The condition $\bar{g}\in G_V^W$ implies that

$$\bar{g}(z^{1/e})^e \,=\, \bar{g}(z)\,\in\, R((z))\,,$$

and, by the previous lemma, it follows that $\sum_i a_i z^{i/e}\in R((z))$, that is, $a_i=0$ if $e$ does not divide $i$. In particular, $g=\pi(\bar{g})$ is the automorphism given by

$$g(z)\,=\,z(\sum_j a_{je}z^j)\,\in\, R((z))\,.$$

By linearity, it is sufficient to check the claim for $z^{l/e}$ where $l\in\mathbb{Z}$,

$$\operatorname{Tr}(\bar{g}(z^{l/e}))\,=\,\operatorname{Tr}(z^{l/e}\cdot(\sum_j a_{je}z^j))^l)\,=\,(\sum_j a_{je}z^j))^l\cdot\operatorname{Tr}(z^{l/e})\,=\,g(\operatorname{Tr}(z^{l/e}))$$

since $\operatorname{Tr}(z^{l/e})=0$ for $e$ not dividing $l$ and $\operatorname{Tr}(z^{l/e})=e\cdot z^{l/e}$ for $e$ dividing $l$.

Conversely. For $\bar{g}\in G_W^0$ commuting with the trace, we have that

$$\begin{gathered}\bar{g}(\operatorname{Tr} z)\,=\,\bar{g}(e\cdot z)\,=\,e\cdot\bar{g}(z^{1/e})^e\\ \operatorname{Tr}(\bar{g}(z))\,\in\, R((z))\end{gathered}$$

Therefore, one has that $\bar{g}(z^{1/e})^e\in R((z))$. The previous lemma implies that $\bar{g}(z^{1/e})$ is of the form $z^{1/e}(\sum_i a_i z^i)$, which belongs to $G_V^W$. □

**Theorem 5.2.** *It holds that*

1. $\operatorname{Lie} G_V\simeq\oplus_{i=1}^r\operatorname{Der}(\mathbb{C}((z_i)),\mathbb{C}((z_i)))\simeq\oplus_{i=1}^r\mathbb{C}((z_i))\frac{\partial}{\partial z_i}$;
2. $\operatorname{Lie} G_W\simeq\oplus_{i=1}^r\oplus_{j=1}^{k_i}\operatorname{Der}(\mathbb{C}((z_i^{1/e_j^{(i)}})),\mathbb{C}((z_i^{1/e_j^{(i)}})))$;
3. $\operatorname{Lie} G_V^W\simeq(\operatorname{Lie} G_W)^{\operatorname{Tr}}$ *(those derivations commuting with the trace; as in (4.3)).*

*Proof.* Recall that Theorem 3.5 of [14] claims that $\operatorname{Lie} G\sim\mathbb{C}((z))\frac{\partial}{\partial z}$. Then, the statements follow from Theorem 5.1, Corollary 5.1 and Lemma 5.3. □

**Theorem 5.3.** *The group $G_V^W$ acts on $\overline{\mathcal{H}}_E^\infty$.*

*Proof.* Recall that $\overline{\mathcal{H}}_E^\infty$ consists of those points $U$ of $\mathcal{M}^\infty(\bar{r})$ such that $\operatorname{Tr}(U)\subseteq U$ and that $G_V^W$ acts on $\mathcal{M}^\infty(\bar{r})$. Therefore, it suffices to show that the condition $\operatorname{Tr}(U)\subseteq U$ implies that $\operatorname{Tr}(\bar{g}U)\subseteq\bar{g}U$ for all $\bar{g}\in G_V^W$.

Let $U$ be a point of $\overline{\mathcal{H}}_E^\infty$. From Lemma 5.3 and from the inclusion $\operatorname{Tr}(U)\subseteq U$, one has

$$\operatorname{Tr}(\bar{g}U)\,=\,\bar{g}(\operatorname{Tr}(U))\,\subseteq\,\bar{g}U$$

and the statement follows. □

**Theorem 5.4.** *Let $(E,n,r,\bar{g},g)$ as in Theorem 3.3. The group $G_V^W$ acts on $\mathcal{H}_E^\infty[\bar{g},g]$ and this action is locally transitive.*

*Proof.* It is straightforward to see that the action on $\overline{\mathcal{H}}_E^\infty$ gives an action on $\mathcal{H}_E^\infty[\bar{g}, g]$. From [14] (Lemma 4.10, Theorem 4.11 and Lemma A.2), the proof is reduced to checking the surjectivity of the map of tangent spaces

$$T_{\mathrm{Id}} G_V^W \longrightarrow T_U \mathcal{H}_E^\infty[\bar{g}, g] \tag{5.3}$$

