STRATEGIC STABILITY IN POISSON GAMES*

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ABSTRACT. In Poisson games, an extension of perfect equilibrium based on perturbations of the strategy space does not guarantee that players use admissible actions. This observation suggests that such a class of perturbations is not the correct one. We characterize the right space of perturbations to define perfect equilibrium in Poisson games. Furthermore, we use such a space to define the corresponding strategically stable sets of equilibria. We show that they satisfy existence, admissibility, and robustness against iterated deletion of dominated strategies and inferior replies.

KEY WORDS. Poisson games, voting, perfect equilibrium, strategic stability, stable sets.

JEL CLASSIFICATION. C63, C70, C72.

1. INTRODUCTION

Poisson games (Myerson [30]) belong to the broader class of games with population uncertainty (Myerson [30], Milchtaich [29]). Not only have these games been used to model voting behavior but also more general economic environments (see, e.g., Satterthwaite and Shneyerov [35], Makris [23, 24], Ritzberger [34], McLennan [25], Jehiel and Lamy [17]). In these models, every player is unaware about the exact number of other players in the population. Each player in the game, however, has probabilistic information about it and, given some beliefs about how the members of such a population behave, can compute the expected payoff that results from each of her available choices. Hence, a Nash equilibrium in this context is a description of behavior for the entire population that is consistent with the players' utility maximizing actions given that they use such a description to form their beliefs about the population's expected behavior.

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Similarly to standard normal form and extensive form games, one can easily construct examples of Poisson games where not every Nash equilibrium is a plausible description of rational behavior. In particular, Nash equilibria in Poisson games can be in dominated strategies. Indeed, many applications of Poisson games (see, e.g., Myerson [31], Maniquet and Morelli [22], Bouton and Castanheira [6], Núñez [32]; among others) focus on undominated strategies in their analysis. In addition, there are also examples in the applied literature of Poisson games that use some other kind of refinements (Hughes [16], Bouton [5], Bouton and Gratton [7]). Hence, it seems worthwhile exploring, also in games with population uncertainty, what can be said from a theoretical standpoint about which Nash equilibria are the most reasonable and to propose a definition that selects such equilibria for us.

Following the main literature on equilibrium refinements, we start focusing our attention on *admissibility*. That is, the principle prescribing players not to play dominated strategies (Luce and Raiffa [21, p.287, Axiom 5]). Furthermore, as in Kohlberg and Mertens [19], we also require that the solution be robust against iterated deletion of dominated actions. Unfortunately, as it is already well known, such an iterative process can lead to different answers depending on which order is chosen to eliminate the dominated strategies. The response to this caveat is defining a set-valued solution concept and requiring that every solution to a Poisson game contain a solution to any game that can be obtained by eliminating dominated strategies. Of course, a definition of such a concept for Poisson games should be guided by the literature on Strategic Stability for finite games (Kohlberg and Mertens [19], Mertens [27, 28], Hillas [14], Govindan and Wilson [13]). In broad terms, a strategically stable set is a subset of Nash equilibria that is robust against every element in some given space of perturbations. The choice of such a space determines the properties that the final concept satisfies and the perturbations are just a means of obtaining the game theoretical properties that we desire (Kohlberg and Mertens [19, p. 1005, footnote 3]). As argued above, a strategically stable set of equilibria should only contain undominated strategies. Furthermore, it should always contain a strategically stable set of any game obtained by eliminating a dominated strategy. However, De Sinopoli and Pimienta [10] show that the main instrument used to define strategic stability in normal form games—i.e. Nash equilibria of strategy perturbed games—fails to guarantee that players only use undominated strategies when applied to Poisson games.

