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# Algebraic solutions of the multicomponent KP hierarchy☆

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**Abstract**

It is shown that it is possible to write down tau functions for the $n$-component KP hierarchy in terms of non-abelian theta functions. This is a generalization of the rank 1 situation; that is, the relation of theta functions of Jacobians and tau functions for the KP hierarchy. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

This paper is concerned with the generalization for higher rank of the relation of theta functions of Jacobians and the KP hierarchy. More precisely, the well-known relation between the theta function of a Jacobian and the tau function of a certain point of an infinite Grassmannian $\mathrm{Gr}^0(k((z)))$ ([17,31,32], see also [24,25,28]) is generalized for non-abelian theta functions and $\mathrm{Gr}^0(k((z))^{\oplus n^2})$. From this point of view, the main result is Theorem 5.1 that shows that it is possible to write down tau functions for the $n$-component KP hierarchy ($n$-KP) in terms of non-abelian thetas. However, further research must be made to obtain explicit expressions.

It is worth remarking that this result is twofold; on the one hand, it shows that non-abelian thetas give solutions for $n$-KP; on the other, it suggests the possibility of characterizing

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non-abelian thetas in terms of differential equations of the $n$-KP (see [24,32] for the rank 1 case).

In a certain sense, Section 2 is the core of the paper. It contains three new results; namely, Lemma 2.7 (Addition formula) and Theorems 2.9 and 2.11 (Residue Bilinear Identity). These results are the key for proving that there exists a one-to-one correspondence between wave functions for the $n$-KP and Baker–Akhiezer (BA) functions of points of an infinite Grassmannian $\mathrm{Gr}^0(k((z))^{\oplus n^2})$ (see Section 3). Then, our strategy is rather simple (see Section 5): a generalization of the well-known Krichever map [17,22,23] is used to show that there is a subscheme:

$$\iota : \hat{\mathcal{U}}(r, d) \hookrightarrow \mathrm{Gr}^0(k((z))^{\oplus n^2})$$

together with a projection onto the moduli space of vector bundles, $\pi : \hat{\mathcal{U}}(r, d) \to \mathcal{U}(r, d)$ (some data must be previously fixed) such that:

$$\iota_F^* \Omega_+ = \pi^* \theta_F, \qquad \iota_F^* \,\mathrm{Det} \tilde{\to} \pi^* \mathcal{O}(\Theta_{[F]})$$

(see Section 5 for notations and precise statements). This proves that non-abelian thetas give rise to tau functions for the $n$-KP.

Before explaning how the paper is organized, let us point out an intermediate result that deserves special mention; namely, Theorem 4.6 that computes the equations defining the subscheme of Gr $(k((z))^{\oplus n}) \times \mathrm{Gr}(k((z))^{\oplus n\cdot r})$ whose set of rational points is:

$$\{(C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta)\},$$

where $C$ is a curve, $p_i \in C$, $\alpha_i : \hat{\mathcal{O}}_{C,p_i} \simeq k[[z]]$, $M$ a rank $r$ torsion-free sheaf, and $\beta : \hat{M}_{\{p_i\}} \tilde{\to} k[[z]]^{\oplus n\cdot r}$.

In Section 2 the approach of [24] to the KP hierarchy, which is based in the "geometry of formal curves" is developed for the $n$-KP; that is, for an infinite Grassmannian of $E((z))$, where $E$ is a finite dimensional $k$-vector space. This enables us to define tau functions for the $n$-KP in terms of global sections of the determinant bundle and to generalize the Addition formula for this situation (see Lemma 2.7). Similarly to the rank 1 case, the BA function of an arbitrary point of $\mathrm{Gr}(E((z)))$ is defined as a certain deformation of the tau function. However, from Theorem 2.9 it follows that our definition agrees with the standard one ([16,17,20], see [19,27,28] for overviews on the subject) for those points coming from algebro-geometric data. This section finishes with the generalization of the Residue Bilinear Identity (Theorem 2.11).

Section 3 recalls the definition of the $n$-KP and shows that there exists a one-to-one correspondence between wave functions for the $n$-KP and BA functions of points of an infinite Grassmannian $\mathrm{Gr}^0(k((z))^{\oplus n^2})$.

Moduli spaces are studied in Section 4 in terms of the Krichever map and infinite Grassmannians. Although it is not needed for our main purpose, we have considered it convenient to include here the equations for these moduli spaces (Theorem 4.6).

The last section unveils the deep relation among tau functions of algebro-geometric points and non-abelian thetas, which is the "expected" generalization of the rank 1 situation.

Finally, the appendix offers two applications of the previously developed techniques; namely, the study of the Weierstrass divisor and of pairs of commuting differential operators. We also hope that our results will help in other problems; particularly, those related with higher rank vector bundles over curves and infinite Grassmannians and with differential operators (e.g. Bäcklund–Darboux transform, etc).

## 2. Infinite Grassmannians

### *2.1. Background*

Let us address the reader to [2,24] for the scheme-theoretic approach to infinite Grassmannians. However, it is convenient to recall some basic facts in order to fix notations and to point out the statements which we shall need.

Since we are concerned with the multicomponent KP hierarchy, we shall not deal with general infinite Grassmannians, but only with that of $V = E((z))$, where $E$ is a $n$-dimensional $k$-vector space. Denote $E[[z]]$ by $V^+$ and consider the linear topology in $V$ given by $\{z^m V^+ | m \in \mathbb{Z}\}$ as a basis of neighborhoods of (0).

Then, we know that there exists a $k$-scheme $\mathrm{Gr}(V)$ locally covered by the open subschemes:

$$F_A(S) := \left\{ \text{sub} - \mathcal{O}_S - \text{modules}\, \mathcal{L} \subset \hat{V}_S \text{ such that } \mathcal{L} \oplus \hat{A}_S = \hat{V}_S \right\},$$

where $S$ is a $k$-scheme and:

- $A \subset V$ is a subspace such that $A \sim V^+$; that is, $\dim((A + V^+)/(A \cap V^+)) < \infty$;
- $\hat{A}_S$ is defined by $\varprojlim_{B \sim V^+} (A/(A \cap B) \otimes \mathcal{O}_S)$ for a subspace $A \subseteq V$. (For instance, when $E = k$ one has $\hat{V}^+_S = \varprojlim_m \mathcal{O}_S[z]/z^m =: \mathcal{O}_S[[z]]$, and $\hat{V}_S = \varinjlim_m \mathcal{O}_S[[z]] =: \mathcal{O}_S((z))$.)

The generalization of some “good” properties of the one-dimensional case requires a choice of a chain of strict inclusions $V_0^+ = V^+ \subset V_1^+ \subset \cdots \subset V_n^+ = z \cdot V^+$ (note that it follows that the inclusions are of codimension 1). Some remarkable facts are:

- the function:

  $$\mathrm{Gr}(V) \to \mathbb{Z}, \qquad L \mapsto \dim(L \cap V^+) - \dim\left(\frac{V}{L + V^+}\right)$$

  gives the decomposition of $\mathrm{Gr}(V)$ in connected components, which will be denoted by $\mathrm{Gr}^n(V)$ $(n \in \mathbb{Z})$;
- there is a natural line bundle on $\mathrm{Gr}(V)$ defined (on the connected component $\mathrm{Gr}^n(V)$) by the determinant of the perfect complex:

  $$\mathcal{L} \oplus (\hat{V}_m^+)_{\mathcal{O}_{\mathrm{Gr}^m(V)}} \to (\hat{V}_m)_{\mathcal{O}_{\mathrm{Gr}^m(V)}},$$

  where $\mathcal{L}$ is the submodule of $\hat{V}_{\mathcal{O}_{\mathrm{Gr}^m(V)}}$ corresponding to the universal object, and $V_m$ is defined as $z^q \cdot V_r^+$ with $m = q \cdot n + r$ and $0 \le r < n$;
- the addition morphism in the above complex gives canonically a global section of the dual of that bundle, $\Omega_+ \in H^0(\mathrm{Gr}(V), \mathrm{Det}^*)$.

