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Addition formulae for non-Abelian theta functions and applications

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Abstract

This paper generalizes for non-Abelian theta functions a number of formulae valid for theta functions of Jacobian varieties. The addition formula, the relation with the Szëgo kernel and with the multicomponent KP hierarchy and the behavior under cyclic coverings are given.

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1. Introduction

Fay's addition formula for theta functions of Jacobians [4] has turned out to be highly relevant in a number of problems: geometric properties of Jacobians (existence of trisecants to their Kummer varieties), infinitesimal behavior of theta functions of Jacobians (KP and KdV equations) and algebraic formulations of certain aspects of conformal field theories. On the other hand, deep relations between moduli spaces of vector bundles and Jacobians varieties [2,11] has been already established. Therefore, it is thus natural to expect similarities between the properties of classical theta functions and those of non-Abelian theta functions, in particular, an analogue of Fay's addition formula.

In fact, the existence of generalized addition formulae for non-Abelian theta functions has been conjectured by Schork (conjecture IV.9 of [18]) when generalizing for higher rank

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Raina's approach to b–c systems [16]. Therefore, this kind of formulae should be useful tools when studying Schork's correlation functions as well as non-Abelian generalizations of multiplicative Ward identities given by Witten [20]. In fact, the case of line bundles has been already worked out completely by Raina [15,16]. Further, in this direction, the relations between theta functions and the Szegö kernel are well known (e.g. Theorem 25 of [7] or Section 6 of [15]) and have been useful in many problems (e.g. [16]).

Moreover, an infinitesimal version of such a formula has been given in [13] when proving that non-Abelian theta functions verify the multicomponent KP hierarchy. Hence, an addition formula may help in the understanding of this result and of its geometric consequences (see [12] for the rank 1 case).

On the other hand, the study of how Jacobian theta functions vary under morphisms of curves has shed light on their properties (e.g. chapters IV and V of [4]). This question is related to the so-called twist structures of b–c systems [16] and is also addressed in p. 844 of [1] for higher rank.

The above problems are treated in this paper as follows. A generalization of the addition formula for non-Abelian theta functions is the main result of Section 3 (Theorem 3.8) which coincides with Corollary 2.19 of [4] in the case of line bundles. This formula will be derived as an identity among global sections of certain isomorphic line bundles. It is worth mentioning some results needed for its proof, Theorems 3.3 and 3.6, which allow us to determine the pullback of the generalized theta divisor by different morphisms which are essentially given by the action of the Jacobian on the moduli space of semistable vector bundles. The latter theorem has been already applied by Schork [19] in the study of correlation functions of generalized b–c systems.

The known relations between theta functions and the Szegö kernel associated to a line bundle are generalized in Section 4. The identity given in Theorem 4.2 could be a first step in the question addressed by Ball and Vinnikov in p. 865 of [1] about the existence of a explicit formula for the Szegö kernel in higher rank. Another methods were used by Fay [5] to give a similar relation for degree 0 stable vector bundles.

Theorem 5.1 of the following section contains a global version of Lemma 2.7 of [13]. Recall that the bilinear identity for the multicomponent KP hierarchy is a consequence of this kind of formulae and that, in particular, the non-Abelian theta function is a τ-function of that hierarchy.

Section 6 studies the behavior of non-Abelian theta functions under direct and inverse image by a cyclic covering (Theorem 6.3 and Proposition 6.7). Since our methods are valid for all $r \geq 1$, some of our results specialize to formulae of the Jacobian case ($r = 1$) given by Fay (see Remark 6.8).

2. Preliminaries

This section fixes notations and summarizes some results concerned with the generalized theta divisor and non-Abelian theta functions (see [2,3,10,14]).

Let C be an irreducible smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. Given two integers, r, d, let $\mathcal{U}_C(r, d)$ (or simply $\mathcal{U}(r, d)$) denote the moduli space of semistable vector bundles on C of rank r and degree d. Let δ be p.g.c.d.(r, d) and $\bar{r}$ be r^2/δ.

Recall that there is a closed subscheme of $\mathcal{U}(r, r(g-1))$ of codimension 1 given by

$$\Theta_r := \{M \in \mathcal{U}(r, r(g-1)) : h^0(C, M) > 0\}.$$

It thus defines a polarization which is called the generalized theta divisor [3]. Moreover, it holds that (Theorem 2 of [2])

$$h^0(\mathcal{U}(r, r(g-1)), \mathcal{O}(\Theta_r)) = 1.$$

We have therefore a global section θ_r, defined up to a constant, of $\mathcal{O}(\Theta_r)$ whose zero divisor is Θ_r. This section is known as the non-Abelian theta function of rank r and degree $r(g-1)$.

From Drezet and Narashimhan [3] and Le Potier [14] we learn that in order to define a polarization on the moduli space $\mathcal{U}(r, d)$ for an arbitrary d, we need to fix a vector bundle $\bar{F}$ of degree $-(d/\delta) + (r/\delta)(g-1)$ and rank r/δ such that there exists a vector bundle $E \in \mathcal{U}(r, d)$ with $h^0(C, E \otimes \bar{F}) = 0$. In particular, one obtains that $\chi(M \otimes \bar{F}) = 0$ for all $M \in \mathcal{U}(r, d)$.

From now on, we will fix a theta characteristic $\mathcal{O}(\eta)$ on C and we write $\bar{F}$ as $F(\eta)$ for a degree $-d/\delta$ rank r/δ vector bundle F. Then, it is known that

$$\Theta_{[F_\eta]} := \{M \in \mathcal{U}(r, d) : h^0(C, M \otimes F(\eta)) > 0\}$$

defines a polarization, that depends only on the class of $F(\eta)$ in the Grothendieck group of coherent algebraic sheaves on C. This divisor is known as the generalized theta divisor on $\mathcal{U}(r, d)$ defined by $F(\eta)$.

Assume that a polarization $\Theta_{[F_\eta]}$ in $\mathcal{U}(r, d)$ is given. Recall that there exists E such that $h^i(C, E \otimes F(\eta)) = 0\ (i = 0, 1)$. Then, by Lemma 2.5 of [14], it follows that F is semistable.

Being F semistable, we can define the following morphism:

$$\beta_{F_\eta} : \mathcal{U}(r, d) \to \mathcal{U}(\bar{r}, \bar{r}(g-1)), \qquad M \mapsto M \otimes F(\eta) \tag{2.1}$$

since the tensor product of semistable vector bundles is semistable (see Theorem 3.1.4 of [9]). It holds that $\beta_{F_\eta}^{-1}(\Theta_{\bar{r}}) = \Theta_{[F_\eta]}$. Then, define the non-Abelian theta function $\theta_{[F_\eta]}$ as the image of $\theta_{\bar{r}}$ by the induced morphism

$$H^0(\mathcal{U}(\bar{r}, \bar{r}(g-1)), \mathcal{O}(\Theta_{\bar{r}})) \to H^0(\mathcal{U}(r, d), \mathcal{O}(\Theta_{[F_\eta]})). \tag{2.2}$$

However, the construction of these divisors as determinantal subvarieties [3,10] turns out to be an essential tool when proving statements about them.

Let S be a scheme and M be a semistable vector bundle on $C \times S$ of rank r and degree d and let ϕ_M be the morphism

$$\phi_M : S \to \mathcal{U}(r, d), \qquad s \mapsto M|_{C\times\{s\}}.$$

Then, the polarization satisfies the following property:

$$\phi_M^*(\mathcal{O}(\Theta_{[F_\eta]})) \simeq \mathrm{Det}(R^\bullet q_*(M \otimes p^*(F(\eta))))^*,$$

where $q : C \times S \to S$ and $p : C \times S \to C$ are the natural projections.

In order to compute this determinant we proceed as follows. Fix an effective divisor D on $C \times S$ such that $R^1 q_*(M(D) \otimes p^* F(\eta)) = 0$. Then, tensor with $M \otimes p^* F(\eta)$ is the following exact sequence on $C \times S$:

$$0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}_D(D) \to 0$$

and consider the induced cohomology sequence on S:

$$\begin{aligned} 0 &\to q_*(M \otimes p^* F(\eta)) \to q_*(M(D) \otimes p^* F(\eta)) \xrightarrow{\alpha} q_*(M \otimes \mathcal{O}_D(D) \otimes p^* F(\eta)) \\ &\to R^1 q_*(M \otimes p^* F(\eta)) \to 0. \end{aligned}$$

The properties of determinants [8] show that (up to a constant)

$$\phi_M^*(\theta_{F_\eta}) = \det(\alpha) \in H^0(S, \phi_M^* \mathcal{O}(\Theta_{[F_\eta]})), \tag{2.3}$$

which is an effective way to deal with non-Abelian theta functions.

Finally, it is worth pointing out that the above construction also applies to the universal bundle of $\mathcal{U}(r, d)$ when r, d are coprime.

3. Addition formula

The first part of this section is devoted to the explicit computation of the pullback of the generalized theta divisor $\Theta_{[F_\eta]}$ by the natural morphism

$$m : \mathcal{U}(r, d) \times J \to \mathcal{U}(r, d), \qquad (M, L) \mapsto M \otimes L, \tag{3.1}$$

where J denotes the Jacobian variety of C, that is, isomorphism classes of degree 0 line bundles.

This calculation requires a number of intermediate results. Let us introduce the following notation. Let J_d denote the variety of isomorphism classes of degree d line bundles on C. The choice of the theta characteristic η gives rise to a principal polarization on J, Θ_J. Denote by $\phi_{\Theta_J} : J \to \mathrm{Pic}^0(J)$ the isomorphism induced by Θ_J.

