

THE ALGEBRAIC FORMALISM OF SOLITON EQUATIONS OVER ARBITRARY BASE FIELDS

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1. INTRODUCTION

The aim of this paper is to offer an algebraic construction of infinite-dimensional Grassmannians and determinant bundles. As an application we construct the τ -function and formal Baker-Akhiezer functions over arbitrary fields, by proving the existence of a “formal geometry” of local curves analogous to the geometry of global algebraic curves.

Recently G. Anderson ([A]) has constructed the infinite-dimensional Grassmannians and τ -functions over p -adic fields; his constructions are basically the same as in the Segal-Wilson paper ([SW]) but he replaces the use of the theory of determinants of Fredholm operators over a Hilbert space by the theory of p -adic infinite determinants (Serre [S]).

Our point of view is completely different and the formalism used is valid for arbitrary base fields; for example, for global number fields or fields of positive characteristic. We begin by defining the functor of points, $\underline{\text{Gr}}(V, V^+)$, of the Grassmannian of a k -vector space

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V (with a fixed k -vector subspace $V^+ \subseteq V$) in such a way that the points $\underline{\mathrm{Gr}}(V, V^+)(\mathrm{Spec}(k))$ are precisely the points of the Grassmannian defined by Segal-Wilson or Sato-Sato ([**SW**], [**SS**]) although the points over an arbitrary k -scheme S have been not previously considered by other authors. This definition of the functor $\underline{\mathrm{Gr}}(V, V^+)$, which is a sheaf in the category of k -schemes, allows us to prove that it is representable by a separated k -scheme $\mathrm{Gr}(V, V^+)$. The universal property of the k -scheme $\mathrm{Gr}(V, V^+)$ implies, as in finite-dimensional Grassmannians, the existence of a universal submodule, \mathcal{L}_V , of π^*V ($\pi : \mathrm{Gr}(V, V^+) \rightarrow \mathrm{Spec}(k)$ being the natural projection). These constructions allow us to use the theory of determinants of Knudsen and Mumford ([**KM**]) to construct the determinant bundle over $\mathrm{Gr}(V, V^+)$. This is one of the main results of the paper because it implies that we can define “infinite determinants” in a completely algebraic way. From this definition of the determinant bundle, we show in §3 that global sections of the dual determinant bundle can be computed in a very natural form. The construction of τ -functions and Baker functions is based on the algebraic version, given in §4, of the group Γ of continuous maps $S^1 \rightarrow \mathbb{C}^*$ defined by Segal-Wilson ([**SW**]) which acts as a group of automorphisms of the Grassmannians. We replace the group Γ by the representant of the following functor over the category of k -schemes

$$S \rightsquigarrow H^0(S, \mathcal{O}_S)((z))^* = H^0(S, \mathcal{O}_S)[[z]][z^{-1}]^*$$

This is one of the points where our view differs essentially from other known expositions ([**A**], [**AD**], [**SW**], [**SS**]). Usually, the elements of Γ are described as developments, of the type $f = \sum_{-\infty}^{+\infty} \lambda_i z^i$ ($\lambda_k \in \mathbb{C}$), but in the present formalism the elements of Γ with values in a k -algebra A are developments $f = \sum_{i \geq -N} \lambda_i z^i \in A((z))$ such that $\lambda_{-1}, \dots, \lambda_{-N}$ are nilpotent elements of A .

In future papers we shall apply the formalism offered here to arithmetic problems (Drinfeld moduli schemes and reciprocity laws) and shall give an algebraic formalism of the theory of KP-equations related to the characterization of Jacobians and Prym varieties. We also hope that this formalism might clarify the algebro-geometric aspects of conformal field theories over base fields different from \mathbb{R} or \mathbb{C} in the spirit of the paper of E. Witten ([**W**]).

2. INFINITE GRASSMANNIANS

Let V be a vector space over a field k .

Definition 2.1. (Tate [T]) Two vector spaces A and B of V are commensurable if $A + B/A \cap B$ is a vector space over k of finite dimension. We shall use the symbol $A \sim B$ to denote commensurable vector subspaces.

Let us observe that commensurability is an equivalence relation between vector subspaces. The addition and intersection of two vector subspaces commensurable with a vector subspace A is also commensurable with A .

Let us fix a vector subspace $V^+ \subseteq V$. The equivalence class of vector subspaces commensurable with V^+ allows one to define on V a topology, which will be called V^+ -topology: a basis of neighbourhoods of 0 in this topology is the set of vector subspaces of V commensurable with V^+ .

V is a Hausdorff topological space with respect to the V^+ -topology.

Definition 2.2. The completion of V with respect to the V^+ -topology is defined by:

$$\widehat{V} = \varprojlim_{A \sim V^+} (V/A)$$

Analogously, given a vector subspace $B \subseteq V$ we define the completions of B and V/B with respect to $B \cap V^+$ and $B + V^+/B$, respectively.

The homomorphism of completion $V \xrightarrow{i} \widehat{V}$ is injective and V is said to be complete if $V \xrightarrow{i} \widehat{V}$ is an isomorphism.

Example 1. • $(V, V^+ = 0)$; V is complete.

• $V = k((t))$, $V^+ = k[[t]]$; V is complete.

• Let (X, \mathcal{O}_X) be a smooth, proper and irreducible curve over the field k , and let V be the ring of adèles of the curve and $V^+ = \prod_p \widehat{\mathcal{O}_p}$

(\mathcal{O}_p being the \mathfrak{m}_p -adic completion of the local ring of X in the point p); V is complete with respect the V^+ -topology.

Proposition 2.3. The following conditions are equivalent:

1. V is complete.
2. V^+ is complete.
3. Each vector subspace commensurable with V^+ is complete.

Proof. This follows easily from the following commutative diagram for every $A \sim V^+$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{V} & \longrightarrow & \widehat{V/A} \longrightarrow 0 \\ & & \uparrow i_A & & \uparrow i_V & & \uparrow \simeq \\ 0 & \longrightarrow & A & \longrightarrow & V & \longrightarrow & V/A \longrightarrow 0 \end{array}$$

□

Definition 2.4. Given a k -scheme S and a vector subspace $B \subseteq V$, we define:

1. $\widehat{V}_S = \varprojlim_{\substack{A \\ A \sim V^+}} (V/A \otimes_k \mathcal{O}_S).$
2. $\widehat{B}_S = \varprojlim_{\substack{A \\ A \sim V^+}} (B/A \cap B) \otimes_k \mathcal{O}_S.$
3. $\widehat{(V/B)}_S = \varprojlim_{\substack{A \\ A \sim V^+}} ((V/A + B) \otimes_k \mathcal{O}_S).$

Proposition 2.5. \widehat{V}_S is a sheaf of \mathcal{O}_S -modules and given $B \sim V^+$, we have:

$$\widehat{(V/B)}_S = \widehat{V}_S / \widehat{B}_S = (V/B) \otimes_k \mathcal{O}_S$$

Proof. This is an easy exercise of linear algebra. □

Let V be a k -vector space and V^+ a vector subspace determining a class of commensurable vector subspaces.

Definition 2.6. A discrete vector subspace of V is a vector subspace, $L \subseteq V$, such that $L \cap V^+$ and $V/L + V^+$ are k -vector spaces of finite dimension.

We aim to define a Grassmannian scheme $\text{Gr}(V, V^+)$, defining its functor of points $\underline{\text{Gr}}(V, V^+)$ and proving that it is representable in the category of k -schemes.

If V is complete, the rational points of our Grassmannian will be precisely the discrete vector spaces of V ; that is, $\underline{\text{Gr}}(V, V^+)(\text{Spec}(k))$ as a set coincides with the usual infinite Grassmannian defined by Pressley and Segal [PS] or M. and Y. Sato [SS].

Definition 2.7. Given a k -scheme S , a discrete submodule of \widehat{V}_S is a sheaf of quasi-coherent \mathcal{O}_S -submodules $\mathcal{L} \subset \widehat{V}_S$ such that $\mathcal{L}_{S'} \subset \widehat{V}_{S'}$ for every morphism $S' \rightarrow S$ and for each $s \in S$, $\mathcal{L} \otimes_{\mathcal{O}_S} k(s) \subset \widehat{V}_S \otimes_{\mathcal{O}_S} k(s)$ and there exists an open neighbourhood U_s of s and a commensurable k -vector subspace $B \sim V^+$ such that: $\mathcal{L}_{U_s} \cap \widehat{B}_{U_s}$ is free of finite type and $\widehat{V}_{U_s} / \mathcal{L}_{U_s} + \widehat{B}_{U_s} = 0$.

Proposition 2.8. With the notations of the above definition, given another commensurable k -vector space, $B' \sim V^+$, such that $B \subseteq B'$, $\mathcal{L}_{U_s} \cap \widehat{B'}_{U_s}$ is locally free of finite type.

Proof. This follows easily from the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widehat{B_{U_s}} & \longrightarrow & \widehat{B'_{U_s}} & \longrightarrow & ((B'/B) \otimes_k \mathcal{O}_{U_s}) = \widehat{B'_{U_s}}/\widehat{B_{U_s}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \widehat{V_{U_s}}/\mathcal{L}_{U_s} & \longrightarrow & \widehat{V_{U_s}}/\mathcal{L}_{U_s} & \longrightarrow & 0 \longrightarrow 0
\end{array}$$

using the snake Lemma. \square

Definition 2.9. Given a k -vector space V and $V^+ \subseteq V$, the Grassmannian functor, $\underline{\text{Gr}}(V, V^+)$, is the contravariant functor over the category of k -schemes defined by

$$\underline{\text{Gr}}(V, V^+)(S) = \left\{ \begin{array}{l} \text{discrete sub-}\mathcal{O}_S\text{-modules of } \widehat{V_S} \\ \text{with respect the } V^+\text{-topology} \end{array} \right\}$$

Remark 1. Note that if V is a finite dimensional k -vector space and $V^+ = (0)$, then $\underline{\text{Gr}}(V, (0))$ is the usual Grassmannian functor defined by Grothendieck [EGA].

Definition 2.10. Given a commensurable vector subspace $A \sim V^+$, the functor \underline{F}_A over the category of k -schemes is defined by:

$$\underline{F}_A(S) = \{ \text{sub-}\mathcal{O}_S\text{-modules } \mathcal{L} \subset \widehat{V_S} \text{ such that } \mathcal{L} \oplus \widehat{A_S} = \widehat{V_S} \}$$

(That is: $\mathcal{L} \cap \widehat{A_S} = (0)$ and $\mathcal{L} + \widehat{A_S} = \widehat{V_S}$).

Lemma 2.11. For every commensurable subspace $B \sim V^+$, the contravariant functor \underline{F}_B is representable by an affine and integral k -scheme F_B .

Proof. Let L_0 be a discrete k -subspace of V such that $L_0 \oplus B = V$; we then have:

$$F_B(S) = \underline{\text{Hom}}_{\mathcal{O}_S}((\mathcal{L}_0)_S, \widehat{B_S}) = \varprojlim_{\substack{A \\ A \sim V^+}} \underline{\text{Hom}}_{\mathcal{O}_S}((\mathcal{L}_0)_S, B/B \cap A \otimes_k \mathcal{O}_S)$$

If we denote by $F_{B/B \cap A}(S)$ the set $\underline{\text{Hom}}_{\mathcal{O}_S}((\mathcal{L}_0)_S, B/B \cap A \otimes_k \mathcal{O}_S)$, it is obvious that the functor $F_{B/B \cap A}(S)$ is representable by an affine and integral k -scheme since $B/B \cap A$ is a finite dimensional k -vector space. But \underline{F}_B is now a projective limit of functors representable by affine schemes, so we conclude that \underline{F}_B is representable by an affine k -scheme. \square

Lemma 2.12. *Let \mathcal{L} be an element in $\underline{\text{Gr}}_{V^+}(V)(S)$ and A and B are two k -subspaces of V commensurable with V^+ . It holds that:*

- a) *if $\widehat{V}_S/\mathcal{L} + \widehat{A}_S = 0$, then $\mathcal{L} \cap \widehat{A}_S$ is a finite type locally free of \mathcal{O}_S -module.*
- b) *$\widehat{V}_S/\mathcal{L} + \widehat{B}_S$ is an \mathcal{O}_S -module locally of finite presentation.*

Proof.