**Remark 2.1.** *Although as abstract scheme* Gr$(V)$ *is independent of the dimension of* $E$*, it is straightforward that the groups acting (naturally) on the Grassmannians do depend on it. The standard procedure to introduce these Grassmannians consists of "re-labeling" the indexes of* $k((z))$ *and taking* $V_m^+ = T^m(V^+)$*, where:*

$$T := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ z & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{2.1}$$

*but in this case the induced* $k((z))$*-module structures are different*.

For simplicity's sake, we will assume that:

- $E = k^{\oplus n}$; that is, we consider a basis $\{e_1, \dots, e_n\}$ of $E$ so that the elements of $E((z))$ may be thought as $E$-valued series in $z$ or as $n$-tuples of series;
- $k$ is an algebraically closed field of characteristic 0.

### *2.2. τ-functions*

Analogous to the one-dimensional case, the formal geometry language [2,24] will be the base of our approach to the definition $\tau$-function. From now on, a pair $(E, \mathcal{T})$ consisting of a $n$-dimensional $k$-vector space and a semisimple commutative subalgebra $\mathcal{T} \subseteq \mathrm{End}(E)$ will be fixed.

Although the constructions and results below hold in greater generality (e.g. $\dim \mathcal{T} \leq \dim E$), we will assume that $\mathcal{T}$ is $n$-dimensional and that there exist a basis $\{T_1, \dots, T_n\}$ such that $T_i(e_j) = \delta_{ij} e_i$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. (The existence follows from [4], Chapter 8, Section 9 no. 3).

Motivated by the fact that the choice of a basis of $\mathcal{T}$ induces an isomorphism of the completion of the symmetric algebra generated by $\mathcal{T}$, and $k[[t_1, \dots, t_n]]$ define the $n$-dimensional formal variety by:

$$\hat{C}^n := \mathrm{Spf}\,(k[[t_1, \dots, t_n]])\,,$$

and let $\Gamma_-^n$ be the direct limit of the symmetric products of $\hat{C}^n$, $\varinjlim_r S^r \hat{C}^n$.

Now, we will study an action of $\Gamma_-^n$ that, roughly said, is induced by that of $\mathcal{T}$ on $V$ at an infinitesimal level.

Analogous to case $n = 1$ (see Theorem 3.6 of [2]), one has the following theorem:

**Theorem 2.2.** $\Gamma_-^n$ *is a formal group scheme whose S-valued points are n-tuples of series on* $z^{-1}$ :

$$\left(1 + \sum_{i>0} a_{i1} z^{-i}, \dots, 1 + \sum_{i>0} a_{in} z^{-i}\right)$$

(*where* $a_{ij} \in H^0(S, \mathcal{O}_S)$ *are nilpotents*) *with componentwise multiplication as composition law*.

Let $\prod^r \hat{C}^n$ be the formal spectrum of $k[[\{t_{ij}\}_{1\leq j\leq n, 1\leq i\leq r}]]$. Then, the ring of $\Gamma_-^n$ is $\varprojlim_{r>0} k[[\{s_{ij}\}_{1\leq j\leq n, 1\leq i\leq r}]]$ (where $s_{ij}$ is to be understood as the $i$th symmetric function on $t_{1j}, t_{2j}, \dots$). Since $\operatorname{char}(k) = 0$, the exponential map gives an isomorphism of $\Gamma_-^n$ with an additive group scheme, the universal element might be written as $\left(\exp\left(\sum_{i>0} s_{i1} z^{-i}\right), \dots, \exp\left(\sum_{i>0} s_{in} z^{-i}\right)\right)$.

Consider the action $\mu : \Gamma_-^n \times \operatorname{Gr}(V) \to \operatorname{Gr}(V)$ induced by that of $\Gamma_-^n$ on $V$ given by componentwise multiplication; or equivalently by:

$$\left(\exp\left(\sum_{i>0} s_{i1} z^{-i}\right), \dots, \exp\left(\sum_{i>0} s_{in} z^{-i}\right)\right) \cdot v := \exp\left(\sum_{j=1}^{n}\sum_{i>0} s_{ij} z^{-i} T_j\right)(v).$$

Note that $\mu$ preserves the determinant bundle; that $(\mu_U^n)^* \operatorname{Det}^*$ is trivial; and that the group structure of $\Gamma_-^n$ gives a trivialization of it.

**Definition 2.3.** The $\tau$-function of a rational point $U \in \operatorname{Gr}(V)$, $\tau_U$, is the inverse image of the global section of the determinant bundle $\Omega_+$ by the morphism $\mu_U : \Gamma_-^n \times \{U\} \to \operatorname{Gr}(V)$.

### 2.3. *Baker–Akhiezer functions*

Recall that the Abel morphism:

$$\operatorname{Spf}(k[[\bar{z}]]) \times \Gamma_- \to \Gamma_-$$

is the associated with the series $(1-(\bar{z}/z))^{-1} \cdot (1+\sum_{i>0} a_i z^{-i})$. Let $\phi_j : \operatorname{Spf}(k[[\bar{z}]]) \times \Gamma_-^n \to \Gamma_-^n$ be the morphism given by the Abel morphism in the $j$th entry and by the identity on the others.

**Definition 2.4.** The Baker–Akhiezer function of a rational point $U \in \operatorname{Gr}^r(V)$ (with $\Omega_+(U) \neq 0$), $\psi_U(z, s)$, is the vector valued function:

$$\psi_U(z, s) := \exp\left(-\sum_{i,j} T_j \frac{s_{ij}}{z^i}\right) \cdot \frac{1}{\tau_U(s)} \cdot \left(\phi_1^*(\tau_U), \dots, \phi_n^*(\tau_U)\right).$$

**Remark 2.5.** *In our definitions of tau and BA functions, the commutativity of* $\mathcal{T}$ *is extremely important. It implies that expressions like* $\prod_{1\leq i\leq N, 1\leq j\leq n}(1 - T_j(t_{ij}/z))$, $\exp\left(-\sum_{i,j} T_j(s_{ij}/z^i)\right)$ *are well defined and, further, that the Abel morphisms* $\phi_j$ *and the morphism* $\mu_U$ *are compatible*.

In order to introduce the adjoint BA function, one must assume that there is a non-degenerate symmetric pairing:

$$T_2 : E \times E \to k.$$

Then, $E((z))$ carries a natural non-degenerate pairing, Res, given by:

$$\operatorname{Res}\left(\sum f_i z^i\right)\cdot\left(\sum g_j z^j\right) := \operatorname{Res}_{z=0}\sum_{i,j} T_2(f_i, g_j) z^{i+j}\,\mathrm{d}z = \sum_i T_2(f_i, g_{-i-1}).$$

When $E$ is the algebra of matrices, $M_{r\times s}(k)$ (including the case $r=1$), we shall consider the pairing given by:

$$(A, B) \mapsto \operatorname{Tr}(A\cdot B^{\mathrm{t}}).$$

**Definition 2.6.** The adjoint BA function of a rational point $U\in \operatorname{Gr}(V)$ is:

$$\psi_U^*(z,s) := \psi_{U^\perp}(z,-s).$$

### 2.4. *Addition formula*

Let $N>0$ be an integer. Let $U\in\operatorname{Gr}(V)$ be a rational point. Consider the morphism:

$$\bar\mu_U^N : \prod^N \hat C^n \to \operatorname{Gr}(V)$$

given by:

$$\left(\prod_{1\le i\le N, 1\le j\le n}\left(1 - T_j\frac{t_{ij}}{z}\right)\right)^{-1}(U)$$

(the ring of the $j$th copy of $\prod^N \hat C$ in $\prod^N \hat C^n$ is $k[[t_{1j},\dots,t_{Nj}]]$).