Consider the morphism

$$\det : \mathcal{U}(r, r(g-1)) \to J_{r(g-1)},$$

which maps a vector bundle to its determinant. Finally, for a line bundle $L \in J$ let T_{L} denote the translation defined by L on the moduli space of vector bundles as well as on the Jacobian variety.

Lemma 3.1. *Let $L \in J$. Then, there is an isomorphism*:

$$T_{\mathrm{L}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \otimes \mathcal{O}_{\mathcal{U}}(-\Theta_r) \simeq \det{}^*(T_{-r\eta}^*(\phi_{\Theta_J}(L))).$$

Proof. Let $\mathcal{SU}(r, \mathcal{O}(r\eta))$ be the moduli space of semistable vector bundles of rank r with determinant isomorphic to $\mathcal{O}(r\eta)$ and let $\bar{\Theta}_r$ be the restriction of Θ_r to $\mathcal{SU}(r, \mathcal{O}(r\eta))$.

Since $\mathrm{Pic}(\mathcal{U}(r, r(g-1))) \simeq \mathrm{Pic}(\mathcal{SU}(r, \mathcal{O}(r\eta))) \oplus \mathrm{Pic}(J_{r(g-1)})$ and $\mathrm{Pic}(\mathcal{SU}(r, \mathcal{O}(r\eta))) \simeq \mathbb{Z}\bar{\Theta}_r$ [3], one has that $T_{\mathrm{L}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \otimes \mathcal{O}_{\mathcal{U}}(-\Theta_r) \simeq \det^*(N)$ for some $N \in \mathrm{Pic}^0(J_{r(g-1)})$ depending on L.

Consider the morphism

$$\bar{m} : \mathcal{SU}(r, \mathcal{O}(r\eta)) \times J \to \mathcal{U}(r, r(g-1)), \qquad (M, L) \mapsto M \otimes L.$$

By Beauville et al. [2], we know that $\bar{m}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \simeq p_{\mathcal{SU}}^*(\mathcal{O}_{\mathcal{SU}}(\bar{\Theta}_r)) \otimes p_J^*(\mathcal{O}_J(r\Theta_J))$, where $p_{\mathcal{SU}}$ and p_J are the natural projections.

Taking the pullback of $T_{\mathrm{L}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \otimes \mathcal{O}_{\mathcal{U}}(-\Theta_r)$ by the map $\bar{m}$, one checks that $T_{r\eta}^*(N) \simeq \phi_{\Theta_J}(L) \otimes \mu$ with μ an r-torsion point of $\mathrm{Pic}^0(J)$ depending on L. Since J is complete and the r-torsion subgroup of $\mathrm{Pic}^0(J)$ is finite, one obtains that μ does not depend on L. Letting $L = \mathcal{O}_C$, one has that $\mu = \mathcal{O}_J$ and the claim follows. □

Now, we consider the morphism (3.1) for the case $d = r(g-1)$. Recall that the Poincaré bundle on $J \times J$ is given by

$$\mathcal{P} := m_J^*(\mathcal{O}_J(\Theta_J)) \otimes p_1^*(\mathcal{O}_J(-\Theta_J)) \otimes p_2^*(\mathcal{O}_J(-\Theta_J)),$$

where $m_J : J \times J \to J$ corresponds to the tensor product of line bundles and p_i is the projection onto the ith factor.

Lemma 3.2. *It holds that*

$$m^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) = p_{\mathcal{U}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \otimes p_J^*(\mathcal{O}_J(r\Theta_J)) \otimes ((T_{-r\eta} \circ \det) \times 1)^*\mathcal{P}.$$

Proof. We consider the bundle on $\mathcal{U}(r, r(g-1)) \times J$ defined by

$$\mathcal{F} := m^*\mathcal{O}_{\mathcal{U}}(\Theta_r) \otimes p_{\mathcal{U}}^*(\mathcal{O}_{\mathcal{U}}(-\Theta_r)) \otimes ((T_{-r\eta} \circ \det) \times 1)^*\mathcal{P}^{-1}.$$

Then, the above lemma implies that

$$\mathcal{F}|_{\mathcal{U}(r,r(g-1))\times\{L\}} = T_{\mathrm{L}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_r)) \otimes \mathcal{O}_{\mathcal{U}}(-\Theta_r) \otimes \det^*(T_{-r\eta}^*(\phi_{\Theta_J}(L)))^* \simeq \mathcal{O}_{\mathcal{U}},$$

where L is a point of J.

Hence, $\mathcal{F}$ is trivial along the fibers of the natural projection $p_J : \mathcal{U}(r, r(g-1)) \times J \to J$. Seesaw theorem allows us to conclude that $\mathcal{F} \simeq p_J^* N$ for some $N \in \mathrm{Pic}(J)$.

If we show that $N \simeq \mathcal{O}_J(r\Theta_J)$, we are done. Recall from Raynaud [17] that there exists a vector bundle $M \in \mathcal{U}(r, r(g-1))$ with $\wedge M := \det(M) = \mathcal{O}(r\eta)$ such that the subscheme $D(M) := \{L \in J : h^0(M \otimes L) > 0\}$ is a divisor of J which is linearly equivalent to $r\Theta_J$. We now have that

$$N \simeq \mathcal{F}|_{\{M\}\times J} = \mathcal{O}_J(D(M)) \otimes \mathcal{P}^{-1}|_{\{(\wedge M)\otimes\mathcal{O}(-r\eta)\}\times J} \simeq \mathcal{O}_J(r\Theta_J).$$ □

We are now ready to compute the pullback of the generalized theta divisor by the morphism (3.1).

Theorem 3.3. *One has that*

$$m^*(\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]})) = p_{\mathcal{U}}^*(\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]})) \otimes p_J^*(\mathcal{O}_J(\bar{r}\Theta_J)) \otimes ((\det \circ \beta_F) \times 1)^*\mathcal{P},$$

where $\beta_F : \mathcal{U}(r, d) \to \mathcal{U}(\bar{r}, 0)$ *corresponds to tensor product by* F.

Proof. It follows from the above lemma and the following commutative diagram: □

$$\begin{tikzcd} \mathcal{U}(r,d) \times J \arrow[r, "m"] \arrow[d, "\beta_{F_\eta} \times 1"'] & \mathcal{U}(r,d) \arrow[d, "\beta_{F_\eta}"] \\ \mathcal{U}(\bar{r}, \bar{r}(g-1)) \times J \arrow[r, "m"] & \mathcal{U}(\bar{r}, \bar{r}(g-1)) \end{tikzcd}$$

Corollary 3.4. *Let* $M \in \mathcal{U}(r, d)$ *and* $\beta_M : J \to \mathcal{U}(r, d)$ *be the morphism which sends L to* $M \otimes L$. *It holds that*

$$\beta_M^*(\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]})) \simeq \mathcal{O}_J(\bar{r}\Theta_J) \otimes \phi_{\Theta_J}(\wedge(M \otimes F)).$$

Proof. It follows from the previous theorem and from the isomorphism $\beta_M^*(\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]})) \simeq m^*(\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]}))|_{\{M\}\times J}$. □

The rest of this section aims at giving explicit formulas for the pullback of non-Abelian theta functions by the morphism

$$\alpha_M : C^{2m} \to \mathcal{U}(r, d), \qquad (x_1, y_1, \ldots, x_m, y_m) \mapsto M\left(\sum_{i=1}^{m}(x_i - y_i)\right),$$

where C^{2m} is the product of $2m$ copies of the curve C and $M \in \mathcal{U}(r, d)$.

Firstly, we will deal with an isomorphism of line bundles on C^{2m} and then it will be applied to obtain an identity among global sections of them. Such a formula can be understood as an addition formula for non-Abelian theta functions. For the rank 1 case and identifying the theta function (as a section) with its classical analytic expression, our formula turns out to coincide with Fay's formula. However, as long as no analytic expressions for non-Abelian theta functions are known, our generalization must be regarded as an identity of sections.

If a point of C^{2m} is denoted by $(x_1, y_1, \ldots, x_m, y_m) \in C^{2m}$, we will call an index odd (resp. even) if it corresponds to a variable x_i (resp. y_j). Finally, let p_i be the projection onto the ith factor and Δ_{ij} the divisor of C^{2m} where the ith and the jth entries coincide.

Lemma 3.5. *Let* $L \in J$ *and consider the following morphism*:

$$\alpha_{\mathrm{L}} : C^{2m} \to J, \qquad (x_1, y_1, \ldots, x_m, y_m) \mapsto L\left(\sum_i (x_i - y_i)\right).$$

Then, one has that

$$\alpha_{\mathrm{L}}^{*}\mathcal{O}_J(\Theta_J) \simeq \mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij}\right) \otimes \left(\bigotimes_{i\ \text{odd}} p_i^{*} L^{*}(\eta)\right) \otimes \left(\bigotimes_{i\ \text{even}} p_i^{*} L(\eta)\right),$$

where the sums involve only i, j *with* $i < j$.

Proof. See Theorem 11.1 in [15]. □

Theorem 3.6. *Let* M *be a rational point of* $\mathcal{U}(r, d)$ *and let* α_M *be the morphism defined by*

$$\alpha_M : C^{2m} \to \mathcal{U}(r, d), \qquad (x_1, y_1, \dots, x_m, y_m) \mapsto M\left(\sum_{i=1}^{m}(x_i - y_i)\right).$$

Then, there is an isomorphism of line bundles on C^{2m}:

$$\alpha_M^{*}\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]}) \simeq \mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij}\right)^{\otimes \bar{r}} \otimes \left(\bigotimes_{i\ \text{odd}} p_i^{*} \wedge (M \otimes F(-\eta))^{*}\right)$$
$$\otimes \left(\bigotimes_{i\ \text{even}} p_i^{*} \wedge (M \otimes F(\eta))\right),$$

where the sums involves only i, j *with* $i < j$.