Let $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}})$ be the data attached to $U \in \mathcal{H}_E^\infty[\bar{g}, g]$. Then, Theorems 4.1, 4.2 and 5.2 and Corollary 5.1 give the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(Y - \bar{y}, \mathbb{T}_Y) \arrow[r] & \operatorname{Lie} G_W^0 \arrow[r] & \operatorname{Der}(U, W/U) \arrow[r] & 0 \\
 & & \operatorname{Lie} G_V^W \arrow[u, hook] \arrow[r, "\psi"] \arrow[d, "\wr"] & \operatorname{Der}(U, W/U)^{\mathrm{Tr}} \arrow[u, hook] \arrow[d, hook] & \\
0 \arrow[r] & H^0(X - \bar{x}, \mathbb{T}_X) \arrow[r] & \operatorname{Lie} G_V^0 \arrow[r] & \operatorname{Der}(\operatorname{Tr} U, V/\operatorname{Tr} U) \arrow[r] & 0
\end{tikzcd}$$

($\mathbb{T}$ denoting the tangent sheaf). From the diagram, we deduce that $\psi$ is surjective. Since $\psi$ coincides with the map (5.3), we conclude. □

**Theorem 5.5.** *Let* $(E, n, r, \bar{g}, g)$ *as in Theorem* 3.3. *Let* $U \in \mathcal{H}_E^\infty[\bar{g}, g]$ *be a rational point. Then, there is an isomorphism*

$$T_U \mathcal{H}_E^\infty[\bar{g}, g] \xrightarrow{\sim} T_{\operatorname{Tr} U} \mathcal{M}^\infty(r)$$

*Proof.* The injectivity follows from Theorem 4.2. The surjectivity is a consequence of the diagram of the previous proof. □

**Remark 5.** This Theorem is the analog of the fact that the map from the classical Hurwitz space to the moduli of curves is étale at those points corresponding to covers where both curves are smooth. Our approach also allows us to study the non-smooth case; however, because of our goals we have only focused on the smooth case.

## 6. Picard schemes

**Definition 6.1.** Let $\underline{\mathrm{Pic}}_E^\infty[\bar{g}, g]$ be the contravariant functor from the category of $\mathbb{C}$-schemes to the category of sets defined by

$$S \rightsquigarrow \{(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}, L, \phi_{\bar{y}})\}$$

where

1. $(Y, X, \pi, \bar{x}, \bar{y}, t_{\bar{x}}, t_{\bar{y}}) \in \mathcal{H}_E^\infty[\bar{g}, g](S)$;
2. $L$ is an invertible sheaf on $Y$ and $\phi_{\bar{y}}$ is a formal trivialization of $L$ along $\bar{y}$; that is, an isomorphism $\phi_{\bar{y}} : \widehat{L}_{\bar{y}} \simeq \widehat{\mathcal{O}}_{Y, \bar{y}}$.
3. two sets of data are said to be equivalent when there is an isomorphism of $S$-schemes $Y \xrightarrow{\sim} Y'$ compatible with all the data.

**Theorem 6.1.** *The functor* $\underline{\mathrm{Pic}}_E^\infty[\bar g, g]$ *is representable by a subscheme* $\mathrm{Pic}_E^\infty[\bar g, g]$ *of* $\mathrm{Gr}(W)$.

*Proof.* Consider the morphism from $\underline{\mathrm{Pic}}_E^\infty[\bar g, g]$ to $\mathrm{Gr}(W)\times\mathrm{Gr}(W)$, which sends the $S$-valued point $(Y, X, \pi, \bar x, \bar y, t_{\bar x}, t_{\bar y}, L, \phi_{\bar y})$ to the following pair of submodules as a point of $\mathrm{Gr}(W)\times\mathrm{Gr}(W)$

$$\left( t_{\bar y}\left(\varinjlim_m (p_*\mathcal{O}_Y(m\cdot\pi^{-1}(\bar x))\right), (t_{\bar y}\circ\phi_{\bar y})\left(\varinjlim_m (p_* L(m\cdot\pi^{-1}(\bar x))\right)\right)$$

where $p : Y \to S$ is the projection.

This map is injective and the image is contained in the subscheme $Z\subset \mathrm{Gr}(W)\times\mathrm{Gr}(W)$ of those pairs $(A, L)$ satisfying

$$\mathbb{C}\subset A \quad , \quad A\cdot A\subseteq A \quad , \quad A\cdot L\subseteq L\ .$$

Let $(\mathcal{A},\mathcal{L})$ be the pair defined by the pullback to $Z$ of the universal submodules.

Applying the converse construction of the Krichever correspondence to the algebra $\mathcal{A}$ we obtain a curve $\mathcal{Y}\to Z$. Let us now consider the subscheme $Z'\subset Z$ defined by the points $z\in Z$ such that $\mathcal{Y}_z$ is smooth.