a) By proposition 2.8, for each point $s \in S$ there exists an open neighbourhood U_s and a commensurable k subspace $A' \sim V^+$ such that: $A \subseteq A'$, $\widehat{V}_{U_s}/\mathcal{L}_{U_s} + \widehat{A}'_{U_s} = 0$ and $\mathcal{L}_{U_s} \cap \widehat{A}'_{U_s}$ is free of finite type. From the exact sequence:

$$0 \rightarrow \mathcal{L} \cap \widehat{A}_S \rightarrow \mathcal{L} \rightarrow (\widehat{V}_S/\widehat{A}_S) = (V/A)_S \rightarrow 0$$

one deduces that $\mathcal{L} \cap \widehat{A}_S$ is quasicoherent and

$$0 \rightarrow (\mathcal{L} \cap \widehat{A}_S)_{U_s} \rightarrow \mathcal{L}_{U_s} \rightarrow (\widehat{V}_{U_s}/\widehat{A}_{U_s}) = (V/A)_{U_s} \rightarrow 0$$

Let us consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L} \oplus \widehat{A}_S & \longrightarrow & \mathcal{L} \oplus \widehat{A}'_S & \longrightarrow & (A'/A)_S \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{V}_S & \longrightarrow & \widehat{V}_S & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

By using the snake lemma we have an exact sequence:

$$0 \rightarrow (\mathcal{L}_S \cap \widehat{A}_S) \rightarrow \mathcal{L}_S \cap \widehat{A}'_S \rightarrow (A'/A)_S \rightarrow \widehat{V}_S/\mathcal{L}_S + \widehat{A}_S \rightarrow \widehat{V}_S/\mathcal{L}_S + \widehat{A}'_S \rightarrow 0$$

In our conditions for A and A' we have:

$$0 \rightarrow (\mathcal{L}_{U_s} \cap \widehat{A}_{U_s}) \rightarrow \mathcal{L}_{U_s} \cap \widehat{A}'_{U_s} \rightarrow (A'/A)_{U_s} \rightarrow 0$$

Then, $(\mathcal{L} \cap \widehat{A}_S)_{U_s} = \mathcal{L}_{U_s} \cap \widehat{A}_{U_s}$ is the kernel of a surjective homomorphism between two free \mathcal{O}_{U_s} -modules of finite type, and we conclude the proof.

b) For a given $s \in S$, let us take $B \subseteq B'$ such that $B' \sim V^+$ is a commensurable subspace, $\mathcal{L}_{U_s} \cap \widehat{B}'_{U_s}$ is free of finite type and $\widehat{V}_{U_s}/\mathcal{L}_{U_s} + \widehat{B}'_{U_s} = 0$. We then have the exact sequence:

$$\mathcal{L}_{U_s} \cap \widehat{B}'_{U_s} \rightarrow B'/B \otimes_k \mathcal{O}_{U_s} \rightarrow \widehat{V}_{U_s}/\mathcal{L}_{U_s} + \widehat{B}_{U_s} \rightarrow 0$$

and so we conclude. \square

Theorem 2.13. *The functor $\underline{\text{Gr}}(V, V^+)$ is representable by a k -scheme $\text{Gr}(V, V^+)$.*

Proof. The proof is modeled on the Grothendieck construction of finite Grassmannians [EGA]; that is:

It is sufficient to prove that $\{\underline{F}_A, A \sim V^+\}$ is a covering of $\underline{\text{Gr}}(V, V^+)$ by open subfunctors:

1) For every $A \sim V^+$, the morphism of functors $\underline{F}_A \rightarrow \underline{\text{Gr}}(V, V^+)$ is representable by an open immersion:

That is, given morphism of functors $X^\bullet \rightarrow \underline{\text{Gr}}(V, V^+)$ (where X is a k -scheme), the functor

$$X^\bullet \times_{\underline{\text{Gr}}(V, V^+)} \underline{F}_A \hookrightarrow X^\bullet$$

is represented by an open subscheme of X . This is equivalent to proving that given $\mathcal{L} \in \underline{\text{Gr}}(V, V^+)(X)$, the set :

$$\mathcal{U}(A, \mathcal{L}) = \{x \in X \text{ such that } \mathcal{L}_x = \mathcal{L} \otimes_{\mathcal{O}_X} k(x) \in F_A(\text{Spec}(k(x)))\}$$

is open in X .

If $\mathcal{L}_x \in F_A(\text{Spec}(k(x)))$, then:

$$\widehat{V}_{k(x)}/\mathcal{L}_x + \widehat{A}_{k(x)} = 0$$

but $\widehat{V}_X/\mathcal{L} + \widehat{A}_X$ is a \mathcal{O}_X -module of finite presentation and, applying the lemma of Nakayama, there exists an open neighbourhood U_x of x , such that

$$\widehat{V}_{U_x}/\mathcal{L}_{U_x} + \widehat{A}_{U_x} = 0$$

By lemma 2.12, $\mathcal{L}_{U_x} \cap \widehat{A}_{U_x}$ is a \mathcal{O}_{U_x} -module coherent. However, bearing in mind that $\mathcal{L}_x \cap \widehat{A}_{k(x)} = 0$, there exists another open neighbourhood of x , $U'_x \subseteq U_x$, such that $\mathcal{L}_{U'_x} \cap \widehat{A}_{U'_x} = 0$ and therefore $\mathcal{L}_{U'_x} \in F_A(U'_x)$.
2) For every k -scheme X and every morphism of functors

$$X^\bullet \rightarrow \underline{\text{Gr}}(V, V^+)$$

the open subschemes $\{\mathcal{U}(A, \mathcal{L}), A \sim V^+\}$ defined above are a covering of X .

That is, given $\mathcal{L} \in \underline{\text{Gr}}(V, V^+)(X)$ and a point $x \in X$, there exists an open neighbourhood, U_x , of x and a commensurable subspace $A \sim V^+$ such that:

$$\mathcal{L}_{U_x} \in F_A(U_x)$$

Let A be a commensurable subspace such that:

$$\mathcal{L}_x \cap \widehat{A}_{k(x)} = 0$$

since $\widehat{V}_{k(x)}/\mathcal{L}_x + \widehat{A}_{k(x)}$ is a $k(x)$ -vector space of finite dimension, we can choose a basis $(\overline{e_1 \otimes 1}, \dots, \overline{e_k \otimes 1})$ of $\widehat{V}_{k(x)}/\mathcal{L}_x + \widehat{A}_{k(x)}$ where $e_i \in V$.

Defining

$$B = A + \langle e_1, \dots, e_k \rangle$$

obviously $B \sim V^+$. One can easily prove that there exists an open subset $U'_x \subseteq U$ such that $\mathcal{L}_{U'_x} \in F_B(U'_x)$ and this completes the proof of the theorem. \square

Lemma 2.14. *Let A, B be two k -vector spaces of V commensurable with V^+ . A necessary and sufficient condition for the existence of $\mathcal{L} \in \underline{\text{Gr}}(V, V^+)(S)$ such that $\mathcal{L} \oplus \widehat{A}_S = \mathcal{L} \oplus \widehat{B}_S = \widehat{V}_S$, is that there should exist an isomorphism of k -vector spaces*

$$\tau : B/A \cap B \xrightarrow{\sim} A/A \cap B$$

Proof. Let us consider the decomposition:

$$\widehat{V}_S = (\widehat{A \cap B})_S \oplus (B/A \cap B)_S \oplus (A/A \cap B)_S \oplus (V/A + B)_S$$

If the isomorphism τ exists, we take:

$$\mathcal{L} = \{(a, b, \tau(b)), \quad a \in (V/A + B)_S, \quad b \in (B/B \cap A)_S\}$$

Conversely, assume that $\mathcal{L} \oplus \widehat{A}_S = \mathcal{L} \oplus \widehat{B}_S = \widehat{V}_S$. We then have:

$$\mathcal{L} \oplus (B/B \cap A)_S \oplus (\widehat{B \cap A})_S \simeq \mathcal{L} \oplus \widehat{B}_S \simeq \mathcal{L} \oplus \widehat{A}_S \simeq \mathcal{L} \oplus (A/A \cap B)_S \oplus (\widehat{B \cap A})_S$$

from which we deduce that:

$$(B/A \cap B)_S \simeq (A/A \cap B)_S$$

\square

Theorem 2.15. *$\text{Gr}(V, V^+)$ is a separated scheme.*

Proof. Let $F_B = \text{Spec}(A_B) = (\mathcal{U}_B)$ be the affine open subschemes of the Grassmannian constructed in lemma 2.11. It suffices to prove that given two commensurable subspaces B' and B such that $F_B \cap F_{B'} \neq \emptyset$ then $F_B \cap F_{B'}$ is affine.

By lemma 2.14

$$F_B \cap F_{B'} \neq \emptyset$$

implies the existence of $\mathcal{L}_0 \in F_B(\text{Spec}(k)) \cap F_{B'}(\text{Spec}(k))$ and bearing in mind that

$$\underline{F_B} \cap \underline{F_{B'}} = \underline{F_B} \times_{\underline{F_{B+B'}}} \underline{F_{B'}}$$

we conclude the proof. \square

Definition 2.16. *The discrete submodule corresponding to the identity*

$$Id \in \underline{\text{Gr}}(V, V^+) (\text{Gr}(V, V^+))$$

will be called the universal module and will be denoted by

$$\mathcal{L}_V \subset \widehat{V}_{\text{Gr}(V, V^+)}$$

Remark 2. In this section we have constructed infinite-dimensional Grassmannian schemes in an abstract way. Since we select particular vector spaces (V, V^+) we obtain different classes of Grassmannians. Two examples are relevant:

1. $V = k((t))$, $V^+ = k[[t]]$. In this case, $\text{Gr}(k((t)), k[[t]])$ is the algebraic version of the Grassmannian constructed by Pressley, Segal, and M. and Y. Sato ([PS], [SS]) and this Grassmannian is particularly suitable for studying problems related to the moduli of curves (over arbitrary fields) and KP-equations.
2. Let (X, \mathcal{O}_X) be a smooth, proper and irreducible curve over the field k and let V be the adeles ring over the curve and $V^+ = \prod_p \widehat{\mathcal{O}_p}$

(Example 1.3.3). In this case $\text{Gr}(V, V^+)$ is an adelic Grassmannian which will be useful for studying arithmetic problems over the curve X or problems related to the classification of vector bundles over a curve (non abelian theta functions...).

Instead of adelic Grassmanians, we could define Grassmanians associated with a fixed divisor on X in an analogous way.

These adelic Grassmanians will be also of interest in the study of conformal field theories over Riemann surfaces in the sense of Witten ([W]).

Remark 3. From the universal properties satisfied by the Grassmannian one easily deduces the well known fact that given a geometric point $W \in \text{Gr}(V)(\text{Spec}(K))$ ($k \hookrightarrow K$ being an extension of fields), the Zariski tangent space to $\text{Gr}(V)$ at the point W is the K -vector space:

$$T_W \text{Gr}(V) = \text{Hom}(W, \widehat{V}_K/W)$$

3. DETERMINANT BUNDLES

In this section we construct the determinant bundle over the Grassmannian following the idea of Knudsen and Mumford ([KM]). This allow us to define determinants algebraically and over arbitrary fields (for example for $k = \mathbb{Q}$ or $k = \mathbb{F}_q$).

Let us set a pair of vector spaces, $V^+ \subseteq V$. As in section 2, we will denote the Grassmannian $\text{Gr}(V, V^+)$ simply by $\text{Gr}(V)$.

Definition 3.1. For each $A \sim V^+$ and each $L \in \text{Gr}(V)(S)$ we define a complex, $\mathcal{C}_A^\bullet(L)$, of \mathcal{O}_S -modules by:

$$\mathcal{C}_A^\bullet(L) \equiv \cdots \rightarrow 0 \rightarrow L \oplus \hat{A}_S \xrightarrow{\delta} \hat{V}_S \rightarrow 0 \rightarrow \cdots$$

δ being the addition homomorphism.

Theorem 3.2. $\mathcal{C}_A^\bullet(L)$ is a perfect complex of \mathcal{O}_S -modules.

Proof. We have to prove that the complex of $\mathcal{C}_A^\bullet(L)$ is locally quasi-isomorphic to a bounded complex of free finitely-generated modules.