Proof. The morphism α_M factors as follows:

$$C^{2m} \xrightarrow{\alpha_{\mathcal{O}}} J \simeq \{M\} \times J \xrightarrow{\beta_M} \mathcal{U}(r, d).$$

Let M' be $M \otimes F$. Recalling Corollary 3.4 and Lemma 3.5 we have that

$$\begin{aligned}
\alpha_M^{*}\mathcal{O}_{\mathcal{U}}(\Theta_{[F_\eta]}) &\simeq \alpha_{\mathcal{O}}^{*}(\mathcal{O}_J(\bar{r}\Theta_J) \otimes \phi_{\Theta_J}(\wedge M')) \simeq \alpha_{\mathcal{O}}^{*}(T_{\wedge M'}^{*}\mathcal{O}_J(\Theta_J) \otimes \mathcal{O}_J(\Theta_J)^{\otimes \bar{r}-1}) \\
&\simeq \alpha_{\wedge M'}^{*}(\mathcal{O}_J(\Theta_J)) \otimes \alpha_{\mathcal{O}_C}^{*}(\mathcal{O}_J(\Theta_J))^{\otimes \bar{r}-1} \\
&\simeq \mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij}\right) \otimes \left(\bigotimes_{i\ \text{odd}} p_i^{*}(\wedge M')^{*}(\eta)\right) \\
&\quad \otimes \left(\bigotimes_{i\ \text{even}} p_i^{*}(\wedge M')(\eta)\right) \otimes \mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij}\right)^{\otimes \bar{r}-1} \\
&\quad \otimes \left(\bigotimes_{i\ \text{odd}} p_i^{*}\mathcal{O}(\eta)^{\otimes \bar{r}-1}\right) \otimes \left(\bigotimes_{i\ \text{even}} p_i^{*}\mathcal{O}(\eta)^{\otimes \bar{r}-1}\right)
\end{aligned}$$

and the theorem follows. □

Remark 3.7. This result has been applied in [19] when proving the relation of determinants of correlation functions of generalized b–c systems and determinants of non-Abelian theta functions. This is a first step of the expected fact that correlation functions of generalized b–c system are determined completely by the geometry of the non-Abelian theta divisor, analogous to the rank 1 case.

Recall from chapter II of [4] that the line bundle $\mathcal{O}(\Delta)$ con $C \times C$ has a unique section $E(x, y)$, which is known as the prime form and that it can be constructed in terms of η. To be consistent with Fay, it will be assumed that the theta characteristic η is odd. In particular, it holds that $E(x, y) = -E(y, x)$.

Theorem 3.8. *Let M be a rational point of $\mathcal{U}(r, d)$ such that $\theta_{F_\eta}(M) \neq 0$. Then, for $(x_1, y_1, \dots, x_m, y_m) \in C^{2m}$, one has that*

$$\frac{\theta_{F_\eta}\left(M\left(\sum_{i=1}^m (x_i - y_i)\right)\right)}{\theta_{F_\eta}(M)} \prod_{i<j} E(x_i, x_j)^{\bar{r}} E(y_i, y_j)^{\bar{r}} = \prod_{i,j} E(x_i, y_j)^{\bar{r}} \det\left(\frac{\theta_{F_\eta}(M(x_i - y_j))}{\theta_{F_\eta}(M) E(x_i, y_j)^{\bar{r}}}\right).$$

Proof. Observe that the r.h.s. of the equality of the statement equals the sum

$$\sum_{\sigma \in \mathcal{S}_m} \operatorname{sign}(\sigma) \prod_{\substack{i,j \\ \sigma(i) \neq j}} E(x_i, y_j)^{\bar{r}} \prod_i \frac{\theta_{F_\eta}(M(x_i - y_{\sigma(i)}))}{\theta_{F_\eta}(M)}.$$

By Theorem 3.6 with $m = 1$, $\theta_{F_\eta}(M(x_i - y_j))/\theta_{F_\eta}(M)$ is a section of the line bundle

$$\mathcal{O}(\Delta)^{\otimes \bar{r}} \otimes p_i^* \wedge (M \otimes F(-\eta))^* \otimes p_j^* \wedge (M \otimes F(\eta)).$$

So, it turns out that each term of the above sum is a global section of

$$\mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij}\right)^{\otimes \bar{r}} \otimes \left(\bigotimes_{i \text{ odd}} p_i^* \wedge (M \otimes F(-\eta))^*\right) \otimes \left(\bigotimes_{i \text{ even}} p_i^* \wedge (M \otimes F(\eta))\right).$$

The l.h.s. is a section of the line bundle:

$$\alpha_M^* \mathcal{O}(\Theta_{[F_\eta]}) \otimes \mathcal{O}\left(\sum_{i+j=\text{even}} \Delta_{ij}\right)^{\otimes \bar{r}}.$$

These two line bundles are isomorphic by Theorem 3.6. Hence, both sides of the equality are to be understood as global sections of the same line bundle. Since C^{2m} is proper, there is no non-constant global section of the trivial bundle. So, if we show that both sections have the same zero divisor, then they coincide up to a multiplicative constant, which will be eventually shown to be 1.

Let D_{R} (resp. D_{L}) denote the zero divisor of the r.h.s. (resp. l.h.s.). Since D_{L} and D_{R} are linearly equivalent, there exists a rational function f on C^{2m} such that

$$D_{\mathrm{R}} - D_{\mathrm{L}} = D(f).$$

Let us consider the following diagram:

$$\begin{array}{ccc} C^{2m} & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \pi} & & \\ C^{2m-1} & & \end{array}$$

where $\pi(x_1, y_1, \ldots, x_m, y_m) := (x_1, x_2, y_2, \ldots, x_m, y_m)$.

Suppose we have proved that there exists $z \in C^{2m-1}$ such that $D_{\mathrm{L}}|_{\pi^{-1}(z)} = D_{\mathrm{R}}|_{\pi^{-1}(z)}$ and $\mathrm{supp}(D_{\mathrm{L}}|_{\pi^{-1}(z)}) \neq \pi^{-1}(z)$. It thus follows that $f|_{\pi^{-1}(z)}$ is a non-zero constant, since $\pi^{-1}(z) \simeq C$ is proper. From the *rigidity lemma* one deduces that f is constant along the fibers of π and, therefore, f has neither poles nor zeroes in C^{2m}. Summing up, f is invertible or, what amounts to the same, $D_{\mathrm{L}} = D_{\mathrm{R}}$. So, there exists a non-zero constant λ such that the l.h.s. equals the r.h.s. multiplied by λ. Letting $x_i = y_i$ for all i, we obtain that $\lambda = 1$.

By the above discussion, it remains to show that there exists z such that $D_{\mathrm{L}}|_{\pi^{-1}(z)} = D_{\mathrm{R}}|_{\pi^{-1}(z)}$ and $\mathrm{supp}(D_{\mathrm{L}}|_{\pi^{-1}(z)}) \neq \pi^{-1}(z)$.

We take $z = (x_1, x_2, y_2, \ldots, x_m, y_m) \in C^{2m-1}$ such that $x_k = y_k$ for $k \neq 1$ and $x_i \neq x_j$ for all $i \neq j$. Then, we have that

$$\left(\frac{\theta_{F_\eta}\left(M\left(\sum_{i=1}^{m}(x_i - y_i)\right)\right)}{\theta_{F_\eta}(M)} \prod_{i<j} E(x_i, x_j)^{\bar{r}} E(y_i, y_j)^{\bar{r}} \right)\Bigg|_{\pi^{-1}(z)}$$
$$= \frac{\theta_{F_\eta}(M(x_1 - y_1))}{\theta_{F_\eta}(M)} \prod_{k\neq 1} E(y_1, y_k)^{\bar{r}} \prod_{i<j} E(x_i, x_j)^{\bar{r}} \prod_{2\leq i<j} E(y_i, y_j)^{\bar{r}}$$

and the r.h.s. restricted to the fiber of z is

$$\det\left(\prod_{k\neq i} E(x_k, y_j)^{\bar{r}} \frac{\theta_{F_\eta}(M(x_i - y_j))}{\theta_{F_\eta}(M)} \right)\Bigg|_{\pi^{-1}(z)} = \prod_{k\neq 1} E(x_k, y_1)^{\bar{r}} \frac{\theta_{F_\eta}(M(x_1 - y_1))}{\theta_{F_\eta}(M)}.$$

Letting $y_1 = x_1$ one checks that both restrictions are not zero. Furthermore, since the first one is equal to the second times a non-zero constant on $\pi^{-1}(z)$, one has that $D_{\mathrm{L}}|_{\pi^{-1}(z)} = D_{\mathrm{R}}|_{\pi^{-1}(z)}$. The theorem is proved. □

4. Addition formula and the Szegö kernel

Now, let us recall briefly the definition and properties of the Szegö kernel associated to a vector bundle $M \in \mathcal{U}(r, d) - \Theta_{[F_\eta]}$. For such a bundle define the Szegö kernel, $S_M(x, y)$, to be the meromorphic section of $p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))$ on $C\times C$ with a simple pole along the diagonal such that its residue along it is 1.

Note that $S_M(x,y)$ might be written as an $r\times r$ matrix, because there is an isomorphism:

$$p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))\simeq \mathrm{Hom}(p_1^*(M\otimes F(-\eta)),p_2^*(M\otimes F(\eta))).$$

On the other hand, observe that the restriction to the diagonal $\Delta\subset C\times C$ induces an isomorphism

$$H^0(C\times C,p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))\otimes \mathcal{O}(\Delta))\simeq H^0(C,\mathrm{End}(M\otimes F))$$

and denote by $S_M^{\mathrm{h}}(x,y)$ the holomorphic global section of the vector bundle $p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))\otimes \mathcal{O}(\Delta)$ whose image by the above isomorphism is the identity.