**Lemma 2.7.** *Let $U\in\operatorname{Gr}^0(V)$ satisfy $\Omega_+(U)\ne 0$. Then, for all $N\gg 0$ the inverse image of $\Omega_+$ by $\bar\mu_U^N$ is given* (*up to a non-zero scalar*) *by the following expression:*

$$\left(\prod_{j=1}^n \Delta_j\right)^{-1}\cdot\det\begin{pmatrix} f_1^1(t_{11}) & \dots & f_1^1(t_{N1}) & \dots & f_n^1(t_{1n}) & \dots & f_n^1(t_{Nn}) \\ \vdots & & & & & & \vdots \\ f_1^M(t_{11}) & \dots & f_1^M(t_{N1}) & \dots & f_n^M(t_{1n}) & \dots & f_n^M(t_{Nn})\end{pmatrix},$$

*where $M := N\cdot n$ and $\{f^i = (f_1^i,\dots,f_n^i)\}_{1\le i\le M}$ is a basis of $V^+\cap z^N U$.*

**Proof.** We have to compute the determinant of the inverse image of the complex:

$$\mathcal{L}\to\frac{V}{V^+}$$

by the morphism $\bar\mu_U^N$, which is $\prod(1-T_j(t_{ij}/z))^{-1}\cdot U\to V/V^+$. It is straightforward that its determinant coincides with that of the following complex:

$$\operatorname{Im}(h)\to\frac{V}{z^N U}$$

where $h$ is the homothety:

$$h : V^+ \to V^+, \qquad (f_1, \dots, f_n) \mapsto \left(\prod_{i=1}^{N}(z - t_{i1}) f_1, \cdots \prod_{i=1}^{N}(z - t_{in}) f_n\right)$$

Recall that $V^+ = E[[z]] = k[[z]] \oplus \cdots^n \oplus k[[z]]$. Consider now the following evaluation map:

$$k[[z]] \oplus \overset{n}{\cdots} \oplus k[[z]] \to \mathcal{M}_{Nn} := \bigoplus_{1\le i\le N, 1\le j\le n} k[[t_{ij}]],$$

$$(f_1(z), \dots, f_n(z)) \mapsto (f_1(t_{11}), \dots, f_1(t_{N1}), \dots, f_n(t_{1n}), \dots, f_n(t_{Nn}))\,,$$

and observe that the cokernel of $h$ is isomorphic to:

$$\frac{k[[z]]}{\prod_i (z - t_{i1})} \oplus \cdots \oplus \frac{k[[z]]}{\prod_i (z - t_{in})}.$$

Now, we construct the following exact sequence of complexes (written vertically):

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Im}(h) & \to & V^+ & \to & \mathrm{Coker}(h) & \to & 0 \\ & & {\scriptstyle\pi}\downarrow & & {\scriptstyle(\pi, v^N)}\downarrow & & {\scriptstyle v^N}\downarrow & & \\ 0 & \to & V/(z^N U) & \to & V/(z^N U) \oplus \mathcal{M}_{Nn} & \to & \mathcal{M}_{Nn} & \to & 0 \end{array}$$

Since the determinant of the morphism in the complex of the left-hand side is precisely $(\bar{\mu}_U^N)^*\Omega_+$, it is sufficient to calculate the others. First, observe that:

$$\det(\bar{v}_N) = \prod_{j=1}^{n} \Delta_j, \quad \text{where } \Delta_j := \prod_{i<k} (t_{ij} - t_{kj}).$$

To compute the determinant of the complex in the middle, note that in the diagram:

$$\begin{array}{ccccccccc} 0 & \to & V^+ \cap z^N U & \to & V^+ & \to & V^+/(V^+ \cap z^N U) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{M}_{Nn} & \to & \mathcal{M}_{Nn} \oplus V/(z^N U) & \to & V/(z^N U) & \to & 0 \end{array}$$

the determinant of the morphism of the right-hand side is a non-zero constant for $N \gg 0$ (since $U$ lies in $\mathrm{Gr}^0(V)$ and $\Omega_+(U) \neq 0$). We conclude now since in the determinant of the morphism of the complex of the left-hand side is:

$$\det \begin{pmatrix} f_1^1(t_{11}) & \dots & f_1^1(t_{N1}) & \dots & f_n^1(t_{1n}) & \dots & f_n^1(t_{Nn}) \\ \vdots & & & & & & \vdots \\ f_1^M(t_{11}) & \dots & f_1^M(t_{N1}) & \dots & f_n^M(t_{1n}) & \dots & f_n^M(t_{Nn}) \end{pmatrix},$$

where $M := N \cdot n$ and $\{f^i = (f_1^i, \dots, f_n^i) | 1 \le i \le M\}$ is a basis of $V^+ \cap z^N U$. □

**Remark 2.8.** *Since Lemma* 2.7 *shows that* $\left(\prod_j \Delta_j\right)\left((\mu_U^N)^*\Omega_+\right)(U)$ *is a determinant for all U, it follows that one can generalize the "Addition formula" of* [30] *to this case.*

**Theorem 2.9.** *Let $U_0 \in \mathrm{Gr}^m(V)$ be a rational point. Then, there exists polynomials $p_{ij} \in k[\{t_{ij}\}]$ and $\zeta_m \in V_m^+$ generating $V_m^+/V_{m-1}^+$ such that for every rational point, U, lying on a (Zarisky) open neighborhood of $U_0$ the following formula holds:*

$$\psi_U(z,s) = z \cdot \zeta_m^{-1} \cdot \sum_{i>0} \left( f_1^i(z) p_{i1}(s), \dots, f_n^i(z) p_{in}(s) \right),$$

*where $\{f^i(z) = (f_1^i(z), \dots, f_n^i(z))\}_{i>0}$ is a basis of U.*

**Remark 2.10.** *Lemma* 2.7 *and Theorem* 2.9 *have straightforward generalizations for the case* $\dim \mathcal{T} \leq \dim E$.

**Proof.** Once we have proved Lemma 2.7, this proof is a generalization of that of Theorem 4.8 of [24].

Let us consider a rational point $U \in \mathrm{Gr}^m(V)$. In order to compute the $j$th entry of its BA function, we will calculate first the expression $\left((\phi_{j,N}^*\Omega_+)/(\phi_N^*\Omega_+)\right)(U)$ for all $N \gg 0$, where $\phi_N : \prod^N \hat{C}^n \times \{U\} \to \mathrm{Gr}(V)$ and $\phi_{j,N}$ consist of composing with the Abel morphism in the $j$th entry. However, we shall use the fact that $\phi_{j,N}$ and the morphism $\prod^N \hat{C}^n \times \{U_j\} \to \mathrm{Gr}(V)$ coincide ($U_j := (1, \dots, (1-(\bar{z}/z))^{-1}, \dots, 1)$ in the $j$th place).

Observe that $U_j \in \mathrm{Gr}^{m+1}(V)$ and choose $\zeta_m$ such that $\zeta_m^{-1} \cdot U_0 \in \mathrm{Gr}^0(V)$ lies on the complementary of the zero locus of the section $\Omega_+$. Then, it is clear that the points $U \in \mathrm{Gr}^m(V)$ such that $\Omega_+(\zeta_m^{-1} \cdot U) \neq 0$ define a (Zariski open) neighborhood of $U_0$, and that Lemma 2.7 might be applied to $\zeta_m^{-1} \cdot U$.