Let us note that the homomorphism of complexes given by the diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L \oplus \hat{A}_S & \xrightarrow{\delta} & \hat{V}_S & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow p_1 & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & L & \longrightarrow & (V/A)_S & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

(p_1 being the natural projection) is a quasi-isomorphism. The problem is local on S , and hence for each $s \in S$ we can assume the existence of an open neighbourhood, U , and a commensurable subspace $B \sim V^+$ such that $A \subseteq B$ and:

$$\hat{V}_U/(L_U, \hat{B}_U) = 0, \quad L_U \cap \hat{B}_U \text{ is free and finitely-generated}$$

We then have the exact sequence:

$$0 \rightarrow L_U \cap \hat{A}_U \rightarrow L_U \cap \hat{B}_U \rightarrow (B/A)_U \rightarrow \hat{V}_U/(L_U + \hat{A}_U) \rightarrow 0$$

from which we deduce that the homomorphism of complexes given by the following diagram is a quasi-isomorphism:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & L_U \cap \hat{B}_U & \longrightarrow & (B/A)_U & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & L_U & \longrightarrow & (V/A)_U & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

That is, $\mathcal{C}_A^\bullet(L)|_U$ is quasi-isomorphic to the complex $0 \rightarrow L_U \cap \hat{B}_U \rightarrow (B/A)_U \rightarrow 0$, which is a complex of free and finitely-generated modules. \square

Definition 3.3. The index of a point $L \in \text{Gr}(V)(S)$ is the locally constant function $i_L: S \rightarrow \mathbb{Z}$ defined by:

$$i_L(s) = \text{Euler-Poincaré characteristic of } \mathcal{C}_{V^+}^\bullet(L) \otimes k(s)$$

$k(s)$ being the residual field of the point $s \in S$. (For the definition of the Euler-Poincaré characteristic of a perfect complex see [KM]).

Remark 4. The following properties of the index are easy to verify:

1. Let $f : T \rightarrow S$ be a morphism of schemes and $L \in \text{Gr}(V)(S)$; then: $i_{f^*L} = f^*(i_L)$.
2. The function i is constantly zero over the open subset F_{V^+} .
3. If $B \sim V^+$, $V^+ \subseteq B$ and $\hat{V}_U/L_U + \hat{B}_U = 0$ over an open subscheme $U \subseteq S$, then $i_L(s) = \dim_{k(s)}(L_s \cap \hat{B}_s) - \dim_{k(s)}(B_s/V_s^+)$.
4. If V is a finite-dimensional k -vector space and $V^+ = V$ and $L \in \text{Gr}(V)(S)$, then $i_L = \text{rank}(L)$.
5. For any rational point $L \in \text{Gr}(V)(\text{Spec}(k))$ one has:

$$i_L = \dim_k(L \cap V^+) - \dim_k(\hat{V}/L + \hat{V}^+)$$

Theorem 3.4. *Let $\text{Gr}^n(V)$ be the subset over which the index takes values equal to $n \in \mathbb{Z}$. $\text{Gr}^n(V)$ are open connected subschemes of $\text{Gr}(V)$ and the decomposition of $\text{Gr}(V)$ in connected components is:*

$$\text{Gr}(V) = \coprod_{n \in \mathbb{Z}} \text{Gr}^n(V)$$

Proof. This is obvious from the properties of the index. \square

Given a point $L \in \text{Gr}(V)(S)$ and $A \sim V^+$, we denote by $\text{Det } \mathcal{C}_A^\bullet(L)$ the determinant sheaf of the perfect complex $\mathcal{C}_A^\bullet(L)$ in the sense of [KM].

Theorem 3.5. *With the above notations the invertible sheaf over S , $\text{Det } \mathcal{C}_A^\bullet(L)$, does not depend on A (up to isomorphisms).*

Proof. Let A and A' be two commensurable subspaces. It suffices to prove that:

$$\text{Det } \mathcal{C}_A^\bullet(L) \xrightarrow{\sim} \text{Det } \mathcal{C}_{A'}^\bullet(L)$$

in the case $A \subseteq A'$. In this case we have a diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{L} \oplus \hat{A}_S & \xrightarrow{\delta} & \hat{V}_S & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & & \downarrow & & \text{Id} \downarrow & & & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{L} \oplus \hat{A}'_S & \xrightarrow{\delta} & \hat{V}_S & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

and by the additivity of the functor $\text{Det}(-)$ we obtain:

$$\text{Det } \mathcal{C}_A^\bullet(L) \otimes \text{Det}(A'/A)_S \xrightarrow{\sim} \text{Det } \mathcal{C}_{A'}^\bullet(L)$$

However A'/A is free and we conclude the proof. \square

Definition 3.6. *The determinant bundle over $\mathrm{Gr}(V)$, Det_V , is the invertible sheaf:*

$$\mathrm{Det} \mathcal{C}_{V^+}^\bullet(\mathcal{L}_V)$$

\mathcal{L}_V being the universal submodule over $\mathrm{Gr}(V)$.

Proposition 3.7. *(Functoriality) Let $L \in \mathrm{Gr}(V)(S)$ be a point given by a morphism $f_L : S \rightarrow \mathrm{Gr}(V)$. There exists a functorial isomorphism:*

$$f_L^* \mathrm{Det}_V \xrightarrow{\sim} \mathrm{Det} \mathcal{C}_A^\bullet(L)$$

We shall denote this sheaf by $\mathrm{Det}_V(L)$.

Proof. The functor $\mathrm{Det}(-)$ is stable under base changes. \square

Remark 5. Let $L \in \mathrm{Gr}(V)(\mathrm{Spec}(K))$ be a rational point and let $A \sim V^+$ such that $L \cap \hat{A}_K$ and $\hat{V}_K/L + \hat{A}_K$ are K -vector spaces of finite dimension. In this case we have an isomorphism:

$$\mathrm{Det}_V(L) \simeq \wedge^{\max}(L \cap \hat{A}_K) \otimes \wedge^{\max}(\hat{V}_K/(L + \hat{A}_K))^*$$

That is, our determinant coincides, over the geometric points, with the determinant bundles of Pressley, Segal, Wilson and M. and Y. Sato ([PS], [SW], [SS]).

We shall now state with precision the connection between the determinant bundle Det_V and the determinant bundle over the finite Grassmannians.

Let $L, L' \in \mathrm{Gr}(V)(\mathrm{Spec}(k))$ such that $L \subseteq L'$. In these conditions, L'/L is a k -vector space of finite dimension. The natural projection $\pi : L' \rightarrow L'/L$ induces an injective morphism of functors:

$$\underline{\mathrm{Gr}}(L'/L) \hookrightarrow \underline{\mathrm{Gr}}(V)$$

defined by:

$$j(M) = \pi^{-1}(M) \quad \text{for each } M \in \underline{\mathrm{Gr}}(L'/L)(S)$$

We then have a morphism of schemes:

$$j : \underline{\mathrm{Gr}}(L'/L) \hookrightarrow \underline{\mathrm{Gr}}(V)$$

It is not difficult to prove that j is a closed immersion.

Theorem 3.8. *With the above notations, there exists a natural isomorphism:*

$$j^* \mathrm{Det}_V \xrightarrow{\sim} \mathrm{Det}_{L'/L}$$

$\mathrm{Det}_{L'/L}$ being the determinant bundle over the finite Grassmannian $\underline{\mathrm{Gr}}(L'/L)$.

Proof. Let \mathcal{L}^f be the universal submodule over $\mathrm{Gr}(L'/L)$. By definition $\mathrm{Det}_{L'/L} = \mathrm{Det}(\mathcal{L}^f \rightarrow L'/L)$, which is isomorphic to $\mathrm{Det}(\pi^{-1}\mathcal{L}^f \rightarrow L')$. By the definition of j , one has $j^*\mathcal{L}_V \xrightarrow{\sim} \pi^{-1}\mathcal{L}^f$ and hence:

$$j^* \mathrm{Det}_V \simeq \mathrm{Det}(\pi^{-1}\mathcal{L}^f \oplus \hat{V}^+ \rightarrow \hat{V})$$

and from the exact sequence of complexes:

$$\begin{array}{ccccc} \pi^{-1}\mathcal{L}^f & \longrightarrow & \pi^{-1}\mathcal{L}^f \oplus \hat{V}^+ & \longrightarrow & \hat{V}^+ \\ \downarrow & & \downarrow & & \downarrow \\ L' & \longrightarrow & \hat{V} & \longrightarrow & \hat{V}/L' \end{array}$$

we deduce that $j^* \mathrm{Det}_V \simeq \mathrm{Det}_{L'/L}$. \square

Corollary 3.9. *Let i be the index function over $\mathrm{Gr}(V)$. For each rational point $M \in \mathrm{Gr}(L'/L)$ one has:*

$$i(j(M)) = i(L') + \dim_k(L'/M + L)$$

Proof. Obvious. \square

3.1. Global sections of the determinant bundles and Plücker morphisms . It is well known that the determinant bundle have no global sections. We shall therefore explicitly construct global sections of the dual of the determinant bundle over the connected component $\mathrm{Gr}^0(V)$ of index zero.

We use the following notations: $\wedge^\bullet E$ is the exterior algebra of a k -vector space E ; $\wedge^r E$ its component of degree r , and $\wedge E$ is the component of higher degree when E is finite-dimensional.

Given a perfect complex \mathcal{C}^\bullet over k -scheme X , we shall write $\mathrm{Det}^* \mathcal{C}^\bullet$ to denote the dual of the invertible sheaf $\mathrm{Det} \mathcal{C}^\bullet$.

To explain how global sections of the invertible sheaf $\mathrm{Det}^* \mathcal{C}^\bullet$ can be constructed, let us begin with a very simple example:

Let $f : E \rightarrow F$ be a homomorphism between finite-dimensional k -vector spaces of equal dimension. This homomorphism induces:

$$\wedge(f) : \wedge E \rightarrow \wedge F$$

and $\wedge(f) \neq 0 \iff f$ is an isomorphism. $\wedge(f)$ can be expressed as a homomorphism:

$$\wedge(f) : k \rightarrow \wedge F \otimes (\wedge E)^*$$

Thus, if we consider $E \xrightarrow{f} F$ as a perfect complex, \mathcal{C}^\bullet , over $\mathrm{Spec}(k)$, we have defined a **canonical section** $\wedge(f) \in H^0(\mathrm{Spec}(k), \mathrm{Det}^* \mathcal{C}^\bullet)$.

Let us now consider a perfect complex $\mathcal{C}^\bullet \equiv (E \xrightarrow{f} F)$ of sheaves of \mathcal{O}_X -modules over a k -scheme X , with Euler-Poincaré characteristic $\mathcal{X}(\mathcal{C}^\bullet) = 0$. Let U be an open subscheme of X over which \mathcal{C}^\bullet

is quasi-isomorphic to a complex of finitely-generated free modules. By the above argument, we construct a canonical section $\det(f|_U) \in H^0(U, \text{Det}^* \mathcal{C}^\bullet)$ and for other open subset, V , there is a canonical isomorphism $\det(f|_U)|_{U \cap V} \simeq \det(f|_V)|_{U \cap V}$ and we therefore have a canonical section $\text{Det}(f) \in H^0(X, \text{Det}^* \mathcal{C}^\bullet)$. If the complex \mathcal{C}^\bullet is acyclic, one has an isomorphism:

$$\begin{aligned} \mathcal{O}_X &\xrightarrow{\sim} \text{Det}^* \mathcal{C}^\bullet \\ 1 &\mapsto \det(f) \end{aligned}$$

(for details see [KM]).

Let $0 \rightarrow H^\bullet \rightarrow \mathcal{C}_1^\bullet \rightarrow \mathcal{C}_2^\bullet \rightarrow 0$ be an exact sequence of perfect complexes. There exists a functorial isomorphism

$$\text{Det}^* \mathcal{C}_1^\bullet \xrightarrow{\sim} \text{Det}^* H^\bullet \otimes_{\mathcal{O}_X} \text{Det}^* \mathcal{C}_2^\bullet$$

If H^\bullet is acyclic, we obtain an isomorphism

$$H^0(X, \text{Det}^* \mathcal{C}_2^\bullet) \xrightarrow{\sim} H^0(X, \text{Det}^* \mathcal{C}_1^\bullet)$$

In the case $H^\bullet \equiv (E \xrightarrow{Id} E)$, $\mathcal{C}_1^\bullet \equiv (V \xrightarrow{f} V)$, $\mathcal{C}_2^\bullet \equiv (F \xrightarrow{f'} F)$, and $\mathcal{X}(\mathcal{C}_i^\bullet) = 0$ ($i = 1, 2$), we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\sim} & \text{Det}^* H^\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X \simeq \mathcal{O}_X \\ \downarrow & & \downarrow \\ \text{Det}^* \mathcal{C}_1^\bullet & \xrightarrow{\sim} & \text{Det}^* H^\bullet \otimes \text{Det}^* \mathcal{C}_2^\bullet \simeq \text{Det}^* \mathcal{C}_2^\bullet \end{array}$$

from which we deduce that $\det(f) = \det(f')$. Moreover, if F is locally free of finite rank, this means that computation of $\det(f)$ is reduced to computation of $\det(f')$, as mentioned above.

Let V be a k -vector space and $V^+ \subseteq V$ and $A \sim V^+$ a commensurable vector subspace. Let us consider the perfect complex $\mathcal{C}_A^\bullet \equiv (\mathcal{L} \oplus \hat{A} \xrightarrow{\delta_A} \hat{V})$ over $\text{Gr}(V)$ defined in 3 (\mathcal{L} being the universal discrete submodule over $\text{Gr}(V)$).