Thus, before defining strategically stable sets of equilibria in Poisson games we need to find the appropriate space of perturbations that guarantees that every member of the stable set is undominated. It turns out that the “right” space of perturbations is of the same nature as the one used in infinite normal-form games (Simon and Stinchcombe [37], Al-Najjar [1], Carbonell-Nicolau [9]) and different from the one used in finite games (Selten [36]) even if players have finite action sets. Once this class of perturbations has been identified, it can be reinterpreted as a collection of perturbations of the best response correspondence. Then, a stable set is defined as a minimal subset of fixed points of the best response correspondence with the property that every correspondence that can be obtained using such perturbations has a fixed point close to it.

As an illustration of stable sets in Poisson games we construct a referendum game with a threshold for implementing a new policy (see Example 5). In this example, every voter prefers the new policy over the status quo but some voters incur a cost in supporting it. Given the parametrization that we use, the game has three equilibria which can be ranked according to the probability of implementation of the new policy: *zero*, *low*, and *high*. We show that the first equilibrium is dominated because, in particular, voters who do not incur any cost do not support the new policy. In the second equilibrium, only voters who incur the cost do not support the new policy, even if they are indifferent between supporting it or not. Furthermore, every such a voter would strictly prefer supporting the new policy and paying the cost if the share of voters supporting the new policy was slightly higher than the equilibrium one. We show that this equilibrium is undominated and perfect but becomes unstable once dominated strategies are eliminated. Hence, the unique stable set of the game is the equilibrium in which the new policy is implemented with *high* probability.

We review the general description of Poisson games in the next Section. We then discuss the admissibility postulate in Section 3 and the definition of perfection in Section 4. The space of perturbations used to define perfect equilibria is used to describe, in Section 5, the stable sets of equilibria in Poisson games. We show that they satisfy existence, admissibility and iterated deletion of dominated strategies. Section 6 contains some applications of stability. In the Appendix we show that, in generic Poisson games, every Nash equilibrium is a singleton stable set.

2. Preliminaries

We begin fixing a *Poisson game* $\Gamma \equiv (n, \mathcal{T}, r, C, (C_t)_{t\in\mathcal{T}}, u)$. The number of *players* is distributed according to a Poisson random variable with parameter n. Hence, the probability that there are k players in the game is equal to

$$P(k \mid n) = \frac{e^{-n} n^k}{k!}.$$

The set $\mathscr{T} = \{1,\dots,T\}$ is the set of player types. The probability that a randomly selected player is of each type is given by the vector $r = (r_1,\dots,r_T) \in \Delta(\mathscr{T})$.[1] That is, a player is of type $t \in \mathscr{T}$ with probability r_t.

The finite set of *actions* is C. However, we allow that not every action be available to type t players. The set of actions that are in fact available to players of type t is $C_t \subset C$.[2] An action profile $x \in Z(C)$ specifies for each action $c \in C$ the number of players $x(c)$ that have chosen that action. The set of *action profiles* is $Z(C) \equiv \mathbb{Z}_+^C$. Players' preferences in the game are summarized by $u = (u_1,\dots,u_T)$. It is assumed that each function $u_t : C_t \times Z(C) \to \mathbb{R}$ be bounded. We interpret $u_t(c,x)$ as the payoff accrued by a type t player when she chooses action c and the realization resulting from the rest of the population's behavior is the action profile $x \in Z(C)$.

The set of *mixed actions* for players of type t is $\Delta(C_t)$. If $\alpha \in \Delta(C_t)$ the *carrier* of α is the subset $\mathscr{C}(\alpha) \subset C_t$ of pure actions that are given strictly positive probability by α. We identify the mixed action that attaches probability one to action $c \in C$ with the pure action c. As in Myerson [30], a *strategy function* σ is an element of $\Sigma \equiv \left\{\sigma \in \Delta(C)^{\mathscr{T}} : \sigma_t \in \Delta(C_t) \text{ for all } t\right\}$. That is, a strategy function maps types to the set of mixed actions available to the corresponding type. We always write strategy functions as bracketed arrays $(\sigma_1,\dots,\sigma_T)$ where $\sigma_t \in \Delta(C_t)$ for $t = 1,\dots,T$. Furthermore, we may also refer to strategy functions simply as *strategies*. The "average" behavior induced by the strategy function σ is represented by $\tau(\sigma) \in \Delta(C)$ and it is defined by $\tau(\sigma)(c) \equiv \sum_{t\in\mathscr{T}} r(t)\sigma_t(c)$. Construct the set $\tau(\Sigma) \equiv \{\tilde\tau \in \Delta(C) : \tilde\tau = \tau(\sigma) \text{ for some } \sigma \in \Sigma\}$.[3] When the population's aggregate behavior is summarized by $\tau \in \tau(\Sigma)$, the probability that the action profile $x \in Z(C)$ is realized is equal to