Then, it is worth noting that $E(x,y)\cdot S_M(x,y)$ is a holomorphic section of $p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))\otimes \mathcal{O}(\Delta)$, because the morphism $\mathcal{O}\to\mathcal{O}(\Delta)$ maps the global section 1 to the global section $E(x,y)$. One checks that $S_M^{\mathrm{h}}(x,y)-E(x,y)\cdot S_M(x,y)$ gives a global section of $p_1^*(M\otimes F(-\eta))^*\otimes p_2^*(M\otimes F(\eta))$. Since this bundle has no non-zero section, then one has that

$$S_M^{\mathrm{h}}(x,y)=E(x,y)\cdot S_M(x,y).$$

If S_M^{h} and S_M are both understood as matrices, then this identity makes sense too.

Remark 4.1. One can show that the rows of $E(x_0,y)\cdot S_M(x_0,y)$ for a fixed point $x_0\in C$ give a basis of $H^0(C,M\otimes F(\eta+x_0))$, because the restriction to $\{x_j\}\times C$ maps S_M^{h} to its rows

$$\begin{aligned}&H^0(C\times C,\mathrm{Hom}(p_1^*(M\otimes F(-\eta)),p_2^*(M\otimes F(\eta))\otimes\mathcal{O}(\Delta)))\\&\quad\to H^0(C,M\otimes F(\eta+x_j))^{\oplus r}.\end{aligned}$$

Now, the relation of the non-Abelian theta function and the Szegö kernel given by Fay for degree 0 stable bundles [5] is generalized for semistable ones by the following theorem.

Theorem 4.2. *Let M be a rational point of $\mathcal{U}(r,d)$ such that $\theta_{F_\eta}(M)\neq 0$. Then, for $(x_1,y_1,\dots,x_m,y_m)\in C^{2m}$, one has that*

$$\frac{\theta_{F_\eta}\left(M\left(\sum_{i=1}^m(x_i-y_i)\right)\right)}{\theta_{F_\eta}(M)}\prod_{i<j}E(x_i,x_j)^{\bar r}E(y_i,y_j)^{\bar r}=\prod_{i,j}E(x_i,y_j)^{\bar r}\det S_M(x,y),$$

where $S_M(x,y)$ is an $\bar r m\times\bar r m$ matrix builded up from the $\bar r\times\bar r$ boxes $S_M(x_i,y_j)$.

Before giving the proof we need some results.

Lemma 4.3. *Both sides of the equality in the statement of Theorem 4.2 are global sections of the isomorphic line bundles on C^{2m}.*

Proof. Note that the matrix $S_M(x, y)$ is a meromorphic section of the bundle:

$$\mathrm{Hom}\left(\underset{i=\mathrm{odd}}{\oplus} p_i^*(M \otimes F(-\eta)), \underset{j=\mathrm{even}}{\oplus} p_j^*(M \otimes F(\eta))\right)$$

with poles along $\sum_{i+j=\mathrm{odd}} \Delta_{ij}$ (odd indexes correspond to x's variables, while even indexes correspond to y's variables). Therefore, the determinant $\det S_M(x, y)$ is a meromorphic section of

$$\left(\underset{i=\mathrm{odd}}{\otimes} p_i^* \wedge (M \otimes F(-\eta))^*\right) \otimes \left(\underset{j=\mathrm{even}}{\otimes} p_j^* \wedge (M \otimes F(\eta))\right).$$

Counting the order of these poles, one concludes that the r.h.s. is a holomorphic section of

$$\left(\underset{i=\mathrm{odd}}{\otimes} p_i^* \wedge (M \otimes F(-\eta))^*\right) \otimes \left(\underset{j=\mathrm{even}}{\otimes} p_j^* \wedge (M \otimes F(\eta))\right) \otimes \mathcal{O}\left(\sum_{i+j=\mathrm{odd}} \Delta_{ij}\right)^{\otimes \bar{r}}.$$

The l.h.s. is a holomorphic global section of

$$\alpha^*(\Theta_{[F_\eta]}) \otimes \mathcal{O}\left(\sum_{i+j=\mathrm{even}} \Delta_{ij}\right)^{\otimes \bar{r}}$$

and the two line bundles above are isomorphic by Theorem 3.6. □

Lemma 4.4. *Let $M \in \mathcal{U}(r, d) - \Theta_{[F_\eta]}$. Then, for $(x, y) \in C \times C$, one has that*

$$\frac{\theta_{F_\eta}(M(x-y))}{\theta_{F_\eta}(M)} = E(x, y)^{\bar{r}} \det S_M(x, y).$$

Proof. Lemma 4.3 implies that both sides are global sections of the same line bundle.

Label the three copies of C in $C \times C \times C$ by 0, 1 and 2. Let Δ_{0i} be the subscheme where the 0th entry coincides with the ith entry. Let p denote the projection from $C \times C \times C$ onto the copy of C labeled with 0. Finally, let $q : C \times C \times C \to C \times C$ be the projection onto the copies labeled with 1 and 2.

The bundle M defines the morphism

$$\alpha_M : C \times C \to \mathcal{U}(r, d), \qquad (x, y) \mapsto M(x - y).$$

By the construction of the polarization it is known that

$$\alpha_M^*(\mathcal{O}(-\Theta_{[F_\eta]})) \simeq \mathrm{Det}(R^\bullet q_* \mathcal{M}),$$

where $\mathcal{M} := p^*(M \otimes F(\eta)) \otimes \mathcal{O}(\Delta_{01} - \Delta_{02})$. Let M' be $M \otimes F(\eta)$.

Let us compute this determinant as well as a section of its dual. Consider the exact sequence on $C \times C \times C$:

$$0 \to \mathcal{O}(\Delta_{01} - \Delta_{02}) \to \mathcal{O}(\Delta_{01}) \to \mathcal{O}(\Delta_{01})|_{\Delta_{02}} \to 0.$$

Tensor with p^*M' and pushing it forward by q one obtains

$$0 \to q_*\mathcal{M} \to q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})) \xrightarrow{\beta} q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{02}}) \to R^1q_*\mathcal{M} \to 0,$$

because $R^1q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})) = 0$.

Observe that the two middle terms of the above sequence are locally free of same rank. Then, it follows that there exists a canonical isomorphism:

$$\alpha_M^*(\mathcal{O}(\Theta_{[F_\eta]})) \simeq \wedge(q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})))^* \otimes \wedge q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{02}}),$$

which, by relation (2.3), maps the global section $\alpha_M^*\theta_{F_\eta}$ to $\det\beta$.

Our task now consists of relating the determinant of β with that of $S_M^{\mathrm{h}}(x, y)$, since $\det S_M^{\mathrm{h}}(x, y) = E(x, y)^{\bar{r}}\det S_M(x, y)$. The fact that it will be shown that the morphism $S_M^{\mathrm{h}}(x, y)$ factorizes as $\beta \circ \phi^{-1}$, where ϕ is a morphism whose determinant equals $\theta_{F_\eta}(M)$.

Let us begin with the morphism ϕ. Analogous arguments as previously applied to the exact sequence:

$$0 \to \mathcal{O} \to \mathcal{O}(\Delta_{01}) \to \mathcal{O}(\Delta_{01})|_{\Delta_{01}} \to 0$$

show that there is an isomorphism

$$q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})) \overset{\phi}{\simeq} q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{01}}).$$

If $\alpha_0 : C \times C \to \mathcal{U}(r, d)$ is the morphism that sends (x, y) to M, then it follows that the isomorphism:

$$\alpha_0^*\mathcal{O}(\Theta_{[F_\eta]}) \simeq \wedge(q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})))^* \otimes \wedge q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{01}}) \simeq \mathcal{O}$$

maps $\alpha_0^*(\theta_{F_\eta}) = \theta_{F_\eta}(M)$ to $\det\phi$.

In order to write down the factorization of $S_M^{\mathrm{h}}(x, y)$ we need the following identifications:

$$\begin{aligned} q_*(p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{02}} &\simeq \iota_2^*(p^*M' \otimes \mathcal{O}(\Delta_{01})) \simeq p_2^*M' \otimes \mathcal{O}(\Delta), \\ q_*(p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{01}} &\simeq \iota_1^*(p^*M' \otimes \mathcal{O}(\Delta_{01})) \simeq p_1^*(M' \otimes \omega_C^*), \end{aligned}$$

where ι_j $(j = 1, 2)$ is the embedding $C \times C \simeq \Delta_{0j} \subset C \times C \times C$ and p_j the projection from $C \times C$ onto its jth factor $(j = 1, 2)$, and Ω_C is the canonical line bundle on C.

These identifications shows that there is a natural map of bundles on $C \times C$:

$$\begin{aligned} p_1^*(M' \otimes \omega_C^*) &\simeq q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{01}}) \\ &\overset{\phi^{-1}}{\simeq} q_*(p^*M' \otimes \mathcal{O}(\Delta_{01})) \xrightarrow{\beta} q_*((p^*M' \otimes \mathcal{O}(\Delta_{01}))|_{\Delta_{02}}) \\ &\simeq p_2^*M' \otimes \mathcal{O}(\Delta). \end{aligned}$$

If we check that this map coincides with $S_M^{\mathrm{h}}(x, y) = E(x, y)S_M(x, y)$, the lemma is proved. To this goal it is enough to verify that the restriction of $\beta \circ \phi^{-1}$ to the diagonal is the identity map and this fact follows from a straightforward calculation. □

Proof of Theorem 4.2. Firstly, observe that Lemma 4.3 implies that both sides of the equality are holomorphic global sections of the same line bundle.