Lemma 3.10. $F_A \subseteq \text{Gr}^0(V)$ if and only if

$$\dim_k(A/A \cap V^+) - \dim_k(V^+/A \cap V^+) = 0$$

Proof. Obvious. □

Corollary 3.11. *The open subschemes F_A with $\dim_k(A/A \cap V^+) - \dim_k(V^+/A \cap V^+) = 0$ are a covering of $\text{Gr}^0(V)$. Given $A, B \sim V^+$ under the assumption $F_A, F_B \subseteq \text{Gr}^0(V)$ one has $\dim_k(A/A \cap B) - \dim_k(B/A \cap B) = 0$.*

Proof. Obvious. □

Given $A \sim V^+$ with $F_A \subseteq \text{Gr}^0(V)$, let us note that $\mathcal{C}_A^\bullet|_{F_A}$ is an acyclic complex. One then has an isomorphism:

$$\begin{aligned} \mathcal{O}_X|_{F_A} &\xrightarrow{\sim} \text{Det}^* \mathcal{C}_A^\bullet|_{F_A} \\ 1 &\mapsto s_A = \det(\delta_A)|_{F_A} \end{aligned}$$

We shall prove that the section $s_A \in H^0(F_A, \text{Det}^* \mathcal{C}_A^\bullet)$ can be extended in a canonical way to a global section of $\text{Det}^* \mathcal{C}_A^\bullet$ over the Grassmannian $\text{Gr}^0(V)$.

Let $B \sim V^+$ be such that $F_B \subseteq \text{Gr}^0(V)$ and let us consider the complex:

$$\mathcal{C}_{AB}^\bullet \equiv (\mathcal{L} \oplus \hat{A} \xrightarrow{\delta_{AB}} \mathcal{L} \oplus \hat{B})$$

where $\delta_{AB} = \delta_B^{-1} \circ \delta_A$. Obviously $\delta_{AB}|_{(0, \hat{A} \cap \hat{B})} = \text{Id}_{\hat{A} \cap \hat{B}}$ and $\delta_{AB}|_{(\mathcal{L}, 0)} = \text{Id}_{\mathcal{L}}$, we then have an exact sequence of complexes:

$$\begin{array}{ccccccc} & & & & A/A \cap B & & \\ & & & & \simeq \downarrow & & \\ 0 & \longrightarrow & \mathcal{L} \oplus (\hat{A} \cap \hat{B}) & \longrightarrow & \mathcal{L} \oplus \hat{A} & \longrightarrow & (\mathcal{L} \oplus \hat{A})/\mathcal{L} \oplus (\hat{A} \cap \hat{B}) \longrightarrow 0 \\ & & \text{Id} \downarrow & & \delta_{AB} \downarrow & & \phi_{AB} \downarrow \\ 0 & \longrightarrow & \mathcal{L} \oplus (\hat{A} \cap \hat{B}) & \longrightarrow & \mathcal{L} \oplus \hat{B} & \longrightarrow & (\mathcal{L} \oplus \hat{B})/\mathcal{L} \oplus (\hat{A} \cap \hat{B}) \longrightarrow 0 \\ & & & & \simeq \downarrow & & \\ & & & & B/A \cap B & & \end{array}$$

and from the discussion at the beginning of this section we have that $\det(\phi_{AB}) = \det(\delta_{AB}) \in H^0(F_B, \text{Det}^* \mathcal{C}_{AB}^\bullet)$ and $\det(\delta_{AB})$ satisfies the cocycle condition:

$$\det(\delta_{AA}) = 1$$

$$\det(\delta_{AB}) \cdot \det(\delta_{BC}) = \det(\delta_{AC}) \quad \text{over } F_B \cap F_C \text{ for any } C \sim V^+$$

Over $F_A \cap F_B$ we have canonical isomorphisms:

$$\begin{aligned} \mathcal{O}_{F_A \cap F_B} &\xrightarrow{\sim} \text{Det}^* \mathcal{C}_A^\bullet|_{F_A \cap F_B} \\ 1 &\mapsto s_A \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{F_A \cap F_B} &\xrightarrow{\sim} \text{Det}^* \mathcal{C}_B^\bullet|_{F_A \cap F_B} \\ 1 &\mapsto s_B \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{F_A \cap F_B} &\xrightarrow{\sim} \text{Det}^* \mathcal{C}_{AB}^\bullet|_{F_A \cap F_B} \\ 1 &\mapsto \det(\delta_{AB}) \end{aligned}$$

which are compatible, therefore:

$$(s_B \cdot \det(\delta_{AB}))|_{F_A \cap F_B} = s_A|_{F_A \cap F_B}$$

$s_B \cdot \det(\delta_{AB})$ being the image of $s_B \otimes \det(\delta_{AB})$ by the homomorphism:

$$H^0(F_B, \text{Det}^* \mathcal{C}_B^\bullet) \otimes H^0(F_B, \text{Det}^* \mathcal{C}_{AB}^\bullet) \rightarrow H^0(F_B, \text{Det}^* \mathcal{C}_A^\bullet)$$

defined by the isomorphism of sheaves:

$$\text{Det} \mathcal{C}_A^\bullet \simeq \text{Det} \mathcal{C}_B^\bullet \otimes \wedge(A/A \cap B) \otimes \wedge(B/A \cap B)^* \simeq \text{Det} \mathcal{C}_B^\bullet \otimes \text{Det} \mathcal{C}_{AB}^\bullet$$

Definition 3.12. *The global section $\omega_A \in H^0(\text{Gr}^0(V), \text{Det}^* \mathcal{C}_A^\bullet)$ defined by:*

$$\{s_B \cdot \det(\delta_{AB})\}_{B \sim V^+}$$

will be called the canonical section of $\text{Det}^ \mathcal{C}_A^\bullet$.*

This result allows us to compute many global sections of $\text{Det}_{V^+}^* = \text{Det} \mathcal{C}_{V^+}^\bullet$ over $\text{Gr}^0(V)$:

Given $A \sim V^+$ such that $F_A \subseteq \text{Gr}^0(V)$ the isomorphism $\text{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \text{Det}_{V^+}^*$ is not canonical, and in fact we have a canonical isomorphism:

$$\text{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \text{Det}_{V^+}^* \otimes \bigwedge(A/A \cap V^+) \otimes \bigwedge(V^+/A \cap V^+)^*$$

Therefore to give an isomorphism $\text{Det}^* \mathcal{C}_A^\bullet \xrightarrow{\sim} \text{Det}_{V^+}^*$ depends on the choice of bases for the vector spaces $A/A \cap V^+$ and $V^+/A \cap V^+$.

3.2. Computations for finite-dimensional Grassmannians. Let V be a d -dimensional k -vector space with a basis $\{e_1, \dots, e_d\}$, let $\{e_1^*, \dots, e_d^*\}$ be its dual basis, and $V^+ = \langle e_{k+1}, \dots, e_d \rangle \subseteq V$. In this case $\text{Gr}^0(V, V^+)$ is the Grassmannian of V classifying k -dimensional vector subspaces of V . Given a family of indexes $1 \leq i_1 < \dots < i_l \leq d$ ($1 \leq l \leq d$), let $A(i_1, \dots, i_l)$ be the vector subspace generated by $\{e_{i_1}, \dots, e_{i_l}\}$. One has that $F_A \subseteq \text{Gr}^0(V)$ is equivalent to saying that $l = d - k$.

Let us set $A = A(i_1, \dots, i_{d-k})$. Now, the canonical section $\omega_A \in H^0(\text{Gr}^0(V), \text{Det}^* \mathcal{C}_A^\bullet)$ is the section whose value at the point $L = \langle l_1, \dots, l_k \rangle \in \text{Gr}^0(V)$ is given by:

$$\omega_A(L) = \pi_A(l_1) \wedge \dots \wedge \pi_A(l_k) \otimes l_1^* \wedge \dots \wedge l_k^* \in \wedge V/A \otimes \wedge L^* = (\text{Det}^* \mathcal{C}_A^\bullet)_L$$

$\{l_1^*, \dots, l_k^*\}$ being the dual basis of $\{l_1, \dots, l_k\}$ and $\pi_A: L \rightarrow V/A$ the natural projection. Note that $\{e_{j_1}, \dots, e_{j_k}\}$ is a basis of V/A where $\{j_1, \dots, j_k\} = \{1, \dots, d\} - \{i_1, \dots, i_{d-k}\}$, and that its dual basis is $\{e_{j_1}^*, \dots, e_{j_k}^*\}$ in $(V/A)^* \subset V^*$. We have now:

$$\omega_A(L) = (e_{j_1}^* \wedge \dots \wedge e_{j_k}^*)(l_1 \wedge \dots \wedge l_k) \cdot e_{j_1} \wedge \dots \wedge e_{j_k} \otimes l_1^* \wedge \dots \wedge l_k^*$$

Observe that the k -vector space

$$\wedge(A/A \cap V^+)^* \otimes \wedge(V^+/A \cap V^+)$$

is generated by:

$$e_A = e_{m_1}^* \wedge \cdots \wedge e_{m_r}^* \otimes e_{n_1} \wedge \cdots \wedge e_{n_r}$$

where

$$\begin{aligned} \{i_1, \dots, i_{d-k}\} - \{k+1, \dots, d\} &= \{m_1, \dots, m_r\} \\ \{k+1, \dots, d\} - \{i_1, \dots, i_{d-k}\} &= \{n_1, \dots, n_r\} \end{aligned}$$

And tensorializing by e_A gives an isomorphism:

$$H^0(\mathrm{Gr}^0(V), \mathrm{Det}^* \mathcal{C}_A^\bullet) \xrightarrow{\otimes e_A} H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*)$$

Let Ω_A be the image of the canonical section ω_A . The explicit expression of Ω_A is:

$$\Omega_A(L) = (e_{j_1}^* \wedge \cdots \wedge e_{j_k}^*)(l_1 \wedge \cdots \wedge l_k) \cdot e_1 \wedge \cdots \wedge e_{d-k} \otimes l_1^* \wedge \cdots \wedge l_k^* \in (\mathrm{Det}_V^*)_L$$

Let $\mathcal{L} \subseteq V_{\mathrm{Gr}^0(V)}$ be the universal submodule. One has a canonical epimorphism:

$$\wedge^k V_{\mathrm{Gr}^0(V)}^* \rightarrow \wedge^k \mathcal{L}^*$$

and bearing in mind the canonical isomorphism $\mathrm{Det}_V^* \simeq \wedge(V/V^+) \otimes \wedge^k \mathcal{L}^*$ one obtains a canonical homomorphism:

$$\begin{aligned} \wedge^k V^* &= H^0(\mathrm{Gr}^0(V), \wedge^k V^*) \rightarrow H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*) \\ e_{j_1}^* \wedge \cdots \wedge e_{j_k}^* &\longmapsto \Omega_{A(i_1, \dots, i_{d-k})} \end{aligned}$$

(where $\{j_1, \dots, j_k\} \coprod \{i_1, \dots, i_{d-k}\} = \{1, \dots, d\}$).

It is well known that this homomorphism is in fact an isomorphism.

3.3. Computations for infinite-dimensional Grassmannians. In 3.2 we discussed well known facts about the determinants of finite-dimensional Grassmannians but have stated these results in an intrinsic language, which can easily be generalized to the infinite-dimensional case.

Let V be a k -vector space. We shall assume that there exists a family of linearly independent vectors $\{e_i, i \in \mathbb{Z}\}$ such that:

1. $\langle \{e_i\}, i \geq 0 \rangle$ is dense in \hat{V}^+ (with respect to the V^+ -topology),
2. $\langle \{e_i\}, i \in \mathbb{Z} \rangle$ is dense in \hat{V} .

Remark 6. The above conditions are satisfied for example by $V = k((t))$ and $V^+ = k[[t]]$.

Definition 3.13. Let \mathcal{S} be the set of sequences $\{s_0, s_1, \dots\}$ of integer numbers satisfying the following conditions:

1. the sequence is strictly increasing,

2. *there exists $s \in \mathbb{Z}$ such that $\{s, s+1, s+2, \dots\} \subseteq \{s_0, s_1, \dots\}$,*
3. $\#(\{s_0, s_1, \dots\} - \{0, 1, \dots\}) = \#(\{0, 1, \dots\} - \{s_0, s_1, \dots\})$.

The sequences of \mathcal{S} are usually called Maya's diagrams or Ferrer's diagrams of virtual cardinal zero (this is condition 3).