Let $\{f^1, \dots, f^{M+1}\}$ be a basis of $\left((1, \dots, z^{-1}, \dots, 1)z^{-N}V^+\right) \cap \zeta_m^{-1}U$ such that $\{f^1, \dots, f^M\}$ is a basis of $z^{-N}V^+ \cap \zeta_m^{-1}U$. Let $\bar{f}^i$ be $z^N \cdot f^i$. Applying Lemma 2.7, one gets:

$$\frac{\phi_{j,N}^*\Omega_+}{\phi_N^*\Omega_+}(U) = \prod_{l=1}^{N} (\bar{z} - t_{lj})^{-1} \cdot \bar{z} \cdot \left( \bar{f}_j^{M+1} \prod_l t_{lj} + \sum_{l=1}^{M} \bar{f}_j^l \cdot \bar{p}_{lj}(t) \right)$$

(up to a scalar). Taking inverse limit in $N$, replacing $\bar{z}$ by $z$, and recalling that the $j$th entry of the BA function is $\prod(1-(t_{lj}/z))^{-1} \cdot (\phi_{j,N}^*\Omega_+)/(\phi_N^*\Omega_+)(U)$, it follows that the $j$th entry is:

$$z \cdot \sum_{l>0} f_j^l(z) p_{lj}(t),$$

where $\{f^l = (f_1^l, \dots, f_n^l) | l > 0\}$ is a basis of $\zeta_m^{-1}U$. Finally, the very construction of the polynomials $p_{lj}(t)$ implies that they are symmetric in $t$ so that they can be expressed in terms of their symmetric functions $s$. The claim is proved. □

## *2.5. Bilinear Identity*

The previous theorem together with the definition of the pairing Res imply easily the following:

**Theorem 2.11.** *Let $U, U' \in \mathrm{Gr}^0(V)$ be two rational points such that $\Omega_+(U) \neq 0$ and $\Omega_+(U') \neq 0$, then the Residue Bilinear Identity holds:*

$$\mathrm{Res}\left(\frac{1}{z}\psi_U(z,s)\right)\cdot\left(\frac{1}{z}\psi^*_{U'}(z,s')\right) = 0$$

*if and only if $U = U'$.*

**Remark 2.12.** *For other approaches to the definitions of tau and BA functions we refer the reader to* [10,14]. *However, the definitions offered here* (*inspired in* [9,24,30]) *imply easily all the "good" properties of the $n = 1$ case; namely Lemma* 2.6, *and Theorems* 2.7 and 2.8. *Furthermore, they agree with the definitions given for "algebro-geometric" points of the Grassmannian* (*see Section* 4 *and, particularly, Remark* 7).

## 3. *n*-Component KP hierarchy

Here, we introduce the $n$-KP in a very concise way. We will define it as a system of Lax equations and follow closely [29] (see also [33]). Another common approach is based on representation theory (see, for instance, [9,14]). The last approach might be "included" in ours by studying the action of the linear group of $E((z))$ on the space of global sections of the determinant bundle (see [26]).

### *3.1. Pseudo-differential operators*

Let us begin this section summarizing some standard definitions and properties of pseudo-differential operators (PDOs).

For a $\mathbb{C}$-algebra $A$ and a $\mathbb{C}$-derivation $\partial : A \to A$, one considers the $A$-module of PDOs:

$$\mathcal{P} := \left\{\sum_{i\leq n} a_i\partial^i \mid a_i \in A, n \in \mathbb{Z}\right\}.$$

The following generalization of the Leibnitz rule:

$$\left(\sum_i a_i\partial^i\right)\left(\sum_j b_j\partial^j\right) := \sum_{i,j}\sum_{k\geq 0}\binom{i}{k} a_i(\partial^k b_j)\partial^{i+j-k}$$

endows $\mathcal{P}$ with a $\mathbb{C}$-algebra structure. Moreover, $\mathcal{P}$ contains a distinguished $\mathbb{C}$-algebra; namely, the algebra $\mathcal{D}$ of differential operators (those elements $\sum_{i\leq n} a_i\partial^i$ such that $a_i = 0$ for all $i \leq 0$).

A PDOs $\sum_{i\leq n} a_i\partial^i$ is called of order $n$ iff $a_n \neq 0$. The subspace of the operators of order less than or equal to $n \in \mathbb{Z}$ will be denoted by $\mathcal{P}(n)$. Since $\mathcal{P} = \mathcal{D} \oplus \mathcal{P}(-1)$, every operator $P$ decomposes as a sum $P_+ + P_-$. Finally, define the adjoint of $P = \sum_{i\leq n} a_i\partial^i$ to be $P^* = \sum_{i\leq n}(-\partial)^i a_i$.

Observe that:

- $\mathcal{P}(n)\mathcal{P}(m) \subseteq \mathcal{P}(n+m)$;
- the Leibnitz rule induces a composition law in the affine subspace $1+\mathcal{P}(-1) \subset \mathcal{P}$;
- $1+\mathcal{P}(-1)$ acts transitively on $\partial + \mathcal{P}(-1)$ by conjugation;
- the stabilizer of $\partial$ consists of those PDOs with constant coefficients. (Here, $a$ is constant iff $\partial a = 0$).

In the particular case of matrix valued functions, we impose one constraint; namely, the leading coefficient, $a_n$, is the identity matrix, I.

**Definition 3.1.** An $n \times n$ matrix-valued oscillating function is a formal expression of the type:

$$\left(\mathrm{I} + \sum_{i<0} M_i(s)\bar{z}^i\right) \cdot \mathrm{e}^{\xi(s,\bar{z})},$$

where $M_i(s)$ are $n \times n$ matrices, and:

$$\xi(s,\bar{z}) := \sum_{i>0} \begin{pmatrix} s_{i1} & & \\ & \ddots & \\ & & s_{in} \end{pmatrix} \bar{z}^i .$$

From now on, we will consider oscillating functions and PDOs over the ring $M_{n\times n}(C[[s]])$ ($s = \{s_{ij}\}_{i>0, 1\le j\le n}$). Define $\partial_{ij} := \mathrm{d}/\mathrm{d}s_{ij}$ and $\partial = \sum_{j=1}^{n} \partial_{1j}$.

### 3.2. *n-component KP*

Let $L, C^{1)}, \dots, C^{n)}$ be PDOs with $n \times n$ matrix coefficients of the form:

$$\begin{aligned} L &= \mathrm{I}\partial + L_1(s)\partial^{-1} + L_2(s)\partial^{-2} + \cdots, \\ C^{(i)} &= E_i + C_1^{(i)}(s)\partial^{-1} + C_2^{(i)}(s)\partial^{-2} + \cdots, \end{aligned} \tag{3.1}$$

where $E_i$ is a matrix whose only non-zero entry is 1 in the $(i,i)$ place.

Then, the $n$-KP is the following set of Lax equations:

$$\partial_{ij} L = [(L^i C^{(j)})_+, L], \quad \partial_{ij} C^{(k)} = [(L^i C^{(j)})_+, C^{(k)}], \quad 1 \le j,\ k \le n,\ i > 0, \tag{3.2}$$

where $LC^{(j)} = C^{(j)}L$, $C^{(j)}C^{(k)} = \delta_{jk}C^{(j)}$ and $\sum_{j=1}^{n} C^{(j)} = \mathrm{I}$.

The above system might be regarded as the compatibility condition of the following system of differential equations:

$$Lw = \bar{z} \cdot w, \qquad C^{(j)}w = w \cdot E_j, \qquad \partial_{ij} w = B_i^{(j)} w \tag{3.3}$$

for a formal oscillating matrix function $w(\bar{z}, s) = (\mathrm{I} + \sum_{l<0} w_l(s)\bar{z}^l)\,\mathrm{e}^{\xi(s,\bar{z})}$ and $B_i^{(j)} := ((LC^{(j)})^i)_+ = (L^i C^{(j)})_+$.

**Theorem 3.2.** *There is a one-to-one correspondence between the set of solutions of the n-KP and the set of rational points of the open subscheme of* $\mathrm{Gr}^0(V)$ *given by* $\Omega_+ \neq 0$ ($V = M_{n\times n}(k((z)))$ *and* $z = 1/\bar{z}$).