$$\mathbf{P}(x \mid \tau) \equiv \prod_{c\in C} \left(e^{-n\tau(c)} \frac{(n\tau(c))^{x(c)}}{x(c)!} \right).$$

The expected payoff to a type t player who plays $c \in C_t$ is computed as usual,

$$U_t(c,\tau) \equiv \sum_{x\in Z(C)} \mathbf{P}(x \mid \tau) u_t(c,x). \tag{2.1}$$

[1] For any finite set K we write $\Delta(K)$ for the set of probability distributions on K.

[2] Given two sets E and F, we use the expression $E \subset F$ allowing for set equality.

[3] In the usual description of Poisson games (Myerson [30]), the set $\tau(\Sigma)$ coincides with $\Delta(C)$ because every type has the same action set. We need to relax this assumption because after eliminating dominated actions different types can end up with different action sets. In these cases, $\tau(\Sigma)$ is a subset of $\Delta(C)$. Take as an example a plurality voting game with three candidates (a, b and c) and two types of voters, each type of voter having the same ex-ante probability. Let type 1 voters have preferences $a \succ_1 b \succ_1 c$ and let type 2 voters have preferences $c \succ_2 b \succ_2 a$. There is no cost of voting and abstention is not possible. Consider the game obtained after eliminating dominated actions so that type 1 voters *cannot* vote for candidate c and type 2 voters *cannot* vote for candidate a. In this game, $\tau(\Sigma) \equiv \{(\tilde\tau(a), \tilde\tau(b), \tilde\tau(c)) \in \Delta(C) : \tilde\tau(a), \tilde\tau(c) \le 1/2\}$.

Note that, for each type $t \in \mathcal{T}$, each action $c \in C_t$ defines a bounded and continuous function $U_t(c,\cdot) : \Delta(C) \to \mathbb{R}$.

Action $c \in C_t$ is a *pure best response* against $\tau \in \Delta(C)$ for players of type t if $c \in \arg\max_{c' \in C_t} U_t(c',\tau)$. The finite set of such actions is written $\mathrm{PBR}_t(\tau)$. The set of *best responses* against τ is $\mathrm{BR}_t(\tau) \equiv \Delta(\mathrm{PBR}_t(\tau))$. We write $\mathrm{BR}(\tau) \subset \Sigma$ for the collection of strategy functions σ that satisfy $\sigma_t \in \mathrm{BR}_t(\tau)$ for every t.

Definition 1 (Nash equilibrium)**.** The strategy function σ is a Nash equilibrium of the Poisson game Γ if $\sigma \in \mathrm{BR}(\tau(\sigma))$.

Since Σ is compact and convex and $\mathrm{BR} \circ \tau$ is upper semicontinuous and convex valued, every Poisson game has a Nash equilibrium (Myerson [30]). Furthermore, once we fix n, $\mathcal{T}$, r, C and $(C_t)_{t \in \mathcal{T}}$, standard arguments show that the Nash equilibrium correspondence (mapping utilities to equilibria) is upper semicontinuous.

3. Admissibility

Consider a referendum where voters have only two options, voting *yes* or *no* to some policy question. For the policy to be implemented the law requires that at least $K > 1$ voters vote *yes*, otherwise the policy is not implemented. Every voter in the game wants the policy to be implemented. The strategy that prescribes every player to vote *no* is a Nash equilibrium, however, it is clear that such a strategy is dominated. Similar examples can be easily constructed.