For each $S \in \mathcal{S}$, let A_S be the vector subspace of V generated by $\{e_{s_i}, i \geq 0\}$. By the condition 3 one has:

$$\dim_k(A_S/A_S \cap V^+) = \dim_k(V^+/A_S \cap V^+)$$

and hence: $A_S \sim V^+$ and $F_{A_S} \subseteq \text{Gr}^0(V)$. Further, $\{F_{A_S}, S \in \mathcal{S}\}$ is a covering of $\text{Gr}^0(V)$.

Let $\{e_i^*\}$ be a dual basis of $\{e_i\}$; that is, elements of V^* given by $e_i^*(e_j) = \delta_{ij}$.

For each finite set of increasing integers, $J = \{j_1, \dots, j_r\}$, let us define $e_J = e_{j_1} \wedge \dots \wedge e_{j_r}$ and $e_J^* = e_{j_1}^* \wedge \dots \wedge e_{j_r}^*$.

Given $S \in \mathcal{S}$, choose $J, K \subseteq \mathbb{Z}$ such that $\{e_j\}_{j \in J}$ is a basis of $A_S/A_S \cap V^+$ and $\{e_k^*\}_{k \in K}$ of $V^+/A_S \cap V^+$. We have seen that tensorializing by $e_J \otimes e_K^*$ defines an isomorphism:

$$H^0(\text{Gr}^0, \text{Det}^* \mathcal{C}_{A_S}^\bullet) \xrightarrow{\otimes(e_J \otimes e_K^*)} H^0(\text{Gr}^0, \text{Det}_V^*)$$

Definition 3.14. *For each $S \in \mathcal{S}$, Ω_S is the global section of Det_V^* defined by:*

$$\Omega_S = \omega_{A_S} \otimes e_J \otimes e_K^*$$

We shall denote by Ω_+ the canonical section of Det_V^ .*

Let $\Omega(\mathcal{S})$ be the k -vector subspace of $H^0(\text{Gr}^0, \text{Det}_V^*)$ generated by the global sections $\{\Omega_S, S \in \mathcal{S}\}$.

We define the Plücker morphism:

$$\begin{aligned} \mathcal{P}_V : \text{Gr}^0(V) &\rightarrow \check{\mathbb{P}}\Omega(\mathcal{S}) \\ L &\mapsto \{\Omega_S(L)\} \end{aligned}$$

as the morphism of schemes defined by the homomorphism of sheaves:

$$\Omega(S)_{\text{Gr}(V)} \rightarrow \text{Det}_V^* \rightarrow 0$$

(by the universal property of $\check{\mathbb{P}}$).

Remark 7. Given $L, L' \in \text{Gr}(V)(\text{Spec}(k))$ such that $L \subseteq L'$, let $j : \text{Gr}^0(L'/L) \hookrightarrow \text{Gr}^0(V)$ be the natural closed immersion. Since $j^* \text{Det}_V \simeq \text{Det}_{L'/L}$, one can easily see that the composition:

$$\text{Gr}^0(L'/L) \xrightarrow{j} \text{Gr}^0(V) \xrightarrow{\mathcal{P}} \check{\mathbb{P}}\Omega(\mathcal{S})$$

factors through the Plücker immersion of the finite-dimensional Grassmannian $\mathrm{Gr}^0(L'/L)$:

$$\mathfrak{p}_{L'/L} : \mathrm{Gr}^0(L'/L) \rightarrow \mathrm{Proj} S^\bullet H^0(\mathrm{Gr}^0(L'/L), \mathrm{Det}_{L'/L}^*)$$

Theorem 3.15. *The Plücker morphism is a closed immersion.*

Proof. Going on with the analogy with finite grassmannians, we will show that this morphism is locally given as the graph of a suitable morphism. Consider the morphism:

$$F_{A_S} \hookrightarrow \mathrm{Gr}^0(V) \xrightarrow{\mathcal{P}} \check{\mathbb{P}}\Omega(S)$$

From the universal property of $\check{\mathbb{P}}$, we deduce a epimorphism:

$$f_S : \Omega(S) \otimes_k B \rightarrow B$$

(where $\mathrm{Spec}(B) = F_{A_S}$, and $\mathrm{Det}_V^*|_{F_{A_S}}$ is a line bundle). Note that it has a section, since the image of Ω_S is a everywhere non null function. That is, there exists a subspace $W \subset \Omega(S)$, and an isomorphism of k -vector spaces:

$$\langle \Omega_S \rangle \oplus W \xrightarrow{\sim} \Omega(S)$$

such that f_S is the projection onto the first factor. In other words, $\mathcal{P}|_{F_{A_S}}$ is the graph of a morphism. \square

Remark 8. Note that considering the chain of finite-dimensional Grassmannians $\mathrm{Gr}^0(L_i/L_{-i})$ (L_i being the subspaces $\langle \{e_j\}_{j \leq i} \rangle$), which are closed subschemes of $\mathrm{Gr}^0(V)$, one easily deduces that $H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}) = k$ from the fact that the homomorphism:

$$H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}) \rightarrow \varprojlim H^0(\mathrm{Gr}^0(L_i/L_{-i}), \mathcal{O}_{\mathrm{Gr}^0(L_i/L_{-i})})$$

is injective.

4. AUTOMORPHISMS OF THE GRASSMANNIAN AND THE “FORMAL GEOMETRY” OF LOCAL CURVES

Let (V, V^+) be a pair of a k -vector space and a vector subspace $V^+ \subseteq V$ and let $\mathrm{Gr}(V)$ denote the corresponding Grassmannian. We shall define the algebraic analogue of the restricted linear group defined by Pressley, Segal and Wilson ([PS], [SW]). This group is too large to be representable by a k -scheme and we therefore define it as a sheaf of groups in the category of k -schemes.

For each k -scheme S , let us denote by $\mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S)$ the group of automorphisms of the \mathcal{O}_S -module \hat{V}_S .

Definition 4.1.

- a) A sub- \mathcal{O}_S -module $\mathcal{B} \subseteq \hat{V}_S$ is said to be locally commensurable with V^+ if for each $s \in S$ there exists an open neighbourhood U_s of s and a commensurable vector subspace $B \sim V^+$ such that $\mathcal{B}|_{U_s} = \hat{B}_{U_s}$.
- b) An automorphism $g \in \text{Aut}_{\mathcal{O}_S}(\hat{V}_S)$ is called bicontinuous with respect to the V^+ -topology if $g(\hat{V}_S^+)$ and $g^{-1}(\hat{V}_S^+)$ are \mathcal{O}_S -modules of \hat{V}_S locally commensurable with V^+ .
- c) The linear group, $\text{Gl}(V)$, of (V, V^+) is the contravariant functor over the category of k -schemes defined by:

$$S \rightsquigarrow \text{Gl}(V)(S) = \{g \in \text{Aut}_{\mathcal{O}_S}(\hat{V}_S) \text{ such that } g \text{ is bicontinuous} \}$$

Theorem 4.2. *There exists a natural action of $\text{Gl}(V)$ over the functor of points of the Grassmannian $\text{Gr}(V)$:*

$$\begin{array}{ccc} \text{Gl}(V) \times \text{Gr}(V) & \xrightarrow{\mu} & \text{Gr}(V) \\ (g, L) & \longmapsto & g(L) \end{array}$$

Proof. Let $g \in \text{Gl}(V)(S)$ and $L \in \text{Gr}(V)(S)$. We have:

$$\hat{V}_S/g(L) + \hat{V}_S^+ \simeq \hat{V}_S/L + g^{-1}\hat{V}_S^+$$

and by definition of bicontinuous automorphisms, for each $s \in S$ there exist an open neighbourhood U_s and a commensurable $A \sim V^+$ such that $g^{-1}\hat{V}^+|_{U_s} = \hat{A}_{U_s}$. Then:

$$\hat{V}_{U_s}/g(L)_{U_s} + \hat{V}_{U_s}^+ \simeq \hat{V}_{U_s}/L_{U_s} + \hat{A}_{U_s}^+$$

from which we deduce that $g(L) \in \text{Gr}(V)(S)$. \square

Theorem 4.3. *There exists a canonical central extension of functors of groups over the category of k -schemes:*

$$0 \rightarrow \mathbb{G}_m \rightarrow \widetilde{\text{Gl}}(V) \rightarrow \text{Gl}(V) \rightarrow 0$$

and a natural action $\bar{\mu}$ of $\widetilde{\text{Gl}}(V)$ over the vector bundle $\mathbb{V}(\text{Det}_V)$ defined by the determinant bundle, such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{\text{Gl}}(V) \times \mathbb{V}(\text{Det}_V^*) & \xrightarrow{\bar{\mu}} & \mathbb{V}(\text{Det}_V^*) \\ \downarrow & & \downarrow \\ \text{Gl}(V) \times \text{Gr}(V) & \xrightarrow{\mu} & \text{Gr}(V) \end{array}$$

Proof. Let us define $\widetilde{\mathrm{Gl}}(V)(S)$ as the set of commutative diagrams:

$$\begin{array}{ccc} \mathbb{V}(\mathrm{Det}_V^*) & \xrightarrow{\bar{g}} & \mathbb{V}(\mathrm{Det}_V^*) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(V) & \xrightarrow{g} & \mathrm{Gr}(V) \end{array}$$

for each $g \in \mathrm{Gl}(V)(S)$, and the homomorphism $\widetilde{\mathrm{Gl}}(V) \rightarrow \mathrm{Gl}(V)$ given by $\bar{g} \mapsto g$. The rest of the proof follows immediately from the fact that $H^0(\mathrm{Gr}^0(V), \mathcal{O}_{\mathrm{Gr}^0(V)}) = k$ (remark 8) and $g^* \mathrm{Det}_V^* \simeq \mathrm{Det}_V^*$ for every $g \in \mathrm{Gl}(V)$. \square

Remark 9. Let G be a commutative subgroup of $\mathrm{Gl}(V)$ (a subfunctor of commutative groups). The central extension of $\mathrm{Gl}(V)$ gives an extension of G :

$$0 \rightarrow \mathbb{G}_m \rightarrow \widetilde{G} \xrightarrow{\pi} G \rightarrow 0$$

and the commutator of \widetilde{G} :

$$\begin{aligned} \widetilde{G} \times \widetilde{G} &\rightarrow \widetilde{G} \\ (\tilde{a}, \tilde{b}) &\mapsto \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} \end{aligned}$$

induces a pairing:

$$\begin{aligned} G \times G &\xrightarrow{[\cdot, \cdot]} \mathbb{G}_m \\ (g_1, g_2) &\mapsto [g_1, g_2] = \tilde{g}_1\tilde{g}_2\tilde{g}_1^{-1}\tilde{g}_2^{-1} \quad (\tilde{g}_i \in \pi^{-1}(g_i)) \end{aligned}$$

When V is a local field or a ring of adeles, this pairing will be of great importance in the study of arithmetic problems because it is connected with the formulation of reciprocity laws.

The same construction of the extensions \widetilde{G} applied to the Lie algebra of G gives an extension of Lie algebras (taking the points of G with values in $k[x]/x^2$):

$$0 \rightarrow \mathbb{G}_a = \mathrm{Lie}(\mathbb{G}_m) \rightarrow \mathrm{Lie}(\widetilde{G}) \xrightarrow{d\pi} \mathrm{Lie}(G) \rightarrow 0$$

and a pairing:

$$\begin{aligned} \mathrm{Lie}(G) \times \mathrm{Lie}(G) &\xrightarrow{R} \mathbb{G}_a \\ (D_1, D_2) &\mapsto R((D_1, D_2)) = [\tilde{D}_1, \tilde{D}_2] = \tilde{D}_1\tilde{D}_2 - \tilde{D}_2\tilde{D}_1 \end{aligned}$$

(\tilde{D}_i being a preimage of D_i).

The pairing R is an abstract generalization of the definition of Tate [T] of the residue pairing. There are several subgroups of special relevance in the application of this theory to the study of moduli problems

and soliton equations. Firstly, we are concerned with the algebraic analogue of the group Γ ([SW] §2.3) of continuous maps $S^1 \rightarrow \mathbb{C}^*$ acting as multiplication operators over the Grassmannian. The main difference between our definition of the group Γ and the definitions offered in the literature ([SW], [PS]) is that in the algebro-geometric setting the elements $\sum_{-\infty}^{+\infty} g_k z^k$ with infinite positive and negative coefficients do not make sense as multiplication operators over $k((z))$.

Let us now consider the case $V = k((t))$, $V^+ = k[[t]]$. The main idea for defining the algebraic analogue of the group Γ is to construct a “scheme” whose set of rational points is precisely the multiplicative group $k((z))^*$.

Definition 4.4. *The contravariant functor, $\underline{k((z))}^*$, over the category of k -schemes with values in the category of commutative groups is defined by:*

$$S \rightsquigarrow \underline{k((z))}^*(S) = H^0(S, \mathcal{O}_S)((z))^*$$

Where for a k -algebra A , $A((z))^*$ is the group of invertible elements of the ring $A((z)) = A[[z]][z^{-1}]$.