We now claim that if $(\mathcal{A},\mathcal{L})$ belongs to $Z'$, then $\mathcal{A}$ can be obtained from $\mathcal{L}$. Indeed, it will be shown that $\mathcal{A}$ is the stabilizer of $\mathcal{L}$.

Consider $(\mathcal{A},\mathcal{L})\in Z'$ and let $A_{\mathcal{L}}$ denote the stabilizer of $\mathcal{L}$, that is, the $\mathcal{O}_S$-algebra

$$A_{\mathcal{L}} := \{w\in \widehat{W}_S \text{ such that } w\cdot\mathcal{L}\subseteq\mathcal{L}\}\ ,\ .$$

Since $\mathcal{A}_z$ corresponds to a smooth curve for all $z\in Z'$ and $\mathcal{A}\subseteq A_{\mathcal{L}}$ are points of $\mathrm{Gr}(W)$, then $A_{\mathcal{L}}$ is a finite $\mathcal{A}$-module such that $\mathcal{A}_z = (A_{\mathcal{L}})_z$ for all $z\in Z'$. Therefore we have that $\mathcal{A} = A_{\mathcal{L}}$.

One now checks that the image of $Z'$ by the projection onto the second factor, $\mathrm{Gr}(W)\times\mathrm{Gr}(W)\to\mathrm{Gr}(W)$, does represent $\underline{\mathrm{Pic}}_E^\infty[\bar g, g]$. □

**Remark 6.** For $\chi\in\mathbb{Z}$, the subfunctor of $\underline{\mathrm{Pic}}_E^\infty[\bar g, g]$ consisting of those points such that $L$ has Euler-Poincaré characteristic equal to $\chi$ is representable by the subscheme $\mathrm{Pic}_E^\infty[\bar g, g]\cap\mathrm{Gr}^\chi(W)$.

Since $\Gamma_W$ represents the group of invertible elements of $W$ and $G_V^W$ is a group of automorphisms of $W$ as an algebra, one has a canonical action of $G_V^W$ on $\Gamma_W$. Therefore, it makes sense to consider the semidirect product $G_V^W\ltimes\Gamma_W$ as follows

$$(g_2,\gamma_2)(g_1,\gamma_1) := (g_2 g_1, g_1^{-1}(\gamma_2)\gamma_1)$$

and the action of $G_V^W\ltimes\Gamma_W$ on the Grassmannian induced by the action on $W$

$$(g,\gamma)w := g(\gamma\cdot w)\ .$$

**Theorem 6.2.** *There are canonical actions of the groups* $G_V^W$, $\Gamma_W$ *and* $\Gamma_W\ltimes G_V^W$ *on* $\mathrm{Pic}_E^\infty[\bar g, g]$.

*Moreover, the action of* $G_V^W\ltimes\Gamma_W$ *is locally transitive.*

*Proof.* Let

$$\Psi\colon \operatorname{Pic}_E^\infty[\bar g, g] \longrightarrow \mathcal{H}_E^\infty[\bar g, g]$$

be the forgetful morphism. Let us consider a rational point $p \in \operatorname{Pic}_E^\infty[\bar g, g]$ corresponding to the geometric data $(Y, X, \pi, \bar x, \bar y, t_{\bar x}, t_{\bar y}, L, \phi_{\bar y})$. Let $A := t_{\bar y}(H^0(Y - \bar y, \mathcal{O}_Y)) \in \mathcal{H}_E^\infty[\bar g, g]$ and let $U := t_{\bar y}(\phi_{\bar y}(H^0(Y - \bar y, L))) \in \operatorname{Pic}_E^\infty[\bar g, g]$.

Considering the map induced b $\Psi$ at the level of tangent spaces and recalling Theorem 5.4, one easily obtains the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Lie}\Gamma_W \arrow[r] \arrow[d] & \operatorname{Lie}(\Gamma_W \ltimes G_V^W) \arrow[r] \arrow[d] & \operatorname{Lie} G_V^W \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_{A-\mathrm{mod}}(U, W/U) \arrow[r] & T_p \operatorname{Pic}_E^\infty[\bar g, g] \arrow[r] & \operatorname{Der}(A, W/A)^{\mathrm{Tr}} \arrow[r] & 0
\end{tikzcd}$$

The snake lemma implies that the middle vertical arrow is surjective. We conclude by similar ideas as in the proof of Theorem 5.4. □

**Remark 7.** In future work and following ideas of [2] and [9] we plan to study how the deformations of a point $\operatorname{Pic}_E^\infty[\bar g, g]$ under the action of $G_V^W \ltimes \Gamma_W$ can be interpreted as isomonodromic deformations.

A related problem, namely, the case of Higgs bundles (or, equivalently, certain line bundles on a given covering) has already been treated in [5]. Finally, the arithmetic Grassmannian introduced in [16] may be a helpful technique when studying families of coverings.

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