Similar arguments to those of the proof of Theorem 3.8 allows us to reduce the proof to check that the statement holds true on the fiber $\pi^{-1}(z) \simeq C$, where $\pi : C^{2m} \to C^{2m-1}$ is the projection that forgets y_1 and z is a point $(x_1, x_2, y_2, \ldots, x_m, y_m) \in C^{2m-1}$ such that $x_i \neq x_j$ for all $i \neq j$ and $y_i = x_i$.

Now, note that the claim restricted to the fibre $\pi^{-1}(z)$ is precisely the statement of Lemma 4.4, which has been already proved. □

Corollary 4.5. *Under the same hypothesis of the previous theorem, one has that*

$$\det \left(\frac{\theta_{F_\eta}(M(x_i - y_j))}{\theta_{F_\eta}(M)E(x_i, y_j)^{\bar{r}}} \right) = \det S_M(x, y).$$

Proof. It follows from Theorems 3.8 and 4.2. □

5. Relation with the multicomponent KP hierarchy

In this section, it will addressed the relation between some properties of non-Abelian theta functions with those of τ-functions of the multicomponent KP hierarchy. The importance of the theorem below comes from the consequences of its infinitesimal version (Lemma 2.7 of [13]), which eventually leads to the bilinear identity in the framework of the multicomponent KP hierarchy. Moreover, it generalizes Proposition 2.16 of [4] for higher rank.

Theorem 5.1. *Let M be a rational point of $\mathcal{U}(r, r(g-1+m))$ (m being a positive integer) such that $h^1(C, M) = 0$.*

Then, the following identity on C^m holds:

$$\theta_r \left(M \left(-\sum_{i=1}^{m} y_i \right) \right) \prod_{i<j} E(y_i, y_j)^r = \lambda \det(s_i(y_j)),$$

where $\lambda \in \mathbb{C}^*$, $\{s_i = (s_i^1, \ldots, s_i^r) | i = 1, \ldots, mr\}$ is a basis of $H^0(C, M)$ and the matrix $(s_i(y_j))$ is

$$\begin{pmatrix} s_1^1(y_1) & \cdots & s_1^r(y_1) & \cdots & s_1^1(y_m) & \cdots & s_1^r(y_m) \\ \vdots & & & & & & \vdots \\ s_{mr}^1(y_1) & \cdots & s_{mr}^r(y_1) & \cdots & s_{mr}^1(y_m) & \cdots & s_{mr}^r(y_m) \end{pmatrix}.$$

Proof. We begin with the $m = 2$ case where the idea of the proof will be clear. For this case we will work with bundles on $C \times C \times C$ and will use again the notations introduced in the proof of Lemma 4.4.

Recall that the sheaf $\mathcal{O}_{\Delta_{01}+\Delta_{02}}$ is the kernel of the difference map $\mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}} \to \mathcal{O}_{\Delta_{01}\cap\Delta_{02}}$ and, therefore, we have the exact sequence

$$0 \to \mathcal{O}_{\Delta_{01}+\Delta_{02}} \to \mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}} \to \mathcal{O}_{\Delta_{01}\cap\Delta_{02}} \to 0.$$

From the following exact sequence:

$$0 \to \mathcal{O}(-\Delta_{01} - \Delta_{02}) \to \mathcal{O} \to \mathcal{O}_{\Delta_{01}+\Delta_{02}} \to 0,$$

one deduces the exactness of

$$\begin{aligned} 0 &\to q_*(p^*M(-\Delta_{01} - \Delta_{02})) \to q_*(p^*M) \xrightarrow{\alpha} \to q_*(p^*M \otimes \mathcal{O}_{\Delta_{01}+\Delta_{02}}) \\ &\to R^1 q_*(p^*M(-\Delta_{01} - \Delta_{02})) \to 0, \end{aligned}$$

which, by relation (2.3), implies that

$$\theta_r(M(-y_1 - y_2)) = \det(\alpha) \quad \forall y_1, y_2 \in C.$$

The statement is thus reduced to compute $\det(\alpha)$ in an alternative way.

Consider the following commutative diagram:

$$\begin{tikzcd} q_*(p^*M) \arrow[r, "\alpha"] \arrow[d, equal] & q_*(p^*M \otimes \mathcal{O}_{\Delta_{01}+\Delta_{02}}) \arrow[d, "\varphi"] \\ H^0(M) \otimes \mathcal{O}_{C\times C} \arrow[r, "ev"] & q_*(p^*M \otimes (\mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}})) \end{tikzcd}$$

where φ is the morphism induced by $\mathcal{O}_{\Delta_{01}+\Delta_{02}} \to \mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}}$ and ev is the evaluation map, that is, at the point (y_1, y_2) is

$$H^0(M) \to M_{y_1} \oplus M_{y_2}, \qquad s \mapsto (s(y_1), s(y_2)).$$

The diagram shows that

$$\det(\varphi)\det(\alpha) = \det(\mathrm{ev})$$

and therefore

$$\det(\varphi) \cdot \theta_r(M(-y_1 - y_2)) = \lambda' \det(s_i(y_j)),$$

where λ' is a constant that depends on the choice of the basis and on the above isomorphisms of line bundles and it will eventually give the constant of the statement.

Since $q : \Delta_{01} + \Delta_{02} \to C$ is finite of degree 2, $R^1 q_*(\mathcal{O}_{\Delta_{01}+\Delta_{02}}) = 0$. It thus follows the exactness of:

$$0 \to q_*(\mathcal{O}_{\Delta_{01}+\Delta_{02}}) \xrightarrow{\varphi_0} q_*(\mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}}) \to q_*(\mathcal{O}_{\Delta_{01}\cap\Delta_{02}}) \to 0.$$

Now, we will show that $\det(\varphi) = \det(\varphi_0)^r$ and that $\det(\varphi_0) = E(y_1, y_2)$.

Let us begin computing $\det(\varphi_0)$. From the theory of determinants [8] one has the following isomorphism:

$$\mathrm{Det}(q_*(\mathcal{O}_{\Delta_{01}+\Delta_{02}}) \to q_*(\mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}})) \simeq \mathrm{Det}(q_*(\mathcal{O}_{\Delta_{01}\cap\Delta_{02}})).$$

Since the above bundles live on $C \times C$ let us rewrite them as follows. From the diagram:

$$\begin{tikzcd} \Delta_{01} \cap \Delta_{02} \arrow[r] \arrow[d] & C \times C \times C \arrow[d, "q"] \\ \Delta \arrow[r] & C \times C \end{tikzcd}$$

($\Delta \subset C \times C$ being the diagonal) one obtains

$$\mathrm{Det}(q_*(\mathcal{O}_{\Delta_{01}\cap\Delta_{02}})) \simeq \mathrm{Det}(\mathcal{O}_\Delta) \simeq \mathrm{Det}(\mathcal{O}(-\Delta) \to \mathcal{O}) = \mathcal{O}(-\Delta),$$

where the second isomorphism follows from the exactness of the following sequence on $C \times C$:

$$0 \to \mathcal{O}(-\Delta) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0.$$

These calculations imply that:

$$\det(\varphi_0) = E(y_1, y_2) \in H^0(C \times C, \mathcal{O}(\Delta)).$$

On the other hand, $\det(\varphi)$ may be computed similarly and we obtain

$$\begin{aligned} &\mathrm{Det}(q_*(p^*M \otimes \mathcal{O}_{\Delta_{01}+\Delta_{02}}) \\ &\quad \to q_*(p^*M \otimes (\mathcal{O}_{\Delta_{01}} \oplus \mathcal{O}_{\Delta_{02}}))) \simeq \mathrm{Det}(M \otimes \mathcal{O}_\Delta) \simeq \mathcal{O}(-r\Delta). \end{aligned}$$

Now, it follows that $\det(\varphi) = \det(\varphi_0)^r = E(y_1, y_2)^r$. The $m = 2$ case is proved.

For arbitrary m we proceed similarly but replacing the morphism φ by

$$\begin{aligned} &q_*(p^*M \otimes \mathcal{O}_{\sum_{i=1}^m \Delta_{0i}}) \to q_*(p^*M \otimes (\mathcal{O}_{\sum_{i=1}^{m-1} \Delta_{0i}} \oplus \mathcal{O}_{\Delta_{0m}})) \to \cdots \\ &\qquad\qquad \to q_*\left(p^*M \otimes \left(\bigoplus_{i=1}^m \mathcal{O}_{\Delta_{0i}}\right)\right), \end{aligned}$$

which has determinant $\prod_{i<j} E(y_i, y_j)^r$. □

6. Cyclic coverings

Let $\gamma : \tilde{C} \to C$ be a cyclic covering of degree n between two irreducible smooth projective curves given by an automorphism σ of $\tilde{C}$ such that $\sigma^n = \mathrm{Id}$, that is, $\tilde{C}/\langle\sigma\rangle = C$.

In this section we will study the relationship between the polarizations of moduli spaces of vector bundles on $\tilde{C}$ and C. This question is related to the twist structures of b–c systems [16] and has been addressed in [1]. The rank 1 case is to be found in [4].

Let us introduce some notation. Let Δ (resp. $\tilde{\Delta}$) denote the diagonal of $C \times C$ (resp. $\tilde{C} \times \tilde{C}$). Let $\tilde{D}_{ij}$ be the inverse image of the diagonal by the morphism $\sigma^i \times \sigma^j : \tilde{C} \times \tilde{C} \to \tilde{C} \times \tilde{C}$. Let E (resp. $\tilde{E}$) be the prime form of C (resp. $\tilde{C}$). Finally, let $R^\gamma = \sum_{\tilde{x} \in \tilde{C}} (n_{\tilde{x}} - 1)\tilde{x}$ be the ramification divisor of γ, where $n_{\tilde{x}}$ is the ramification index at $\tilde{x}$.