For proving the theorem we shall need the following generalization of the Lemma of [9] proved in [14]:

**Lemma 3.3.** *Let P,Q be two PDOs with matrix coefficients. If:*

$$\mathrm{Res}_{z=0} P(s,\partial)\,\mathrm{e}^{\xi(s,z)}\cdot Q^{\mathrm{t}}(s',\partial')\,\mathrm{e}^{\xi(s',z)}\,\mathrm{d}z=0,$$

*then* $(P\cdot Q^*)_-=0$. (*Here, the superscript* t *denotes the transpose*.)

**Proof.** It is well-known that the system (3.2) has a solution $w$ if and only if there exists a PDOs $P=\mathrm{I}+\sum_{i<0}P_i(s)\partial^i$ satisfying:

$$LP=P\partial,\qquad C^{(j)}P=P\cdot E_j,\qquad \partial_{ij}P=-(L^iC^{(j)})_-P. \tag{3.4}$$

If there exists $P$ solving the system above, then $w_i(s)=P_i(s)$, $L=P\partial P^{-1}$ and $C^{(i)}=PE_iP^{-1}$. Further:

$$\partial_{ij}w(s,\bar z)=\bar z^iP(s,\bar z)E_j\,\mathrm{e}^{\xi(s,\bar z)}-(L^iC^{(j)})_-w(s,\bar z)$$

holds for $w(s,\bar z)=P(s,\bar z)\,\mathrm{e}^{\xi(s,\bar z)}$. And therefore:

$$\partial_{ij}w(s,\bar z)|_{s=0}=\bar z^i(0,\ldots,\overset{(j)}{1},\ldots,0)+\sum_{k<i}\bar z^kw_k,$$

where $w_k$ are certain vector valued functions. Summing up, given a solution $P$ we have shown that the vector space, $U$, generated by $w(s,\bar z)$ (as the variables $s$ vary) belongs to $\mathrm{Gr}^0(V)$. It is easy to check that $\Omega_+(U)\neq 0$.

Conversely, given a point $U\in\mathrm{Gr}^0(V)$ with $\Omega_+(U)\neq 0$, let $P$ be a PDOs such that $\psi_U(z,s)=P(s,\partial)\,e^{\xi(s,\bar z)}$. Theorem 2.11 implies that its BA functions satisfy the following relation:

$$\mathrm{Res}\left(\frac{1}{z}(\partial_{ij}-B_i^{(j)})\psi_U(z,s)\right)\cdot\left(\frac{1}{z}\psi_U^*(z,s')\right)=0,\qquad \forall\,1\le j\le n,\quad i\ge 1,$$

so, the lemma implies $((\partial_{ij}P-B_i^{(j)}P)\cdot P^*)_-=0$.

Observe that $(\partial_{ij}-B_i^{(j)})P$ is of negative order since:

$$\begin{aligned}(\partial_{ij}-B_i^{(j)})\psi_U(z,s)&=(\partial_{ij}P+z^iPE_j-B_i^{(j)}P)\mathrm{e}^{\xi(s,\bar z)}\\&=(\partial_{ij}P+P\partial^iE_j-B_i^{(j)}P)\mathrm{e}^{\xi(s,\bar z)}\\&=(\partial_{ij}P+L^iC^{(j)}P-B_i^{(j)}P)\mathrm{e}^{\xi(s,\bar z)}\\&=(\partial_{ij}P+(L^iC^{(j)})_-P)\mathrm{e}^{\xi(s,\bar z)}.\end{aligned}$$

Then, it follows that $\partial_{ij}P-B_i^{(j)}P$ must be identically zero; that is, $\psi_U$ is a solution of the $n$-KP. □

**Remark 3.4.** *Observe that Eqs.* (3.2) *determine P up to right multiplication by an operator* $\mathrm{I}+\sum_{i<0}A_i\partial^i$ *with constant coefficients.*

Let us finish this section with a brief comment on Wronskian solutions. This "Wronskian method" permits us to construct solutions of the $n$-KP starting with $n$ solutions of the KP; equivalently (in terms of Grassmannians), it is a procedure to construct a point of $\mathrm{Gr}(k((z))^{\oplus n^2})$ from a point of $\mathrm{Gr}(k((z))^{\oplus n})$.

Consider the "Wronskian embedding":

$$\mathcal{W} : \mathrm{Gr}(k((z))^{\oplus n}) \hookrightarrow \mathrm{Gr}(k((z))^{\oplus n^2}), \qquad U \mapsto U \oplus U^{1)} \oplus \cdots \oplus U^{n-1)},$$

where:

$$U^{i)} := \left\{ \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^i f(z) | f(z) \in U \right\}.$$

Now, an easy calculation shows the following relation between the BA functions:

$$\psi_{\mathcal{W}(U)} = \mathrm{Wronskian}(\psi_U),$$

that is, if $\psi_U$ is the vector valued function $(\psi_U^{(1)}, \ldots, \psi_U^{(n)})$ then $\psi_{\mathcal{W}(U)}$ is the determinant of the matrix $\{(\mathrm{d}/\mathrm{d}z)^{i-1}\psi_U^{(j)}\}_{1\leq i,j\leq n}$.

## 4. Moduli spaces

Now, the results of [24] may be generalized in order to give equations for the moduli space of algebraic data:

$$\hat{\mathcal{M}}^r_{g,n} = \{(C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta)\},$$

where $C$ is a curve, $p_i \in C, \alpha_i : \hat{\mathcal{O}}_{C,p_i} \simeq k[[z]], M$ a rank $r$ torsion-free sheaf, and $\beta : \hat{M}_{\{p_i\}} \tilde{\to} k[[z]]^{\oplus n\cdot r}$. (Here, $a$â denotes the completion of a sheaf along a divisor.) But, let us be more precise.

Given a flat curve $\pi : C \to S$ and a Cartier divisor $D$ denote:

$$\hat{\mathcal{O}}_{C,D} = \lim_{\substack{\rightarrow \\ m}} \left(\frac{\mathcal{O}_C}{\mathcal{O}_C(-n)}\right),$$

where $\mathcal{O}_C(-1)$ is the ideal sheaf of $D$. Assume that $D$ is of finite degree, flat and smooth over $S$. (Here smooth over $S$ means that for every closed point $x \in D$ there exists an open neighborhood $U$ of $x$ in $C$ such that the morphism $U \to S$ is smooth). Then, $\hat{\mathcal{O}}_{C,D}$ is a sheaf of $\mathcal{O}_S$-algebras. We also define the following sheaf of $\mathcal{O}_S$-algebras:

$$\hat{\Sigma}_{C,D} = \lim_{\substack{\rightarrow \\ m}} \hat{\mathcal{O}}_{C,D}(m).$$

**Definition 4.1.** *Let S be a k-scheme. Define the functor* $\tilde{\mathcal{M}}^r_{g,n}$ *over the category of k-schemes by:*

$$S \rightsquigarrow \tilde{\mathcal{M}}^r_{g,n}(S) = \{\text{families } (C, p_1, \ldots, p_n, \alpha_1, \ldots, \alpha_n, M, \beta) \text{ over } S\},$$

*where these families satisfy:*

1. $\pi : C \to S$ *is a proper flat morphism, whose geometric fibers are reduced curves of arithmetic genus g;*
2. $p_i : S \to C (1 \le i \le n)$ *is a section of* $\pi$, *such that* (*when considered as a Cartier Divisor over* $C$, *also denoted by* $p_i$) *it is of relative degree* 1, *flat and smooth over S;*
3. *for each irreducible component of C there is at least one divisor* $p_i$ *lying on it;*
4. $\alpha_i (1 \le i \le n)$ *is an isomorphism of* $\mathcal{O}_S$*-algebras* $\hat{\sum}_{C,p_i} \tilde{\to} \mathcal{O}_S((z))$;
5. *M is a torsion-free rank r bundle on C;*
6. $\beta$ *is a direct sum of n isomorphisms of* $\mathcal{O}_S$-modules $\hat{M}_{p_i} \tilde{\to} \mathcal{O}_S[[z]]^{\oplus r} (1 \le i \le n)$.