Lemma 4.5. *For each k -scheme S and $f \in \underline{k((z))}^*(S)$, the function:*

$$\begin{aligned} S &\rightarrow \mathbb{Z} \\ s &\mapsto v_s(f) = \text{order of } f_s \in k(s)((z)) \end{aligned}$$

is locally constant.

Proof. We can assume that $S = \text{Spec}(A)$, A being a k -algebra. Let $f = \sum_{i \geq n} a_i z^i$ be an element of $A((z))^*$ ($n \in \mathbb{Z}$). There then exists another element $g = \sum_{i \geq -m} b_i z^i$ ($m \in \mathbb{Z}$) such that $f \cdot g = 1$. This implies the following relations (from now on we assume $n = 0$ to simplify the calculations):

$$\begin{aligned} 0 &= b_{-m} a_0 \\ 0 &= b_{-m} a_1 + b_{-m+1} a_0 \\ &\dots \\ 0 &= b_{-m} a_{m-1} + \dots + b_{-1} a_0 \\ 1 &= b_{-m} a_m + \dots + b_0 a_0 \end{aligned} \tag{4.6}$$

Let us distinguish two cases:

a) b_{-m} is not nilpotent in A : from 4.6 we obtain:

$$b_{-m} a_0 = b_{-m}^2 a_1 = \dots = b_{-m}^m a_{m-1} = 0$$

That is, a_0, \dots, a_{m-1} are equal zero in the ring $A_{(b_{-m})}$ and for each $s \in \text{Spec}(A) - (b_{-m})_0$ one has $b_{-m}(s) a_m(s) = 1$ and therefore $v_s(f) = m$.

We conclude by proving that in this case $(b_{-m})_0$ is also an open subset of $\text{Spec}(A)$:

From the equations 4.6 we deduce:

$$\begin{aligned} (b_{-m}, a_{m-1}, \dots, a_0)_0 &= (b_{-m})_0 \cap (a_{m-1})_0 \cap \dots \cap (a_0)_0 = \emptyset \\ (b_{-m})_0 \cup (a_i)_0 &= \text{Spec}(A) \quad , \quad i = 0, \dots, n-1 \end{aligned}$$

and hence:

$$(b_{-m})_0 \cup \left(\bigcap_{i=0}^{n-1} (a_i)_0 \right) = \text{Spec}(A)$$

b) Let us assume that b_{-m}, \dots, b_{-r-1} are nilpotent elements of A and that b_{-r} is not nilpotent. The same argument as in case a) proves that $v_s(f)$ is constant in the closed subscheme $(b_{-r})_0$ and that its complementary in $\text{Spec}(A)$ is $\bigcap_{i=0}^{r-1} (a_i)_0$, from which we conclude the proof. \square

Corollary 4.7. *For an affine irreducible k -scheme $S = \text{Spec}(A)$ one has that:*

1. v_s is a constant function over S ,
- 2.

$$\{f \in A((z))^* \mid v(f) = n\} = \left\{ \begin{array}{l} \text{series } a_{n-r} z^r + \dots + a_n z^n + \dots \text{ such that} \\ a_{n-r}, \dots, a_{n-1} \text{ are nilpotent and } a_n \in A^* \end{array} \right\}$$

3. If A is also a reduced k -algebra:

$$A((z))^* = \coprod_{n \in \mathbb{Z}} \left\{ \sum_{i \geq n} a_i z^i \mid a_i \in A \text{ y } a_n \in A^* \right\}$$

Proof. This is obvious from lemma 4.5. \square

Theorem 4.8. *The subfunctor $\underline{k((z))}_{red}^*$ of $\underline{k((z))}^*$ defined by:*

$$S \rightsquigarrow \underline{k((z))}_{red}^*(S) = \coprod_{n \in \mathbb{Z}} \left\{ z^n + \sum_{i > n} a_i z^i \mid a_i \in H^0(S, \mathcal{O}_S) \right\}$$

is representable by a group k -scheme whose connected component of the origin will be denote by Γ_+ .

Proof. It suffices to observe that the functor:

$$S \rightsquigarrow \left\{ z^n + \sum_{i > n} a_i z^i \mid a_i \in H^0(S, \mathcal{O}_S) \right\}$$

is representable by the scheme:

$$\text{Spec}(\varinjlim_l k[x_1, \dots, x_l]) = \varprojlim_l \mathbb{A}_k^l$$

and the group law is given by the multiplication of series. \square

Theorem 4.9. Let $\underline{k((z))}_{\text{nil}}^*$ be the subfunctor of $k((z))^*$ defined by:

$$S \rightsquigarrow \underline{k((z))}_{\text{nil}}^*(S) = \coprod_{n>0} \left\{ \begin{array}{l} \text{finite series } a_n z^{-n} + \cdots + a_1 z^{-1} + 1 \text{ such that} \\ a_i \in H^0(S, \mathcal{O}_S) \text{ are nilpotent and } n \text{ arbitrary} \end{array} \right\}$$

There exists a formal k -scheme Γ_- representing $\underline{k((z))}_{\text{nil}}^*$, that is:

$$\text{Hom}_{\text{for-sch}}(S, \Gamma_-) = \underline{k((z))}_{\text{nil}}^*(S)$$

for every k -scheme S .

Proof. Let us define the ring of “infinite” formal series in infinite variables (which is different from the ring of formal series in infinite variables) by:

$$k\{\{x_1, \dots\}\} = \varprojlim_n k[[x_1, \dots, x_n]]$$

the morphisms of the projective system being:

$$\begin{aligned} k[[x_1, \dots, x_{n+1}]] &\rightarrow k[[x_1, \dots, x_n]] \\ x_i &\mapsto x_i \quad \text{for } i = 1, \dots, n-1 \\ x_{n+1} &\mapsto 0 \end{aligned}$$

Note that:

$$k\{\{x_1, \dots\}\} = \varprojlim_n k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n$$

It is therefore an admissible linearly topological ring ([EGA] 0.7.1) and there therefore exists its formal spectrum $\text{Spf}(k\{\{x_1, \dots\}\})$. Let us denote by J_n the kernel of the natural projection $k\{\{x_1, \dots\}\} \rightarrow k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n$ and $J = \varprojlim (x_1, \dots, x_n)$.

Let us now prove that $\Gamma_- = \text{Spf}(k\{\{x_1, \dots\}\})$:
For every k -scheme S , considering over $H^0(S, \mathcal{O}_S)$ the discrete topology, we have:

$$\begin{aligned} \text{Hom}_{\text{for-sch}}(S, \Gamma_-) &= \text{Hom}_{\text{cont-}k\text{-alg}}(k\{\{x_1, \dots\}\}, H^0(S, \mathcal{O}_S)) = \\ &= \left\{ f \in \text{Hom}_{k\text{-alg}}(k\{\{x_1, \dots\}\}, H^0(S, \mathcal{O}_S)) \text{ such that} \right. \\ &\quad \left. \text{there exists } n \in \mathbb{N} \text{ satisfying } J_n \subseteq f^{-1}((0)) \right\} \end{aligned}$$

However the condition $J_n \subseteq f^{-1}((0))$ is equivalent to saying that $f(x_1), \dots, f(x_n)$ are nilpotent and $f(x_i) = 0$ for $i > n$, from which one concludes the proof. \square

Remark 10. Note that Γ_- is the inductive limit in the category of formal schemes ([EGA] I.10.6.3) of the schemes which represent the subfunctors:

$$S \rightsquigarrow \Gamma_-^n(S) = \left\{ \begin{array}{l} a_n z^{-n} + \cdots + a_1 z^{-1} + 1 \text{ such that} \\ a_i \in H^0(S, \mathcal{O}_S) \text{ and the } n^{\text{th}} \text{ power} \\ \text{of the ideal } (a_1, \dots, a_n) \text{ is zero} \end{array} \right\}$$

Remark 11. Group laws of Γ_+ and Γ_- The group law of $\Gamma_+ = \text{Spec}(k[x_1, \dots])$ is given by:

$$\begin{aligned} k[x_1, \dots] &\rightarrow k[x_1, \dots] \otimes_k k[x_1, \dots] \\ x_i &\mapsto x_i \otimes 1 + \sum_{j+k=i} x_j \otimes x_k + 1 \otimes x_i \end{aligned}$$

The group law of $\Gamma_- = \text{Spf } k\{\{x_1, \dots\}\}$ is given by:

$$\begin{aligned} k\{\{x_1, \dots\}\} &\rightarrow k\{\{x_1, \dots\}\} \hat{\otimes}_k k\{\{x_1, \dots\}\} \\ x_i &\mapsto x_i \otimes 1 + \sum_{j+k=i} x_j \otimes x_k + 1 \otimes x_i \end{aligned}$$

Let be $\underline{k((z))}_0^*$ be the connected component of the origin in the functor of groups $\underline{k((z))}^*$.

Theorem 4.10. *The natural morphism of functors of groups over the category of k -schemes:*

$$\underline{\Gamma_-} \times \underline{\mathbb{G}_m} \times \underline{\Gamma_+} \rightarrow \underline{k((z))}_0^*$$

is injective and for $\text{char}(k) = 0$ gives an isomorphism with $\underline{k((z))}_0^$. $\underline{k((z))}_0^*$ is therefore representable by the (formal) k -scheme:*

$$\Gamma = \Gamma_- \times \mathbb{G}_m \times \Gamma_+$$

Proof. The morphism from $\underline{\mathbb{G}_m}$ to $\underline{k((z))}_0^*$ is the one induced by the natural inclusion $H^0(S, \mathcal{O}_S)^* \hookrightarrow H^0(S, \mathcal{O}_S)((z))^*$.

The injectivity of $\underline{\Gamma} \hookrightarrow \underline{k((z))}_0^*$ follows from the fact that $\Gamma_- \cap \Gamma_+ = \{1\}$. The rest of the proof is trivial from the above results and from the properties of the exponential map we shall see below. \square

Remark 12. Our group scheme Γ is the algebraic analogue of the group Γ of Segal-Wilson [SW]. Note that the indexes “-” and “+” do not coincide with the Segal-Wilson notations. Replacing $k((z))$, by $k((z^{-1}))$ we obtain the same notation as in the paper of Segal-Wilson.

Let us define the exponential maps for the groups Γ_- and Γ_+ . Let \mathbb{A}_n be the n dimensional affine space over $\text{Spec}(k)$ with the additive group law, and $\hat{\mathbb{A}}_n$ the formal group obtained as the completion of \mathbb{A}_n at the origin. We define $\hat{\mathbb{A}}_\infty$ as the formal group $\varinjlim_n \hat{\mathbb{A}}_n$. Obviously $\hat{\mathbb{A}}_\infty$ is the formal scheme:

$$\hat{\mathbb{A}}_\infty = \text{Spf } k\{\{y_1, \dots\}\}$$

with group law:

$$\begin{aligned} k\{\{y_1, \dots\}\} &\rightarrow k\{\{y_1, \dots\}\} \hat{\otimes}_k k\{\{y_1, \dots\}\} \\ y_i &\longmapsto y_i \otimes 1 + 1 \otimes y_i \end{aligned}$$

Definition 4.11. *If the characteristic of k is zero, the exponential map for Γ_- is the following isomorphism of formal group schemes:*

$$\begin{aligned} \hat{\mathbb{A}}_\infty &\xrightarrow{\exp} \Gamma_- \\ \{a_i\}_{i>0} &\mapsto \exp\left(\sum_{i>0} a_i z^{-i}\right) \end{aligned}$$

This is the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1, \dots\}\} &\xrightarrow{\exp^*} k\{\{y_1, \dots\}\} \\ x_i &\mapsto \text{coefficient of } z^{-i} \text{ in the series } \exp\left(\sum_{j>0} y_j z^{-j}\right) \end{aligned}$$

Definition 4.12. *If the characteristic of k is $p > 0$, the exponential map for Γ_- is the following isomorphism of formal schemes:*

$$\begin{aligned} \hat{\mathbb{A}}_\infty &\rightarrow \Gamma_- \\ \{a_i\}_{i>0} &\mapsto \prod_{i>0} (1 - a_i z^{-i}) \end{aligned}$$

which is the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1, \dots\}\} &\xrightarrow{\exp^*} k\{\{y_1, \dots\}\} \\ x_i &\mapsto \text{coefficient of } z^{-i} \text{ in the series } \prod_{i>0} (1 - a_i z^{-i}) \end{aligned}$$

Note that this latter exponential map is not a isomorphism of groups. Considering over $\hat{\mathbb{A}}_\infty$ the law group induced by the isomorphism, \exp , of formal schemes, we obtain the Witt formal group law.