Let us begin with some computations for the ideal sheaf of the diagonal.

Lemma 6.1. *Let γ_1 be $\gamma \times \gamma$ and R^1 be $(R^\gamma \times \tilde{C}) \cup (\tilde{C} \times R^\gamma)$.*
Then, there is an exact sequence on $\tilde{C} \times \tilde{C}$:

$$0 \to \gamma_1^* \mathcal{O}(\Delta) \to \mathcal{O}\left(\sum_j \tilde{D}_{0j}\right) \to \mathcal{O}_{R^1} \to 0$$

and a canonical isomorphism of line bundles:

$$\gamma_1^* \mathcal{O}(\Delta) \simeq \mathcal{O}\left(\sum_j \tilde{D}_{0j}\right) \otimes \tilde{p}_1^* \mathcal{O}(-R^\gamma) \otimes \tilde{p}_2^* \mathcal{O}(-R^\gamma),$$

where $\tilde{p}_i : \tilde{C} \times \tilde{C} \to \tilde{C}$ are the natural projections.

Proof. Since there is an inclusion $\gamma_1^{-1}(\Delta) \subseteq \sum_j \tilde{D}_{0j}$, it follows the exact sequence:

$$0 \to \gamma_1^* \mathcal{O}(\Delta) \to \mathcal{O}\left(\sum_j \tilde{D}_{0j}\right) \to \mathcal{O}_T \to 0. \tag{6.1}$$

Let us compute $\mathcal{O}_T$. If $S = \mathrm{supp}(R^\gamma)$ is the support of R^γ and U the open subscheme $(\tilde{C} - S) \times (\tilde{C} - S)$, one checks easily that $\gamma_1^{-1}(\Delta)|_U = \sum_j \tilde{D}_{0j}|_U$ and T is therefore contained in

$$\tilde{C} \times \tilde{C} - U = \bigcup_{\tilde{x} \in R^\gamma} (\{\tilde{x}\} \times \tilde{C} \cup \tilde{C} \times \{\tilde{x}\}).$$

By symmetry, it is enough to show that the length of T at $\{\tilde{x}\} \times \tilde{C}$ $(\tilde{x} \in S)$ is $n_{\tilde{x}} - 1$. Recalling the exact sequence (6.1), one observes that this can be done by comparing the zero divisors of $\gamma_1^* E$ and $\prod_j (\mathrm{Id} \times \sigma^j)^* \tilde{E}$ as global sections of $\mathcal{O}\left(\sum_j \tilde{D}_{0j}\right)$. One checks now that if $(\tilde{x}, \tilde{y}) \in S \times \tilde{C}$, then $(\tilde{x}, \tilde{y})$ is a simple zero of $\gamma_1^* E$ and a zero of order $n_{\tilde{x}}$ of $\prod_j (\mathrm{Id} \times \sigma^j)^* \tilde{E}$.

For the second claim, it suffices to take determinants in the exact sequence of the first claim. □

Lemma 6.2. *Let $\tilde{M}$ be a vector bundle on $\tilde{C}$ of rank $\tilde{r}$.*
Then, there is an exact sequence:

$$0 \to \gamma^*(\gamma_* \tilde{M}) \to \bigoplus_{k=0}^{n-1} (\sigma^k)^* \tilde{M} \to (\mathcal{O}_{(1/2)nR^\gamma})^{\oplus \tilde{r}} \to 0$$

and a canonical isomorphism

$$\wedge \gamma^*(\gamma_* \tilde{M}) \simeq \bigotimes_{k=0}^{n-1} \wedge (\sigma^k)^* \tilde{M} \otimes \mathcal{O}\left(-\frac{1}{2} \tilde{r} n R^\gamma\right).$$

Finally, if $\tilde{M}$ has degree $\tilde{d}$, then

$$\deg \gamma_* \tilde{M} = \tilde{d} - \tilde{r}(\tilde{g} - 1 - n(g-1)),$$

where $\tilde{g}$ (resp. g) is the genus of $\tilde{C}$ (resp. C).

Proof. The adjunction formula gives a morphism $\gamma^*(\gamma_*\tilde{M}) \to \tilde{M}$ and, since $\gamma^*(\gamma_*\tilde{M})$ is invariant under σ, there is also a morphism to $(\sigma^k)^*\tilde{M}$ for $0 \leq k < n$; that is

$$\gamma^*(\gamma_*\tilde{M}) \to \bigoplus_{k=0}^{n-1} (\sigma^k)^*\tilde{M}.$$

Since this is a morphism between two locally free sheaves of the same rank which is an isomorphism at the stalk of any point $\tilde{x} \in \tilde{C} - R^\gamma$, it follows that there is an exact sequence:

$$0 \to \gamma^*(\gamma_*\tilde{M}) \to \bigoplus_{k=0}^{n-1} (\sigma^k)^*\tilde{M} \to \mathcal{O}_T \to 0, \tag{6.2}$$

where $\mathrm{supp}(T) \subseteq \mathrm{supp}\, R^\gamma$.

Note that the computation of T is a local problem, so it can be assumed $\tilde{M}$ to be $\mathcal{O}_C^{\oplus\tilde{r}}$. Furthermore, observe that $\mathcal{O}_T = \mathcal{O}_{T'}^{\oplus\tilde{r}}$, where

$$0 \to \gamma^*(\gamma_*\mathcal{O}_{\tilde{C}}) \to \bigoplus_{k=0}^{n-1} \mathcal{O}_{\tilde{C}} \to \mathcal{O}_{T'} \to 0. \tag{6.3}$$

For the case $\tilde{M} = \mathcal{O}_{\tilde{C}}$ some results on cyclic coverings are known. From Theorem 3.2 of [6] we learn that the covering $\gamma : \tilde{C} \to C$ is defined by a line bundle L on C and a divisor $D = \sum a_i q_i$ on C, where $1 \leq a_i < n$, $L^{\otimes n} \simeq \mathcal{O}_C(D)$, and q_i is a branch point of γ. Furthermore, all the points on the fibre of a q_i have the same multiplicity, say m_i, and $s_i := n/m_i = \mathrm{g.c.d}(a_i, n)$ is the number of distinct points in $\gamma^{-1}(q_i)$.

Moreover, if $[a]_n$ denotes the remainder of a divided by n and D_k is $\sum_i [ka_i]_n q_i$, it then holds that the coefficients of $\gamma^{-1}(D_k)$ are multiple of n (Section 2 of [6]) and that

$$\gamma^*(\gamma_*\mathcal{O}_{\tilde{C}}) \xrightarrow{\sim} \bigoplus_{k=0}^{n-1} \mathcal{O}_{\tilde{C}}\left(-\frac{1}{n}\gamma^{-1}(D_k)\right).$$

Now, one checks that the morphism (6.3) is given by the divisors $-(1/n)\gamma^{-1}(D_k)$, in particular, $\mathrm{supp}(T') \subseteq \bigcup_k \mathrm{supp}\,\gamma^{-1}(D_k) = \mathrm{supp}\, R^\gamma$.

It only remains to compute the length of T' at a ramification point. Let $p_i \in \gamma^{-1}(q_i)$ be given.

The length of the cokernel of the sequence (6.3) at p_i is given by

$$\sum_{k=1}^{n-1} [ka_i]_n \frac{m_i}{n} = \frac{m_i s_i}{n} \sum_{k=1}^{m_i - 1} [ka_i]_n = s_i \sum_{k'=1}^{m_i-1} [k']_n = s_i \frac{m_i(m_i-1)}{2} = \frac{n(m_i-1)}{2}.$$

Thus, $T' = (1/2)nR^\gamma$ and the conclusion follows. Observe that the coefficients of nR^γ are even. $\square$

Fix a line bundle L_γ on $\tilde{C}$ satisfying

$$L_\gamma := \mathcal{O}_{\tilde{C}}(\tfrac{1}{2}(nR^\gamma) - mR^\gamma) \quad \text{if } \; n = 2m+1,$$
$$L_\gamma \text{ such that } \; L_\gamma^{\otimes 2} \simeq \mathcal{O}_{\tilde{C}}(R^\gamma) \quad \text{if } \; n = 2m.$$

Then, the following two conditions hold:

$$L_\gamma^{\otimes 2} \simeq \mathcal{O}_{\tilde{C}}(R^\gamma), \qquad L_\gamma^{\otimes n} \simeq \mathcal{O}_{\tilde{C}}(\tfrac{1}{2}(nR^\gamma))$$

and L_γ has degree $(\tilde{g}-1) - n(g-1)$.

We also fix theta characteristics η on C and $\tilde{\eta}$ on $\tilde{C}$, where $\tilde{\eta}$ is defined by $\mathcal{O}_{\tilde{C}}(\tilde{\eta}) := \gamma^* \mathcal{O}_C(\eta) \otimes L_\gamma$.

Let $\tilde{d} := \tilde{r}(\tilde{g} - 1 - n(g-1))$. Since p.g.c.d.$(\tilde{r}, \tilde{d}) = \tilde{r}$, the line bundle $F = L_\gamma^*$ may be used to define a polarization $\tilde{\Theta}_{[F_{\tilde{\eta}}]}$ in $\mathcal{U}_{\tilde{C}}(\tilde{r}, \tilde{d})$.

Note that the theta characteristic η also defines a polarization $\Theta_{[\eta]}$ on the moduli space $\mathcal{U}_C(r, 0)$.