On the set $\tilde{\mathcal{M}}^r_{g,n}(S)$ one can define an equivalence relation, $\sim$: $(C, \{p_i, \alpha_i\}, M, \beta)$ and $(C', \{p'_i, \alpha'_i\}, M', \beta')$ are said to be equivalent, if there exists an isomorphism $C \to C'$ (over $S$) such that the first family goes to the second under the induced morphisms.

**Definition 4.2.** *The moduli functor of* $\hat{\mathcal{M}}^r_{g,n}$ *is the functor over the category of k-schemes defined by the sheafication of the functor:*

$$S \rightsquigarrow \tilde{\mathcal{M}}^r_{g,n}(S)/\sim .$$

**Theorem 4.3.** *There is an injective map of functors:*

$$\hat{\mathcal{M}}^r_{g,n} \to \mathrm{Gr}\left(k((z))^{\oplus n}\right) \times \mathrm{Gr}\left(k((z))^{\oplus n\cdot r}\right)$$

$$(C, \{p_i, \alpha_i\}, M, \beta) \mapsto \Big(H^0(C - \{p_1, \dots, p_n\}, \mathcal{O}_C), H^0(C - \{p_1, \dots, p_n\}, M)\Big)$$

*Moreover,* $\hat{\mathcal{M}}^r_{g,n}$ *is representable by a closed subscheme of* $\mathrm{Gr}^{1-g}(k((z))^{\oplus n}) \times \mathrm{Gr}(k((z))^{\oplus n\cdot r})$.

**Proof.** Let $(C, \{p_i, \alpha_i\}, M, \beta)$ be a point of $\hat{\mathcal{M}}^r_{g,n}$. Define the divisor $D$ by $p_1 + \cdots + p_n$ and the sheaf $M(m)$ by $M \otimes \mathcal{O}_C(mD)$. Proceeding as in Proposition 6.3 of [24] one shows that the sheaf $\varinjlim_m \pi_* M(m)$ is an $S$-valued point of $\mathrm{Gr}(\hat{\sum}^{\oplus n\cdot r}_{C,\{p_i\},})$. Moreover, it lies on the component of index $\deg(M) + r(1-g)$.

Now, the morphism of the statement maps $(C, \{p_i, \alpha_i\}, M, \beta) \in \hat{\mathcal{M}}^r_{g,n}(S)$ to the pair:

$$\left(\varinjlim_m \pi_* \mathcal{O}_C(m), \varinjlim_m \pi_* M(m)\right),$$

which are understood as submodules of $\mathcal{O}_S((z))^{\oplus n}$ and of $\mathcal{O}_S((z))^{\oplus n\cdot r}$ via $\alpha_1 \oplus \cdots \oplus \alpha_n$ and $\beta$, respectively.

It is clear that the image lies on the subset of $\mathrm{Gr}^{1-g}(k((z))^{\oplus n}) \times \mathrm{Gr}(k((z))^{\oplus n\cdot r})$ defined by:

$$\{(A, B) \text{ such that } A \cdot A = A \text{ and } A \cdot B = B\}, \tag{4.1}$$

which is actually a closed subscheme (by the same arguments of the proof of Theorem 6.5 of [24]). Here, the composition laws are given as follows: an element $(f_1, \dots, f_n)$ of $k((z))^{\oplus n}$ is represented as the diagonal matrix:

$$\begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{pmatrix},$$

an element of $k((z))^{\oplus n\cdot r}$ is thought as an $n \times r$ matrix; then, the composition laws are those induced by the multiplication of matrices.

Starting with a pair $(A, B)$ of that subscheme, a well-known procedure (see, for instance, Theorem 6.4 in [24]) enables us to construct algebro-geometric data $(C, \{p_i, \alpha_i\}, M, \beta) \in \hat{\mathcal{M}}^r_{g,n}$ whose image is $(A, B)$. Further, both constructions are inverse of each other. □

**Remark 4.4.** *A direct consequence of Theorem* 2.7 *is that* (*with a suitable normalization*) *the restriction of our BA function to a rational point corresponding to* $(C, \{p_i, \alpha_i\}, M, \beta)$ *gives the standard notions of Ahkiezer functions in all its flavors: vector valued, matrix valued, multi-punctured* ([16,18,20], *see also* [19,27,28,31]).

In this case, the Bilinear Residue Identity is geometrically meaningful; it is equivalent to the fact that the sum of the residues must vanish.

**Remark 4.5.** *It is worth mentioning how the notion of Heisenberg algebras turns up in our picture. Given the data* $(C, p, \alpha, M, \beta)$ *we can consider a suitable covering* $\pi : C' \to C$ *and a line bundle* $L$ *on* $C'$ *such that* $\pi_* L \xrightarrow{\sim} M$ (*the spectral cover*). *Then, from the structure of the fiber* $\pi^{-1}(p)$ *arises naturally a Heisenberg algebra* (*see Section* 5.5 *of* [1] *and Section* 6.2 *of* [11] *for a detailed study*) *which is "generated" by matrices of the type* (2.1). *Furthermore, this construction relates the Heisenberg flows* (*associated to a partition*) *and the flows of the multicomponent KP hierarchy* (*see Section* 8 *of* [1] *and Section* 6.1 *of* [11]).

**Theorem 4.6.** *Let* $(A, B)$ *be a rational point of* $\mathrm{Gr}^{1-g}(k((z))^{\oplus n}) \times \mathrm{Gr}(k((z))^{\oplus n\cdot r})$ *and let* $\zeta_A \in V$ *such that* $\Omega_+((\zeta_A/z)\cdot A) \neq 0$. *Then* $(A, B)$ *lies in* $\hat{\mathcal{M}}^r_{g,n}$ *if and only if their BA functions satisfy:*

$$\begin{aligned} &\mathrm{Res}\,((1,\dots,1))\cdot\left(\frac{\zeta_A}{z}\tilde{\psi}^*_A(z,s)\right) = 0,\\ &\mathrm{Res}\left(\frac{\zeta_A}{z}\tilde{\psi}_A(z,s)\cdot\frac{1}{z}\tilde{\psi}_A(z,s')\right)\cdot\left(\frac{1}{z}\tilde{\psi}^*_A(z,s'')\right) = 0,\\ &\mathrm{Res}\left(\frac{\zeta_A}{z}\tilde{\psi}_A(z,s)\cdot\frac{1}{z}\tilde{\psi}_B(z,s')\right)\cdot\left(\frac{1}{z}\tilde{\psi}^*_B(z,s'')\right) = 0, \end{aligned}$$

*where* $\tilde{\psi}_A(z,s) := \psi_A(z,s)|_{s_{ij}=s_i}$.

**Proof.** Firstly, recall that the pairing considered in $k^{\oplus n\cdot r}$ is $(A, B) \mapsto \mathrm{Tr}(A\cdot B^{\mathrm{t}})$. Using Theorem 2.9, it is easy to check that $\tilde{\psi}_A(z,s) = z\zeta_A^{-1}\sum_i f_i(z)p_i(s)$, where $\{f_i(z)\}$ is a basis of $A$. Therefore, the equations defining the subscheme (4.1) are those given in the statement. □

Note that the equations given in this theorem can be transformed into a set of differential equations for $\tau$-functions that might be used to characterize certain sections of bundles over $\hat{\mathcal{M}}^r_{g,n}$. (See [24] for how this transformation is made.) On the other hand, these equations can also be written as algebraic relations among sections of the determinant bundle.

## 5. Solutions to the *n*-KP and non-abelian theta functions

Fix two positive integers $r, d$ such that $(r, d) = 1$. Let us fix a smooth curve $C$ and a vector bundle $F$ such that $\operatorname{rank}(F) = r$ and $\deg(F) = -d + r(g-1)$.