Analogously, we define the exponential maps for the group Γ_+ :

Definition 4.13. Let \mathbb{A}^∞ be the group scheme over k defined by $\varprojlim \mathbb{A}_n$ (where $\mathbb{A}_{n+1} = \operatorname{Spec} k[x_1, \dots, x_{n+1}] \rightarrow \mathbb{A}_n = \operatorname{Spec} k[x_1, \dots, x_n]$ is the morphism defined by forgetting the last coordinate) with its additive group law. The exponential map when $\operatorname{char}(k) = 0$ is the isomorphism of group schemes:

$$\begin{aligned} \mathbb{A}^\infty &\rightarrow \Gamma_+ \\ \{a_i\}_{i>0} &\mapsto \exp\left(\sum_{i>0} a_i z^i\right) \end{aligned}$$

If $\operatorname{char}(k) = p \neq 0$, the exponential map is the isomorphism of schemes:

$$\begin{aligned} \mathbb{A}^\infty &\rightarrow \Gamma_+ \\ \{a_i\}_{i>0} &\mapsto \prod_{i>0} (1 - a_i z^i) \end{aligned}$$

which is not a morphism of groups.

(See [B] for the connection of these definitions and the Cartier-Dieudonné theory).

It should be noted that the formal group scheme Γ_- has properties formally analogous to the Jacobians of the algebraic curves: one can define formal Abel maps and prove formal analogues of the Albanese property of the Jacobians of smooth curves (see [KSU, C]).

Let $\hat{C} = \operatorname{Spf}(k[[t]])$ be a formal curve. We define the Abel morphism of degree 1 as the morphism of formal schemes:

$$\phi_1 : \hat{C} \rightarrow \Gamma_-$$

given by $\phi_1(t) = (1 - \frac{t}{z})^{-1} = 1 + \sum_{i>0} \frac{t^i}{z^i}$; that is, the morphism induced by the ring homomorphism:

$$\begin{aligned} k\{\{x_1, \dots\}\} &\rightarrow k[[t]] \\ x_i &\mapsto t^i \end{aligned}$$

Note that the Abel morphism is the algebro-geometric version of the function $q_\xi(z)$ used by Segal and Wilson ([SW] page 32) to study the Baker function.

Let us explain further why we call ϕ_1 the “Abel morphism” of degree 1. If $\operatorname{char}(k) = 0$, composing ϕ_1 with the inverse of the exponential map, we have:

$$\bar{\phi}_1 : \hat{C} \xrightarrow{\phi_1} \Gamma_- \xrightarrow[\exp^{-1}]{} \hat{\mathbb{A}}_\infty$$

and since $(1 - \frac{t}{z})^{-1} = \exp(\sum_{i>0} \frac{t^i}{iz^i})$ (see [SW] page 33), $\bar{\phi}_1$ is the morphism defined by the ring homomorphism:

$$\begin{aligned} k\{\{y_1, \dots\}\} &\rightarrow k[[t]] \\ y_i &\mapsto \frac{t^i}{i} \end{aligned}$$

or in terms of the functor of points:

$$\begin{aligned} \hat{C} &\xrightarrow{\bar{\phi}_1} \hat{\mathbb{A}}_\infty \\ t &\mapsto \{t, \frac{t^2}{2}, \frac{t^3}{3}, \dots\} \end{aligned}$$

Observe that given the basis $\omega_i = t^i dt$ of the differentials $\Omega_{\hat{C}} = k[[t]]dt$, $\bar{\phi}_1$ can be interpreted as the morphism defined by the “abelian integrals” over the formal curve:

$$\bar{\phi}_1(t) = \left(\int \omega_0, \int \omega_1, \dots, \int \omega_i, \dots \right)$$

which coincides precisely with the local equations of the Abel morphism for smooth algebraic curves over the field of complex numbers. In general, for each integer number $n > 0$, we define the Abel morphism of degree n as the morphism of formal schemes:

$$\bar{\phi}_n : \hat{C} \times \hat{C} \times \dots \times \hat{C} = \hat{C}^n \rightarrow \Gamma_-$$

given by $\bar{\phi}_n(t_1, \dots, t_n) = \prod_{i=1}^n (1 - \frac{t_i}{z})^{-1}$; that is, the morphism induced by the ring homomorphism:

$$k\{\{x_1, \dots\}\} \rightarrow k[[t_1]] \hat{\otimes} \dots \hat{\otimes} k[[t_n]]$$

$$x_i \mapsto \text{coefficient of } z^{-i} \text{ in the series } \prod_{i=1}^n (1 - \frac{t_i}{z})^{-1}$$

Note that $\bar{\phi}_n$ factorizes through a morphism, ϕ_n from the n^{th} -symmetric product of \hat{C} to Γ_- , which is the true Abel morphism; moreover ϕ_n is an immersion.

Theorem 4.14. *(Γ_-, ϕ_1) satisfies the Albanese property for \hat{C} ; that is, every morphism $\psi : \hat{C} \rightarrow X$ in a commutative group scheme (which sends the unique rational point of \hat{C} to the $0 \in X$) factors through the Abel morphism and a homomorphism of groups $\Gamma_- \rightarrow X$.*

Proof. Let $\psi : \hat{C} \rightarrow X$ be a morphism from the formal scheme \hat{C} to a group scheme X such that $\psi(\text{rational point}) = 0$. For each $n > 0$, one constructs a morphism:

$$\hat{C}^n \xrightarrow{\bar{\psi}_n} X$$

which is the composition of $\psi \times \cdots \times \psi : \hat{C} \times \cdot^n \times \hat{C} \rightarrow X \times \cdot^n \times X$ and the addition morphism $X \times \cdot^n \times X \rightarrow X$. Observe that $\bar{\psi}_n$ factors through a morphism:

$$S^n \hat{C} \xrightarrow{\psi_n} X$$

and bearing in mind that $\Gamma_- = \varinjlim_n S^n \hat{C}$ (as formal group schemes)

we conclude the proof of the existence of a homomorphism of groups $\bar{\psi} : \Gamma_- \rightarrow X$ satisfying the desired condition. \square

5. τ -FUNCTIONS AND BAKER FUNCTIONS

This section is devoted to algebraically defining the τ -functions and the Baker functions over an arbitrary base field k .

Following on with the analogy between the groups Γ and Γ_- and the Jacobian of the smooth algebraic curves, we shall make the well known constructions for the jacobians of the algebraic curves for the formal curve \hat{C} and the group Γ : Poincaré bundle over the dual jacobian and the universal line bundle over the jacobian. In the formal case these constructions are essentially equivalent to defining the τ -functions and the Baker functions.

Using the notations of section 4, let us consider the Grassmannian $\text{Gr}(V)$ of $V = k((z))$ and the group

$$\Gamma = \Gamma_- \times \mathbb{G}_m \times \Gamma_+$$

acting on $\text{Gr}(V)$ by homotheties.

As we have shown in 9, there exists a central extension of Γ :

$$0 \rightarrow \mathbb{G}_m \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 0$$

given by a pairing:

$$\Gamma \times \Gamma \rightarrow \mathbb{G}_m \tag{5.1}$$

Proposition 5.2. *The extension $\tilde{\Gamma}_+$ of Γ_+ is trivial.*

Proof. We will construct a section s (as groups) of $\tilde{\Gamma}_+ \rightarrow \Gamma_+$; that is, for an element $g \in \Gamma_+$ we give $s(g) \in \tilde{\Gamma}_+$ such that s is a morphism of groups.

Denote by $\mu : \Gamma \times \text{Gr}(V) \rightarrow \text{Gr}(V)$ the action of Γ on $\text{Gr}(V)$ and by μ_g the automorphism of $\text{Gr}(V)$ induced by the homothety $\cdot g : V \rightarrow V$ for $g \in \Gamma$.

Fix $g \in \Gamma_+$. Observe that there exists a quasi-isomorphism of complexes:

$$\begin{array}{ccc} \mathcal{L} & \longrightarrow & V/V^+ \\ \cdot g \downarrow & & > \downarrow \cdot g \\ \mu_g^*(\mathcal{L}) & \longrightarrow & V/V^+ \end{array}$$

since $g \cdot V^+ \simeq V^+$. We have thus an isomorphism $\text{Det}_V^* \simeq \mu_g^* \text{Det}_V^*$ in a canonical way, and hence a well-defined element $s(g) \in \widetilde{\Gamma}_+$. Since this construction is canonical and $\mu_{g'} \circ \mu_g = \mu_{g' \cdot g}$ it follows easily that $s(g') \cdot s(g) = s(g' \cdot g)$. \square

Proposition 5.3. *For a rational point $U \in \text{Gr}(V)$, let μ_U be the morphism $\Gamma \times \{U\} \rightarrow \text{Gr}(V)$ induced by μ . Then, the line bundle $\mu_U^* \text{Det}_V^*|_{\Gamma_-}$ is trivial, and the extension $\widetilde{\Gamma}_-$ is thus trivial.*

Proof. Assume $U \in F_{V^+}$ (the general case is analogous). It is no difficult to obtain the following equality for $g \in \Gamma_-$:

$$(\mu_U^* \Omega_+)(g) = \Omega_+(g \cdot U) = \Omega_+(U) + \sum_S \chi_S(g) \cdot \Omega_S(U)$$

where the sum is taken over the set of Young diagrams and χ_S is the Schur polynomial (in the coefficients of g) corresponding to S . Since $\Omega_+(U) \neq 0$ and the coefficients of g are nilpotents, it follows that $\mu_U^* \Omega_+$ is a no-where vanishing section of $\mu_U^* \text{Det}_V^*$, and this bundle is therefore trivial.

Observe now that since $\widetilde{\Gamma}_-$ can be thought as the sheaf of automorphisms of $\mu_U^* \text{Det}_V^*$ one has that $\widetilde{\Gamma}_-$ is a trivial extension. \square

Corollary 5.4. *The restrictions of the pairing 5.1 to the subgroups Γ_- and Γ_+ are trivial.*

We define the Poincaré bundle over $\Gamma \times \text{Gr}(V)$ as the invertible sheaf:

$$\mathfrak{P} = \mu^* \text{Det}_V^*$$

For each point $U \in \text{Gr}(V)$, let us define the Poincaré bundle over $\Gamma \times \Gamma$ associated with U by:

$$\mathfrak{P}_U = (1 \times \mu_U)^* \mathfrak{P} = m^*(\mu_U^* \text{Det}_V^*)$$

where $m : \Gamma \times \Gamma \rightarrow \Gamma$ is the group law.

The sheaf of τ -functions of a point $U \in \text{Gr}(V)$, $\widetilde{\mathcal{L}}_\tau(U)$, is the invertible sheaf over $\Gamma \times \{U\}$ defined by:

$$\widetilde{\mathcal{L}}_\tau(U) = \mathfrak{P}|_{\Gamma \times \{U\}}$$

Let us note that the sheaf $\widetilde{\mathcal{L}}_\tau(U)$ is defined for arbitrary points of the Grassmannian and not only for geometric points.

The restriction homomorphism induces the following homomorphism between global sections:

$$H^0(\Gamma \times \text{Gr}(V), \mu^* \text{Det}_V^*) \rightarrow H^0(\Gamma \times \{U\}, \widetilde{\mathcal{L}}_\tau(U)) \quad (5.5)$$

Definition 5.6. *The τ -function of the point U over Γ is defined as the image $\widetilde{\tau}_U$ of the section $\mu^* \Omega_+$ by the homomorphism 5.5 (Ω_+ being the global section defined in 3.14).*

Obviously $\widetilde{\tau}_U$ is not a function over $\Gamma \times \{U\}$ since the invertible sheaf $\widetilde{\mathcal{L}}_\tau(U)$ is not trivial.

The algebraic analogue of the τ -function defined by M. and Y. Sato, Segal and Wilson ([SS], [SW]) is obtained by restricting the invertible sheaf $\widetilde{\mathcal{L}}_\tau(U)$ to the formal subgroup $\Gamma_- \subset \Gamma$.

To see this, fix a rational point $U \in \text{Gr}(V)$ and define:

$$\mathcal{L}_\tau(U) = \widetilde{\mathcal{L}}_\tau(U)|_{\Gamma_- \times \{U\}}$$

which is a trivial invertible sheaf over Γ_- . To obtain a trivialization of $\mathcal{L}_\tau(U)$ which will allow us to identify global sections with functions over Γ_- we must fix a global section of $\mathcal{L}_\tau(U)$ without zeroes in Γ_- .

Recall that $\widetilde{\Gamma}_-$ is a trivial extension of Γ_- and it has therefore a section s . It follows that the group Γ_- acts on $\mathcal{L}_\tau(U)$ (through s) and on Γ_- by translations. One has easily that the morphism:

$$\mathbb{V}(\mathcal{L}_\tau(U)^*) \rightarrow \Gamma_-$$

is equivariant with respect to these actions.