Assume, we are given a vector bundle $\tilde{M} \in \mathcal{U}_{\tilde{C}}(\tilde{r}, \tilde{d})$ whose direct image is a semistable vector bundle on C. Then, Lemma 6.2 implies that $\gamma_* \tilde{M} \in \mathcal{U}(r, 0)$, where $r := n \cdot \tilde{r}$. Further, we have the morphisms:

$$\tilde{\alpha}_{\tilde{M}} : \tilde{C}^{2nm} \to \mathcal{U}_{\tilde{C}}(\tilde{r}, \tilde{d}), \qquad (\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_{nm}, \tilde{y}_{nm}) \mapsto \tilde{M}\left(\sum_i \tilde{x}_i - \tilde{y}_i\right)$$

and

$$\alpha_{\gamma_* \tilde{M}} : C^{2m} \to \mathcal{U}_C(r, 0), \qquad (x_1, y_1, \ldots, x_m, y_m) \mapsto \gamma_* \tilde{M} \otimes \mathcal{O}_C\left(\sum_i x_i - y_i\right).$$

In order to study the relation of the corresponding non-Abelian theta functions, we consider the following diagram:

$$\begin{tikzcd}
 & (\Pi^n(\tilde{C} \times \tilde{C}))^m = \tilde{C}^{2nm} \arrow[r, "\tilde{\alpha}_{\tilde{M}}"] & \mathcal{U}_{\tilde{C}}(\tilde{r}, \tilde{d}) \\
(\tilde{C} \times \tilde{C})^m \arrow[ur, "\rho_m"] \arrow[dr, "\gamma_m"'] & & \\
 & (C \times C)^m = C^{2m} \arrow[r, "\alpha_{\gamma_* \tilde{M}}"] & \mathcal{U}_C(r, 0)
\end{tikzcd}$$

where γ_m denotes the map $\tilde{C}^{2m} \to C^{2m}$ given by γ on each component, and ρ_m is the embedding induced by the morphism

$$\rho_1 : \tilde{C} \times \tilde{C} \to \prod^n (\tilde{C} \times \tilde{C}), \qquad (\tilde{x}, \tilde{y}) \mapsto (\tilde{x}, \tilde{y}, \sigma(\tilde{x}), \sigma(\tilde{y}), \ldots, \sigma^{n-1}(\tilde{x}), \sigma^{n-1}(\tilde{y})).$$

Let us denote by p_i (resp. $\tilde{p}_i$) the projection of C^{2m} (resp. $\tilde{C}^{2m}$) onto its ith factor.

The following theorem gives the relation between the pullbacks of the polarizations by the above diagram.

Theorem 6.3. *There is an isomorphism of line bundles on* $\tilde{C}^{2m}$:

$$(\alpha_{\gamma_*\tilde{M}} \circ \gamma_m)^*\mathcal{O}(\Theta_{[\eta]}) \simeq \rho_m^*((\tilde{\alpha}_{\tilde{M}})^*\mathcal{O}(\tilde{\Theta}_{[F_{\tilde{\eta}}]})) \otimes \left(\bigotimes_{i \text{ all}} \tilde{p}_i^* L_\gamma^*\right)^{\otimes 3r},$$

where $F = L_\gamma^*$ *and* $r = \tilde{r}n$.

Proof. The statement follows from the comparison of the pullbacks $(\alpha_{\gamma_*\tilde{M}} \circ \gamma_m)^*\mathcal{O}(\Theta_{[\eta]})$ and $(\tilde{\alpha}_{\tilde{M}} \circ \rho_m)^*\mathcal{O}(\tilde{\Theta}_{[F_{\tilde{\eta}}]})$, which will be done with the help of Theorem 3.6 and Lemmas 6.1 and 6.2.

To begin with, we compute the pullback by γ_m of $(\alpha_{\gamma_*\tilde{M}})^*\mathcal{O}(\Theta_{[\eta]})$. Note that $L_\gamma^*(-\tilde{\eta})$ is invariant by σ. Therefore, by Lemma 6.2 and the properties of L_γ, one has that

$$\begin{aligned}
&\gamma_m^*\left(\bigotimes_{i \text{ odd}} p_i^* \wedge (\gamma_*\tilde{M} \otimes \mathcal{O}(-\eta))^*\right)\\
&\quad\simeq \bigotimes_{i \text{ odd}} \tilde{p}_i^*(\gamma^* \wedge (\gamma_*\tilde{M} \otimes \mathcal{O}(-\eta))^*) \simeq \bigotimes_{i \text{ odd}} \tilde{p}_i^* \wedge (\gamma^*\gamma_*\tilde{M} \otimes \gamma^*\mathcal{O}(-\eta))^*\\
&\quad\simeq \bigotimes_{i \text{ odd}} \tilde{p}_i^*(\wedge\gamma^*\gamma_*\tilde{M} \otimes L_\gamma(-\tilde{\eta})^{\otimes r})^*\\
&\quad\simeq \bigotimes_{i \text{ odd}} \tilde{p}_i^*\left(\left(\bigotimes_{j=0}^{n-1} \wedge (\sigma^j)^*\tilde{M}\right) \otimes \mathcal{O}\left(-\frac{1}{2}rR^\gamma\right) \otimes L_\gamma(-\tilde{\eta})^{\otimes r}\right)^*.
\end{aligned}$$

Recalling that $(L_\gamma^*)^{\otimes r} \simeq \mathcal{O}(-(1/2)rR^\gamma)$ and that $L_\gamma^*(-\tilde{\eta})$ is invariant under σ, the above expression is isomorphic to

$$\begin{aligned}
&\bigotimes_{i \text{ odd}} \tilde{p}_i^*\left(\left(\bigotimes_{j=0}^{n-1} \wedge (\sigma^j)^*\tilde{M}\right) \otimes L_\gamma^*(-\tilde{\eta})^{\otimes r} \otimes L_\gamma^{\otimes r}\right)^*\\
&\quad\simeq \bigotimes_{i \text{ odd}} \tilde{p}_i^*\left(\left(\bigotimes_{j=0}^{n-1} \wedge (\sigma^j)^*(\tilde{M} \otimes L_\gamma^*(-\tilde{\eta}))\right) \otimes L_\gamma^{\otimes r}\right)^*\\
&\quad\simeq \rho_m^*\left(\bigotimes_{i \text{ odd}} \bar{p}_i^* \wedge (\tilde{M} \otimes L_\gamma^*(-\tilde{\eta}))^*\right) \otimes \left(\bigotimes_{i \text{ odd}} \tilde{p}_i^*(L_\gamma^*)^{\otimes r}\right),
\end{aligned}$$

where $\bar{p}_i$ are the natural projections of $\tilde{C}^{2nm}$.

Similarly, the pullback of

$$\bigotimes_{i \text{ even}} p_i^* \wedge (\gamma_*\tilde{M} \otimes \mathcal{O}(\eta))$$

by γ_m^* is

$$\rho_m^*\left(\bigotimes_{i \text{ even}} \bar{p}_i^* \wedge (\tilde{M} \otimes L_\gamma^*(\tilde{\eta}))\right) \otimes \left(\bigotimes_{i \text{ even}} \tilde{p}_i^*(L_\gamma^*)^{\otimes r}\right).$$

Note that $n\sum_i \tilde{D}_{0i} = \sum_{i,j}\tilde{D}_{ij}$ and $\rho_1^{-1}(\tilde{\Delta}_{ij}) = \tilde{D}_{ij}$. Then, from Lemma 6.1, it follows an isomorphism on $\tilde{C}\times\tilde{C}$

$$\begin{aligned}\gamma_1^*\mathcal{O}(n\Delta) &\simeq \rho_1^*\mathcal{O}\left(\sum_{i,j}^{n}\tilde{\Delta}_{ij}\right)\otimes \tilde{p}_1^*\mathcal{O}(-nR^\gamma)\otimes \tilde{p}_2^*\mathcal{O}(-nR^\gamma)\\ &\simeq \rho_1^*\mathcal{O}\left(\sum_{i,j}^{n}\tilde{\Delta}_{ij}\right)\otimes \tilde{p}_1^*(L_\gamma^*)^{\otimes 2n}\otimes \tilde{p}_2^*(L_\gamma^*)^{\otimes 2n}.\end{aligned}$$

Finally, a lengthy but straightforward calculation shows that

$$\begin{aligned}\gamma_m^*\mathcal{O}\left(\sum_{\substack{i<j\\ i+j=\text{odd}}}\Delta_{ij} - \sum_{\substack{i<j\\ i+j=\text{even}}}\Delta_{ij}\right)^{\otimes r} \simeq \rho_m^*\mathcal{O}\left(\sum_{\substack{i<j\\ i+j=\text{odd}}}\tilde{\Delta}_{ij} - \sum_{\substack{i<j\\ i+j=\text{even}}}\tilde{\Delta}_{ij}\right)^{\otimes\tilde{r}}\\ \otimes\left(\bigotimes_{i\text{ all}}\tilde{p}_i^*(L_\gamma^*)^{\otimes 2r}\right).\end{aligned}$$

Comparing these results with the expression of $\rho_m^*(\tilde{\alpha}_{\tilde{M}}^*\mathcal{O}(\tilde{\Theta}_{[F_{\tilde{\eta}}]}))$ given by Theorem 3.6, the statement follows. □

Observe that $L_\gamma = \mathcal{O}_{\tilde{C}}$ when γ is non-ramified. Then, in this situation, a consequence of the above theorem is the following identity between global sections of the line bundles in the previous statement.