We now introduce the standard concept of dominated actions and dominated strategies.

Definition 2 (Dominated actions)**.** Action α is dominated by β for players of type t if $U_t(\alpha,\tau) \le U_t(\beta,\tau)$ for every $\tau \in \tau(\Sigma)$ and $U_t(\alpha,\tau') < U_t(\beta,\tau')$ for some $\tau' \in \tau(\Sigma)$.

That is, an action α is dominated if there is another action such that, *regardless of what other players do*, always gives higher utility than α and, sometimes, strictly higher. We say that an action α is *strictly dominated* by β if the inequality is strict for every $\tau \in \tau(\Sigma)$. Following from this concept, there is a definition of dominated strategies.

Definition 3 (Dominated strategies)**.** The strategy function σ is dominated if there is a $t \in \mathcal{T}$ such that σ_t is a dominated action for players of type t.

Likewise, a strategy function is *strictly dominated* if it prescribes a strictly dominated action for some type. De Sinopoli and Pimienta [10] prove that every Poisson game has a Nash equilibrium in undominated strategies.

In an attempt to capture undominated behavior, we can also give a straightforward extension of the definition of perfection to Poisson games. If E is a finite set, let us denote by $\Delta^\circ(E)$ the set of completely mixed probability distributions on E. This is the set of distributions that give strictly positive probability to every element in E. We now define a perturbation as a pair $(\varepsilon,\sigma^\circ)$ where $\varepsilon > 0$ and σ° is a completely mixed, i.e., a strategy function such that $\sigma^\circ_t \in \Delta^\circ(C_t)$ for every $t \in \mathscr{T}$. In a perturbed game and under the perturbation $(\varepsilon,\sigma^\circ)$, if the strategy function σ is played then, for each type t, the action σ_t is substituted by $(1-\varepsilon)\sigma_t + \varepsilon\sigma^\circ_t$. Given a *strategy-perturbation* $(\varepsilon,\sigma^\circ)$ we denote the corresponding *strategy-perturbed Poisson game* by $\Gamma_{\varepsilon,\sigma^\circ}$. We can now give the usual definition of perfect equilibrium. For the time being, we call it *inner-perfection*.

Definition 4 (Inner-perfection)**.** The strategy function σ is an *inner-perfect equilibrium* if there is a sequence of perturbations $\{(\varepsilon^k,\sigma^k)\}_k$ and a sequence of strategy functions $\{\varsigma^k\}_k$ such that $\{\varepsilon^k\}_k$ converges to zero, $\{\varsigma^k\}_k$ converges to σ, and ς^k is a Nash equilibrium of $\Gamma_{\varepsilon,\sigma^k}$ for every k.

Using standard arguments, De Sinopoli and Pimienta [10] show that every Poisson game has an inner-perfect equilibrium and that the usual alternative definitions (based on, e.g., ε-perfect equilibria) are also equivalent in the context of Poisson games. It is also showed there that, contrary to well known results for normal form games, inner-perfect equilibria can be in dominated strategies. The following example illustrates why.

Example 1. Let Γ be a Poisson game with expected number of players equal to $n = 2$, set of types $\mathscr{T} = \{1\}$, set of actions $C = \{a,b\}$, and utility function

$$\begin{aligned} u(a,x) &= \quad e^{-2} \quad \text{for every } x \in Z(C),\\ u(b,x) &= \begin{cases} 1 & \text{if } x(a) = x(b) = 1, \text{ and}\\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that e^{-2} is the probability that $x(a) = x(b) = 1$ under the strategy $\sigma = (\frac{1}{2}a + \frac{1}{2}b)$. Also notice that action b is dominated by action a, the former only does as good as the latter against the strategy $\sigma = (\frac{1}{2}a + \frac{1}{2}b)$, and does strictly worse for any other strategy $\sigma' \neq \sigma$. The action $\gamma = \frac{1}{2}a + \frac{1}{2}b$ is also dominated by a. Nevertheless, it is a best response against σ. Finally, since σ is completely mixed, we can conclude that the dominated strategy σ is an inner-perfect equilibrium.