Denote by $\mathcal{U}_s(r, d)$ the moduli space of rank $r$ degree $d$ stable vector bundles on $C$. Recall that $\mathcal{U}_s(r, d)$ carries a natural line bundle associated to the Weil divisor given by:

$$\{M \ \text{such that} \ h^0(M \otimes F) > 0\}.$$

The most important properties of this bundle follows from its construction as a determinant. Let us recall the contruction following [12,21].

Let $\mathcal{P}$ be a Poincaré bundle on $C \times \mathcal{U}_s(r, d)$ and $p_i (i = 1, 2)$ the projection onto the $i$th factor. Let us consider a resolution of $\mathcal{Q} := \mathcal{P} \otimes p_1^* F$ by locally free sheaves:

$$\begin{array}{ccccccccc} 0 & \to & V_1 & \xrightarrow{\alpha} & V_0 & \to & 0 \\ & & & & \downarrow & & \\ & & & & \mathcal{Q} & & \end{array}$$

such that $p_{2*}(V_0) = 0$. Since the vertical arrow is a quasi-isomorphism, it induces an isomorphism:

$$\operatorname{Det}(Rp_{2*}\mathcal{Q}) \simeq \operatorname{Det}(Rp_{2*}V).$$

Note that $\det(Rp_{2*}V) \simeq \wedge(R^1 p_{2*}(V_0)) \otimes \wedge(R^1 p_{2*}(V_1))^{-1}$ (where $\wedge$ denotes the top exterior algebra). It is now easy to check that the morphism $\alpha$ induces a section:

$$\theta_F := \det(\alpha) \in H^0\left(\mathcal{U}_s(r, d), \operatorname{Det}(Rp_{2*}\mathcal{Q})^{-1}\right).$$

Lemma 2.1 of [21] proves that: (a) the above construction does not depend on the resolution; (b) the line bundle $\operatorname{Det}(Rp_{2*}\mathcal{Q})$ does depend only on the equivalence class of $F$, $[F]$, in the Grothendieck group of algebraic coherent sheaves over $C$, $K(C)$ (see also [12]); and (c) the zero locus of the section $\det(\alpha)$ is the subscheme whose closed points are those bundles $M$ such that $h^0(C, M \otimes F) \neq 0$. Moreover, Lemma 1.2 of [21] shows that $\operatorname{Det}(Rp_{2*}\mathcal{Q})$ does not depend on the choice of a universal bundle (up to isomorphisms) provided that $d \operatorname{rank}(F) + r\chi(F) = 0$; or, equivalently, $\chi(M \otimes F) = 0$ for all $M \in \mathcal{U}_s(r, d)$ (this is why we have taken $\deg(F) = -d + r(g-1)$).

Summing up, given a class $[F] \in K(C)$ (such that $d\,\mathrm{rank}(F) + r\chi(F) = 0$) there is an associated line bundle on $\mathcal{U}_s(r,d)$, $\mathcal{O}(\Theta_{[F]}) := \mathrm{Det}(Rp_{2*}\mathcal{Q})$. For each element $F' \in [F]$, there is a section of the dual of this bundle, $\theta_{F'}$, whose zero locus is $\{M$ such that $h^0(C, M \otimes F') \neq 0\}$.

Let us now relate the above picture with infinite Grassmannians. As before, some data must be fixed.

Let $r, d$ be two positive integers such that $(r,d) = 1$. Let us fix $(C, p, \alpha, F, \gamma) \in \mathcal{M}^r_{g,1}$ such that $C$ is smooth and $\deg(F) = -d + r(g-1)$.

From Section 4 we know that:

$$\hat{\mathcal{U}}(r,d) := \{(M,\beta) \text{ such that} (C, p, \alpha, M, \beta) \in \hat{\mathcal{M}}^r_{g,1} \text{ and } \deg(M) = d\}$$

is a scheme. Let $\hat{\mathcal{U}}_s(r,d)$ be the open subscheme consisting of those points of $\hat{\mathcal{U}}(r,d)$ with $M$ stable.

If $\mathcal{M}$ is an universal bundle of $\hat{\mathcal{U}}_s(r,d)$ and $V = M_{r\times r}(k((z)))$, then the submodule:

$$\varinjlim_m \pi_*(\mathcal{M}(m) \otimes F) \hookrightarrow V \hat{\otimes} \mathcal{O}_{\hat{\mathcal{U}}(r,d)}$$

(via $\beta \otimes \gamma$ and $\alpha$) corresponds to the morphism of schemes:

$$\iota_F : \hat{\mathcal{U}}_s(r,d) \hookrightarrow \mathrm{Gr}^0(V),$$

whose expression for rational points is given by:

$$(M,\beta) \mapsto H^0(C-p, M \otimes F).$$

The above discussion and the exactness of the following sequence:

$$0 \to \pi_*(M \otimes F) \to \varinjlim_m \pi_*(M(m) \otimes F) \to \frac{V}{V^+} \to R^1\pi_*(M \otimes F) \to 0,$$

show the following theorem.

**Theorem 5.1.** Let $\pi : \hat{\mathcal{U}}_s(r,d) \to \mathcal{U}_s(r,d)$ be the canonical projection. It holds that:
- $(\iota_F^*\Omega_+)(M,\beta) = 0 \quad \Leftrightarrow \quad h^0(C-p, M \otimes F) \neq 0$;
- $\iota_F^*\Omega_+ = \pi^*\theta_F$ (up to a non-zero constant);
- $\iota_F^*\mathrm{Det} \tilde{\to} \pi^*\mathcal{O}(\Theta_{[F]})$.

Roughly said, the theorem states that, analogously the rank 1 case (see [17,32]), it is theoretically possible to give solutions for the $n$-KP in terms of certain non-abelian theta functions. However, further research must be made in this direction to obtain explicit expressions.

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## Appendix A. Moduli of pointed curves

Similar to the case of moduli spaces of vector bundles (see Section 5), the relation of infinite Grassmannian and moduli spaces of curves is also remarkable.

Let us point out only one construction; namely, that of the "Weierstrass divisor". By "Weierstrass divisor" we refer to the closure of the locus of Weierstrass points of smooth curves, $\bar{\mathcal{W}} \subset \bar{\mathcal{C}}_g$, where $\bar{\mathcal{C}}_g$ is the coarse moduli space of pointed stable algebraic curves over $\mathbb{C}$.

Let us construct $\bar{\mathcal{W}}$ following [8]. Let $\pi : C \to B$ be a family of semistable curves with $C$ smooth, $C_s \subset C$ be the locus of singular points of fibers, $B_s \subset B$ be $\pi(C_s)$, and $\omega_{C/B}$ be the relative dualizing sheaf.

Assume the fibers of $\pi$ to be of arithmetic genus $g$.

Consider the complex of rank $g$ bundles on $C - C_s$:

$$C_B^{\cdot} \equiv \pi^*\pi_*\omega_{C/B} \xrightarrow{e_g} p_{2*}\left(p_1^*\omega_{(C-C_s)/B} \otimes \frac{\mathcal{O}_{C\times_B C}}{\Delta^g}\right)$$

induced by $\mathcal{O}_{C\times_B C} \to \mathcal{O}_{C\times_B C}/\Delta^g$, where $\Delta$ is the ideal of the diagonal. Then, we have obtained a section:

$$\det(e_g) \in H^0(C - C_s, \mathrm{Det}(C_B^{\cdot})^{-1}).$$

Now, let $\mathcal{W}_\pi$ be the divisor of zeroes of $\det(e_g)$; that is, $\mathcal{O}_{C-C_s}(\mathcal{W}_\pi) = \mathrm{Det}(C_B^{\cdot})^{-1}|_{C-C_s}$. Since $\mathcal{W}_\pi \cap \pi^{-1}(b)$ consists of the Weierstrass points of the curve $\pi^{-1}(b)$ (provided that $\pi^{-1}(b)$ is smooth), it is natural to define:

$$\bar{\mathcal{W}}_\pi := \text{closure of } \mathcal{W}_{\pi'} \text{ in } C - C_s,$$

where $\pi' : C - \pi^{-1}(B_s) \to B - B_s$.