Theorem 6.4. *Let γ be non-ramified and $\tilde{M}\in\mathcal{U}(\tilde{r},\tilde{d})-\tilde{\Theta}_{\tilde{\eta}}$ such that $\gamma_*\tilde{M}\in\mathcal{U}(r,0)$. Then, for $(\tilde{x}_1,\tilde{y}_1,\dots,\tilde{x}_m,\tilde{y}_m)\in\tilde{C}^{2m}$, it holds that*

$$\frac{\theta_\eta(\gamma_*\tilde{M}\otimes\mathcal{O}(Z))}{\theta_\eta(\gamma_*\tilde{M})} = \frac{\tilde{\theta}_{\tilde{\eta}}(\tilde{M}\otimes\gamma^*\mathcal{O}(Z))}{\tilde{\theta}_{\tilde{\eta}}(\tilde{M})},$$

where Z is the divisor $\sum_{i=1}^m(\gamma(\tilde{x}_i)-\gamma(\tilde{y}_i))$ on C.

Proof. First of all, observe that

$$\tilde{\theta}_{\tilde{\eta}}(\tilde{M}\otimes\gamma^*\mathcal{O}(Z)) = \tilde{\theta}_{\tilde{\eta}}(\tilde{\alpha}_{\tilde{M}}(\rho_m(\tilde{x}_1,\tilde{y}_1,\dots,\tilde{x}_m,\tilde{y}_m))),$$

because $\gamma^{-1}(\gamma(\tilde{x}_i)) = \rho_1(\tilde{x}_i)$. Then, the r.h.s. of the formula is a holomorphic global section of $\rho_m^*(\tilde{\alpha}_{\tilde{M}}^*\mathcal{O}(\tilde{\Theta}_{[\tilde{\eta}]}))$. On the other hand, the l.h.s. is a holomorphic global section of $\gamma_m^*(\alpha_{\gamma_*\tilde{M}}^*\mathcal{O}(\Theta_{[\eta]}))$. Hence, by Theorem 6.3, both sides are global sections of isomorphic line bundles on $\tilde{C}^{2m}$.

Similar arguments to those of the proof of Theorem 3.8 reduce the proof to check that the statement holds true when restricted to a fibre $\pi^{-1}(z)$, where $\pi:\tilde{C}^{2m}\to\tilde{C}^{2m-1}$ is the projection that forgets $\tilde{y}_1$ and z is a point $(\tilde{x}_1, \tilde{x}_2, \tilde{y}_2, \dots, \tilde{x}_m, \tilde{y}_m) \in \tilde{C}^{2m-1}$ such that $x_i \neq x_j$ for all $i \neq j$ and $y_i = x_i$. For the sake of notation, we define x to be $\gamma(\tilde{x}_1)$.

Let us denote by $\tilde{p}$ and $\tilde{q}$ the projections of $\tilde{C} \times \tilde{C}$ onto its first and second factors, respectively. Consider the bundle

$$\tilde{\mathcal{M}} := \tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \otimes \mathcal{O}(-\tilde{D})$$

on $\tilde{C} \times \tilde{C}$, where $\tilde{D} := \sum_j \tilde{D}_{0j}$. Using the sequence defined by the effective divisor $\tilde{D}$, we obtain the following exact sequence:

$$0 \to \tilde{\mathcal{M}} \to \tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \to \tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \otimes \mathcal{O}_{\tilde{D}} \to 0. \tag{6.4}$$

Since $R^1 q_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta}))) = 0$, the restriction of the r.h.s. to $\pi^{-1}(z)$ is given by the determinant of the morphism

$$\tilde{q}_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta}))) \to \tilde{q}_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \otimes \mathcal{O}_{\tilde{D}})$$

induced by the latter sequence.

We now compute the restriction of the l.h.s. in a similar way. Let us denote by p and q the projections of $C \times \tilde{C}$ onto its first and second factors, respectively. Let $\mathcal{M}$ be the bundle $p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta)) \otimes \mathcal{O}(-\Gamma)$ on $C \times \tilde{C}$, where Γ is the graph of the map γ. The exact sequence associated to the divisor Γ implies the exactness of the sequence

$$0 \to \mathcal{M} \to p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta)) \to p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta)) \otimes \mathcal{O}_\Gamma \to 0. \tag{6.5}$$

Being $R^1 q_*(p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta))) = 0$, it follows that the restriction of the l.h.s. to $\pi^{-1}(z)$ is the determinant of the induced morphism:

$$q_*(p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta))) \to q_*(p^*(\gamma_* \tilde{M} \otimes \mathcal{O}(x + \eta)) \otimes \mathcal{O}_\Gamma).$$

Bearing in mind the commutativity of the diagram:

$$\begin{tikzcd}
\tilde{C} \times \tilde{C} \arrow[rr, "\gamma \times \mathrm{Id}"] \arrow[dr, "\tilde{q}"'] & & C \times \tilde{C} \arrow[dl, "q"] \\
& \tilde{C} &
\end{tikzcd}$$

it will suffice to conclude to show that the direct image by $\gamma \times \mathrm{Id}$ of the sequence (6.4) is the sequence (6.5).

The direct image of the sequence (6.4) by $\gamma \times \mathrm{Id}$ is

$$\begin{aligned} 0 &\to (\gamma \times \mathrm{Id})_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \otimes \mathcal{O}(-\tilde{D})) \to (\gamma \times \mathrm{Id})_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta}))) \\ &\to (\gamma \times \mathrm{Id})_*(\tilde{p}^*(\tilde{M}(\gamma^{-1}(x) + \tilde{\eta})) \otimes \mathcal{O}_{\tilde{D}}) \to 0, \end{aligned}$$

because the map $\gamma \times \mathrm{Id}$ is finite.

Recalling that $(\gamma \times \mathrm{Id})^{-1}(\Gamma) = \tilde{D}$, $\gamma^{-1}(\eta) = \tilde{\eta}$ and using the projection formula and the base change theorem for the case:

$$\begin{array}{ccc} \tilde{C} \times \tilde{C} & \xrightarrow{\gamma \times \mathrm{Id}} & C \times \tilde{C} \\ \tilde{p} \downarrow & & \downarrow p \\ \tilde{C} & \xrightarrow{\gamma} & C \end{array}$$

we conclude that the latter sequence coincides with the sequence (6.5).

Then, we know that both sections are equal up to a constant on $\tilde{C}^{2m}$. This constant might be evaluated on $\pi^{-1}(z)$ by letting $\tilde{y}_1 = \tilde{x}_1$, and it follows that it is equal to 1. □

Remark 6.5. Observe that the above theorem may be generalized for the ramified case. This would require to know sections of $L_\gamma^{\otimes n} = \mathcal{O}((1/2)nR^\gamma)$. Besides, Lemma 6.1 allows us to give a section of it in terms of the prime forms E and $\tilde{E}$ because $(1/2)nR^\gamma$ is effective.

We finish with a similar study for the inverse image. Let $M \in \mathcal{U}_C(r, 0)$ be a vector bundle on C. From Lemma 3.2.2 of [9] it turns out that $\gamma^* M \in \mathcal{U}_{\tilde{C}}(r, 0)$. Consider the following diagram:

$$\begin{array}{ccc} \tilde{C}^{2m} & \xrightarrow{\tilde{\alpha}_{\gamma^* M}} & \mathcal{U}_{\tilde{C}}(r,0) \\ \gamma_m \downarrow & & \\ C^{2m} & \xrightarrow{\alpha_M} & \mathcal{U}_C(r,0) \end{array}$$

Theorem 6.6. *It holds that*

$$(\alpha_M \circ \gamma_m)^* \mathcal{O}(\Theta_{[\eta]}) \otimes \gamma_m^* \mathcal{O}\left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij}\right)^{\otimes r}$$
$$\simeq (\tilde{\alpha}_{\gamma^* M})^* \mathcal{O}(\tilde{\Theta}_{[\tilde{\eta}]}) \otimes \mathcal{O}\left(\sum_{i+j=\text{odd}} \tilde{\Delta}_{ij} - \sum_{i+j=\text{even}} \tilde{\Delta}_{ij}\right)^{\otimes r} \otimes \left(\bigotimes_{i \text{ all}} (\tilde{p}_i^* L_\gamma^*)^{\otimes r}\right).$$

Proof. The claim follows from Theorem 3.6 and from the fact that $\gamma^* \mathcal{O}_C(\eta) \simeq \mathcal{O}_{\tilde{C}}(\tilde{\eta}) \otimes L_\gamma^*$. □

Proposition 6.7. *Suppose that γ is non-ramified and let $M \in \mathcal{U}_C(r, 0)$ such that $\theta_\eta(M) \neq 0$. Then, for $(\tilde{x}_1, \tilde{y}_1, \ldots, \tilde{x}_m, \tilde{y}_m) \in \tilde{C}^{2m}$, it holds that*

$$\frac{\tilde{\theta}_{\tilde{\eta}}\left(\gamma^* M\left(\sum_{i=1}^m (\tilde{x}_i - \tilde{y}_i)\right)\right)}{\tilde{\theta}_{\tilde{\eta}}(\gamma^* M)} \prod_{i<j} \prod_{k=1}^{n-1} (\tilde{E}(\tilde{x}_i, \sigma^k(\tilde{x}_j)) \tilde{E}(\tilde{y}_i, \sigma^k(\tilde{y}_j)))^r$$
$$= \frac{\theta_\eta\left(M\left(\sum_{i=1}^m (\gamma(\tilde{x}_i) - \gamma(\tilde{y}_i))\right)\right)}{\theta_\eta(M)} \prod_{i,j} \prod_{k=1}^{n-1} \tilde{E}(\tilde{x}_i, \sigma^k(\tilde{y}_j))^r.$$

Proof. One proceed similarly as in Theorem 6.4. □

Remark 6.8. It is worth pointing out that Proposition 5.1 of [4] follows from Theorem 6.6 when γ is ramified, $\deg \gamma = 2$ and $r = 1$.

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