Note now that:

$$\text{Hom}_{\Gamma_- \text{-equiv}}(\Gamma_-, \mathbb{V}(\mathcal{L}_\tau(U)^*)) \subseteq \text{Hom}_{\Gamma_- \text{-esq}}(\Gamma_-, \mathbb{V}(\mathcal{L}_\tau(U)^*)) = H^0(\Gamma_-, \mathcal{L}_\tau(U))$$

Let δ be a non-zero element in the fibre of $\mathbb{V}(\mathcal{L}_\tau(U)^*)$ over the point 1 of Γ_- (1 being the identity of Γ_-). Let σ_0 be the unique morphism $\Gamma_- \rightarrow \mathbb{V}(\mathcal{L}_\tau(U)^*)$ Γ_- -equivariant such that $\sigma_0(1) = \delta$, and denote again by σ_0 the corresponding section of $\mathcal{L}_\tau(U)$.

Observe that σ_0 is a constant section and since it has no zeros it gives a trivialization of $\mathcal{L}_\tau(U)$. Through this trivialization, the global section of $\mathcal{L}_\tau(U)$ defined by $\widetilde{\tau}_U$ is identified with the function $\tau_U \in \mathcal{O}(\Gamma_-) = k\{\{x_1, \dots\}\}$ given by Segal-Wilson [SW]:

$$\tau_U(g) = \frac{\widetilde{\tau}_U(g)}{\sigma_0(g)} = \frac{\mu^* \Omega_+(g)}{\sigma_0(g)} = \frac{\Omega_+(gU)}{\delta}$$

Finally, if $U \in F_{V+}$ then one can choose $\delta = \Omega_+(U)$.

Observe that the τ -function τ_U is not a series of infinite variables but an element of the ring $k\{\{x_1, \dots\}\}$.

The subgroup Γ_+ of Γ acts freely over $\text{Gr}(V)$. Accordingly the orbits of the rational points of $\text{Gr}(V)$ under the action of Γ_+ are isomorphic, as schemes, to Γ_+ .

Let X be the orbit of $V^- = z^{-1} \cdot k[z^{-1}] \subset V$ under Γ_+ . The restrictions of Det_V and Det_V^* to X are trivial invertible sheaves. Bearing in mind that the points of X are k -vector subspaces of V whose intersection with V^+ is zero, one has that the section Ω_+ of Det_V^* defines a canonical trivialization of Det_V^* over X .

Theorem 5.7. *The restriction homomorphism $\text{Det}_V^* \rightarrow \text{Det}_V^*|_X$ induces a homomorphism between global sections:*

$$B : H^0(\text{Gr}(V), \text{Det}_V^*) \longrightarrow H^0(X, \text{Det}_V^*|_X) \simeq \mathcal{O}(\Gamma_+) = k[x_1, \dots]$$

which is an isomorphism between the k -vector subspace $\Omega(S)$ defined in 3.14 and $\mathcal{O}(\Gamma_+)$. The isomorphism $H^0(X, \text{Det}_V^|_X) \xrightarrow{\sim} \mathcal{O}(\Gamma_+)$ is the isomorphism induced by the trivialization defined by Ω_+ .*

In the literature, the isomorphism $B : \Omega(S) \xrightarrow{\sim} \mathcal{O}(\Gamma_+)$ is usually called the bosonization isomorphism.

Proof. All one has to prove is that $B(\Omega_S) = F_S(x)$, Ω_S being the Plücker sections of Det_V^* defined in 3.14 and $F_S(x_1, x_2, \dots)$ being the Schur functions. Proof of the identity $B(\Omega_S) = F_S(x)$ is essentially the same as in the complex analytic case; see [SW] and [PS].

In some of the literature, the τ function of a point $U \in \text{Gr}(V)$ is defined as the Plücker coordinates of the point U . Let us therefore explain in which sense both definitions are equivalent.

The canonical homomorphism:

$$H^0(\text{Det}_V^*) \otimes \mathcal{O}_{\text{Gr}(V)} \longrightarrow \text{Det}_V^* \rightarrow 0$$

induces a homomorphism:

$$\text{Det}_V = \text{Det}_V^{**} \xrightarrow{\bar{\tau}} H^0(\text{Det}_V^*)^* \otimes \mathcal{O}_{\text{Gr}(V)}$$

□

Definition 5.8. *Given a point $\tilde{U} \in \text{Det}_V$ in the fibre of $U \in \text{Gr}(V)$, the $\bar{\tau}$ -function of \tilde{U} is defined as the element $\bar{\tau}(U) \in H^0(\text{Det}_V^*)^* \otimes k(U)$ ($k(U)$ being the residual field of U). This is essentially the definition of τ -functions given in the papers of M. and Y. Sato, Arbarello and De Concini, and Kawamoto and others ([SS], [AD], [KNTY]).*

Lemma 5.9. *There exists an isomorphism of k -vector spaces:*

$$\mathcal{O}(\Gamma_+)^* \rightarrow \mathcal{O}(\Gamma_-)$$

Proof. Recall that $\mathcal{O}(\Gamma_+) = k[x_1, \dots]$ and that $\mathcal{O}(\Gamma_-) = k\{\{x_1, x_2, \dots\}\}$. Now think that x_i is the i -symmetric function of other variables, say t_1, t_2, \dots . It is known that the Schur polynomials $\{F_S\}$ (where S is a partition) of the t 's are polynomials in the x 's and are in fact a basis of the k -vector space $k[x_1, \dots]$. The isomorphism is the induced by the pairing:

$$\begin{aligned} \mathcal{O}(\Gamma_+) \times \mathcal{O}(\Gamma_-) &\longrightarrow k \\ (F_S, F_{S'}) &\longmapsto \delta_{S,S'} \end{aligned}$$

(see [Mc]). □

The composition of the homomorphism B^* (the dual homomorphism of B) and the isomorphism of the above lemma gives an homomorphism:

$$\tilde{B}^* : \mathcal{O}(\Gamma_+)^* = k\{\{x_1, x_2, \dots\}\} \longrightarrow H^0(\text{Det}_V^*)^*$$

The connection between τ_U and $\bar{\tau}(\tilde{U})$ is the following:

$$\tilde{B}^*(\tau_U) = \lambda \cdot (\bar{\tau}(\tilde{U}))$$

λ being a non-zero constant. (Of course, if U is not rational but a point with values in a scheme S , $\lambda \in H^0(S, \mathcal{O}_S^*)$).

The connection of the τ -functions with autoduality (in the sense of group schemes) properties of the group $\Gamma = \Gamma_- \times \mathbb{G}_m \times \Gamma_+$ implicit in the above discussion, is studied with detail in [C,P]. L. Breen in [B2] outlines also some of these properties from another point of view.

Once we have algebraically defined the τ -functions, we can define the Baker functions using formula 5.14. of [SW]; this is the procedure used by several authors. However, we prefer to continue with the analogy with the classical theory of curves and jacobians and define the Baker functions as a formal analogue of the universal invertible sheaf of the Jacobian.

Let us consider the composition of morphisms:

$$\tilde{\beta} : \hat{C} \times \Gamma \times \text{Gr}(V) \xrightarrow{\phi \times Id} \Gamma \times \Gamma \times \text{Gr}(V) \xrightarrow{m \times Id} \Gamma \times \text{Gr}(V)$$

$\phi : \hat{C} = \text{Spf } k[[z]] \rightarrow \Gamma$ being the Abel morphism (taking values in $\Gamma_- \subset \Gamma$) and $m : \Gamma \times \Gamma \rightarrow \Gamma$ the group law.

Definition 5.10. *The sheaf of Baker-Akhiezer functions is the invertible sheaf over $\hat{C} \times \Gamma \times \text{Gr}(V)$ defined by:*

$$\tilde{\mathcal{L}}_B = (\phi \times Id)^*(m \times Id)^*\mathfrak{P}$$

Let us define the sheaf of Baker functions at a point $U \in \text{Gr}(V)$ as the invertible sheaf:

$$\widetilde{\mathcal{L}}_B(U) = \widetilde{\mathcal{L}}_B|_{\hat{C} \times \Gamma \times \{U\}} = \widetilde{\beta}_U^* \widetilde{\mathcal{L}}_\tau(U)$$

(where $\widetilde{\beta}_U^*$ is the following homomorphism between global sections:

$$H^0(\Gamma \times \{U\}, \widetilde{\mathcal{L}}_\tau(U)) \xrightarrow{\widetilde{\beta}_U^*} H^0(\hat{C} \times \Gamma \times \{U\}, \widetilde{\mathcal{L}}_B(U))$$

By the definitions, $\widetilde{\mathcal{L}}_B(U)|_{\hat{C} \times \Gamma_- \times \{U\}} = \mathcal{L}_B(U)$ is a trivial invertible sheaf over $\hat{C} \times \Gamma_-$.

Observe that for each element $u \in \Gamma_-(S) \subseteq \underline{k((z))}^*(S) = H^0(S, \mathcal{O}_S)((z))^*$ we can define a fractionary ideal of the formal curve \widehat{C}_S by:

$$I_u = u \cdot \mathcal{O}_S((z))$$

in such a way that we can interpret the formal group Γ_- as a kind of Picard scheme over the formal curve. The universal element of Γ_- is the invertible element of $\underline{k((z))}^*(\Gamma_-)$ given by:

$$v = 1 + \sum_{i \geq 1} x_i z^{-i} \in k((z)) \hat{\otimes} k\{\{x_1, x_2, \dots\}\}$$

This universal element will be the formal analogue of the universal invertible sheaf for the formal curve \hat{C} .

Definition 5.11. *The Baker function of a point $U \in \text{Gr}(V)$ is $\psi_U = v^{-1} \cdot \beta_U^*(\widetilde{\tau}_U)$, where*

$$\beta_U^* : H^0(\Gamma \times \{U\}, \widetilde{\mathcal{L}}_\tau(U)) \longrightarrow H^0(\hat{C} \times \Gamma \times \{U\}, \widetilde{\mathcal{L}}_B(U))$$

is the homomorphism induced by $\widetilde{\beta}_U^*$.

Observe that the Baker function of $V^- = z^{-1} k[z^{-1}]$ is the universal invertible element v^{-1} .

Note that, analogously to the case of τ -function, we can choose a trivialization of $\widetilde{\mathcal{L}}_B(U)$ over $\hat{C} \times \Gamma_- \times \{U\}$ in such a way that the function asociated to the section $v^{-1} \cdot \beta_U^*(\tau_U)$ is:

$$\psi_U(z, g) = v^{-1} \cdot \frac{\tau_U(g \cdot \phi_1)}{\tau_U(g)} \quad (5.12)$$

which is the classical expression for the Baker function.

When the characteristic of the base field k is zero, we can identify Γ_- with the additive group scheme $\hat{\mathbb{A}}_\infty$ through the exponential

and expression 5.12 is the classical expression for the Baker functions ([SW] 5.16):

$$\psi_U(z, t) = \left(\frac{\tau_U(t + [z])}{\tau_U(t)} \right) \cdot \exp\left(-\sum t_i z^{-i}\right)$$

where $[z] = (z, \frac{1}{2}z^2, \frac{1}{3}z^3, \dots)$ and $t = (t_1, t_2, \dots)$ and $v = \exp(\sum t_i z^{-i})$ through the exponential map.

For the general case, we obtain explicit expressions for ψ_U as a function over $\hat{C} \times \hat{\mathbb{A}}_\infty$ but considering in $\hat{\mathbb{A}}_\infty$ the group law induced by the exponential 4.12:

$$\psi_U(z, g) = v(z, g)^{-1} \cdot \frac{\tau_U(t * \phi(z))}{\tau_U(t)}$$

(* being the group law of $\hat{\mathbb{A}}_\infty$).

The classical properties characterizing the Baker functions (for example proposition 5.1 of [SW]) can be immediately generalized for the Baker functions over arbitrary fields.

Remark 13. Note that our definitions of τ -functions and Baker functions are valid over arbitrary base fields and that can be generalized for \mathbb{Z} . One then has the notion of τ -function and Baker functions for families of elements of $\text{Gr}(V)$ and, if we consider the Grassmannian of $\mathbb{Z}((z))$ one then has τ -functions and Baker functions of the rational points of $\text{Gr}(\mathbb{Z}((z)))$ and the geometric properties studied in this paper have a translation into arithmetic properties of the elements of $\text{Gr}(\mathbb{Z}((z)))$. The results stated by Anderson in [A] are a particular case of a much more general setting valid not only for p -adic fields but also for arbitrary global field numbers. Our future aims are to study the arithmetic properties related to these constructions.

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