Let us repeat this construction from the point of view of infinite Grassmannians. Let $\mathrm{Gr}(V)$ denote an infinite Grassmannian associated to the pair $(V := k((z))\,\mathrm{d}z, V^+ := z^g k[[z]]\,\mathrm{d}z)$. Let us consider the morphism of schemes given by:

$$\iota : \hat{\mathcal{M}}_{g,1} \to \mathrm{Gr}^0(V), \qquad (C, p, \alpha) \mapsto H^0(C - p, \omega_C)$$

Now, the restriction of the determinant bundle on the Grassmannian, $\mathrm{Det} := \mathrm{Det}(\mathcal{L} \to V/V^+)$ induces a line bundle on $\hat{\mathcal{M}}_{g,1}$, and the restriction of the global section $\Omega_+$ gives a section of its dual:

$$\omega_g \in H^0(\hat{\mathcal{M}}_{g,1}, \iota^*\mathrm{Det}^{-1}).$$

Given a rational point $(C, p, \alpha) \in \hat{\mathcal{M}}_{g,1}(\mathbb{C})$ with $C$ smooth, the section $\omega_g$ vanishes on it if and only if $p$ is a Weierstrass point.

It is now easy to prove that for a $B$-valued point $(C, p, \alpha) \in \hat{\mathcal{M}}_{g,1}(B)$ such that $\pi : C \to B$ is a family of semistable curves with $C$ smooth,

$$\mathcal{O}_{C-C_s}(\mathcal{W}_\pi) \simeq (\iota \circ f)^*\mathrm{Det} \quad \text{on } C - C_s,$$

holds, where $f$ is the morphism induced by the universal property of $\hat{\mathcal{M}}_{g,1}$; in other words, it is given by the following diagram:

$$\begin{array}{ccc} & \overset{f}{\nearrow} & \widehat{\mathcal{M}}_{g,1} \\ & & \downarrow \\ C = \mathcal{C}_g \times_{\mathcal{M}_g} B & \longrightarrow & \mathcal{C}_g \\ \pi\downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{M}_g \end{array}$$

Geometrically, this means that the Weierstrass divisor (of $\bar{\mathcal{C}}_g$) is essentially a hyperplane section of the Pücker embedding of the Grassmannian [26] restricted to $\hat{\mathcal{M}}_{g,1}$.

A similar relation for the sections $\det(e_g)$ and $\omega_g$ follows.

As in the vector bundle case, expressions for $\det(e_g)$ will give rise to explicit solutions of the KP hierarchy. From another point of view, the above construction might be used for studying the Picard group of $\hat{\mathcal{M}}_{g,1}$, as well as higher Weierstrass points.

## Appendix B. Commutative rings of differential operators

Here, we shall sketch another application of the $n$-KP; namely, the study of pairs of commutating differential operators.

Consider a pair of differential operators (DOs) of the following form:

$$P = \mathrm{I}\partial^p + \sum_{i=0}^{p-1} P_i(s)\partial^i, \qquad Q = \mathrm{I}\partial^q + \sum_{i=0}^{q-1} Q_i(s)\partial^i \tag{B.1}$$

($s$ and $\partial$ as in Section 3), where $P_i$ *and* $Q_i$ are $n \times n$ matrices whose entries are Laurent series in $x$ with coefficients in $\mathbb{C}$. The order of $P$ is defined to be the integer $p$. The gcd of $p$ and $q(p, q)$ is called the rank, $r$.

The first results on this problem go back to [5–7] ($n = 1$). Modern techniques of algebraic curves have been successfully applied by Krichever for the rank 1 case, who obtained a complete classification. See [17,23] for the case $n = 1, r = 1$, [16,18] for $r = 1$ and $n$ arbitrary, [28] for $n = 1, r \geq 1$ and [15] for $n \geq 1, r = 1$. These papers will be our main references.

However, we do believe that our approach, particularly the moduli spaces $\hat{\mathcal{M}}^r_{g,n}$ (see Section 4), might help in the understanding of the general case.

Let us consider a pair of commuting DOs as in (B.1) and show how some algebraic data is constructed from it.

Firstly, observe that most results of [5–7] holds true for the $n \geq 1$ situation. In particular, the pair satisfy the so-called "characteristic identity"; that is, there exists a polynomial $F(x, y)$ such that:

$$F(P, Q) \equiv 0.$$

Let $C$ be the affine algebraic curve defined by $F(x, y) = 0$ and $\bar{C}$ be its completion. A closer look to this identity shows that $\bar{C} - C$ consists of $n$ smooth points $\{\infty_1, \dots, \infty_n\}$ and that $x^{1/p}$ induces formal parameters at them. We have obtained $X(P, Q) := (C, \{\infty_i, \alpha_i\})$ so far (see also [18]).

Our next task is to build a bundle on $\bar{C}$. Define a root of $P$ as a (vector valued) function $f$ such that $Pf \equiv 0$. The kernel of $P$ is the vector space consisting of all roots of $P$. We will consider a bundle whose fiber over a point $(x, y) \in \bar{C}$ is the vector space:

$$\{\text{common roots of } P - x \text{ and } Q - y\}.$$

It is not difficult to prove that this definition gives rise to a torsion-free sheaf of rank $r$ over $\bar{C}$, $M(P, Q)$.

Let $R$ be a PDOs such that $RPR^{-1}$, $RQR^{-1}$ are do. Then, it is trivial that the operators $RPR^{-1}$, $RQR^{-1}$ commute and that they satisfy the same identity:

$$F(RPR^{-1}, RQR^{-1}) \equiv 0.$$

Moreover, the data $X(P, Q)$ and $X(RPR^{-1}, RQR^{-1})$ coincide. However, the associated rank $r$ bundles do not coincide.

For simplicity's sake, from now on it will be assumed that $\bar{C}$ is smooth.

When $n = r = 1$, there is a distinguish class of DOs, $R$, as above. For these DOs, the transformation $(P, Q) \mapsto (RPR^{-1}, RQR^{-1})$ is called transference by Burchnall and Chaundy [5–7]. There, this notion is used to show that the commutative pairs of DOs (of fixed orders) satisfying the identity $F$ is isomorphic to the Jacobian variety of $\bar{C}$. This result was also proved by Krichever in [17]. Let us recall how such DOs may be constructed. Given a point $c = (x(c), y(c)) \in \bar{C}$, let $\nu$ be a common root of $P - x(c)$ and $Q - y(c)$, which is uniquely determined up to a constant since $n = r = 1$ (that is, $< \nu >= \ker(P - x(c)) \cap \ker(Q - y(c))$). Then, the transformation induced by the operator $\partial - (\nu'/\nu)$ is called linear transference. An order $r$ transference consists of $r$ successive linear transferences. Geometrically, it might be thought as the endomorphism of the Jacobian given by $L \mapsto L \otimes \mathcal{O}_{\bar{C}}(c - \infty)$ (where $\bar{C} = C \cup \{\infty\}$).

In general, we call this kind of transformations "Bäcklund–Darboux transformations". If we fix a trivialization $\beta$ of $M(P, Q)$ such that it determines a point in $\mathrm{Gr}(k((z))^{\oplus n \cdot r})$, then these transformations might be interpreted in terms of the Grassmannian (see [3,13] for the case $n = 1$).

A very interesting problem consists of giving a geometric interpretation of these transformations as flows on the moduli space of rank $r$ vector bundles over $\bar{C}$. This would generalize the fact that the KP hierarchy gives flows on Jacobian varieties.

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