In order to see where the difference with respect to normal form games is coming from, it is useful to plot how the players' utility varies as the opponents change their behavior. We do that in Figure 1, where we represent utilities with respect to the probability attached to action a by an average member of

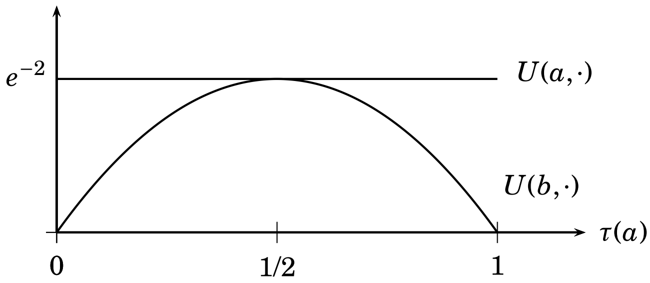

FIGURE 1. Utility functions in Example 1.

the population. (There is only one type of player so, in this example, the sets Σ and $\tau(\Sigma)$ coincide.) The first thing to notice is that $U(b,\cdot)$ is not linear in Σ and that it attains its maximum at the completely mixed strategy $\sigma = (\frac{1}{2}a + \frac{1}{2}b)$. At that point, $U(a,\cdot)$ coincides with $U(b,\cdot)$. If we were to integrate $U(a,\cdot)$ and $U(b,\cdot)$ over the domain of strategies, the integral of $U(a,\cdot)$ would always be larger than the integral of $U(b,\cdot)$. Of course, not only is this true when we integrate with respect to the Lebesgue measure, but also when we integrate with respect to any Borel probability measure that does not give probability one to $\{\sigma\}$. Hence, if we approach σ by an arbitrary sequence of "sufficiently mixed" Borel probability measures over Σ, action b would always be an inferior response to every element of such a sequence. In the next section we formalize and generalize this intuition.

4. PERFECTION

The set $\tau(\Sigma)$ is equipped with the Euclidean distance d, so $(\tau(\Sigma), d)$ is a compact metric space. The distance between τ and an arbitrary subset $A \subset \tau(\Sigma)$ is $d(\tau, A) \equiv \inf\{d(\tau, a) : a \in A\}$.

We let $\mathscr{B}$ denote the σ-algebra of Borel sets in $\tau(\Sigma) \subset \Delta(C)$. The set of all Borel probability measures over the measurable space $(\tau(\Sigma), \mathscr{B})$ is denoted $\mathscr{M}$. We topologize $\mathscr{M}$ with the weak* topology. This topology is characterized by the following: a sequence of measures $\{\mu^k\} \subset \mathscr{M}$ converges (weakly) to μ if for every continuous function $f : \tau(\Sigma) \to \mathbb{R}$ the sequence of real numbers $\int_{\tau(\Sigma)} f \, d\mu^k$ converges to $\int_{\tau(\Sigma)} f \, d\mu$. It can be showed (Billingsley [3, pg. 239]) that $\mathscr{M}$ is a compact metrizable space and that a sequence $\{\mu^k\}$ converges to μ if and only if it converges with respect to the Prokhorov metric.

Let $\delta : \tau(\Sigma) \to \mathscr{M}$ be the function that maps each $\tau \in \tau(\Sigma)$ to the Dirac measure $\delta(\tau) \in \mathscr{M}$ that assigns probability one to $\{\tau\}$. With abuse of notation, if $\sigma \in \Sigma$ we write $\delta(\sigma)$ instead of $\delta(\tau(\sigma))$. Denote by $\mathscr{M}^\circ$ the subset of measures $\mu \in \mathscr{M}$ that satisfy $\mu(O) > 0$ for every nonempty open set $O \subset \tau(\Sigma)$.

We extend the domain of the utility functions to $\mathscr{M}$:

$$\overline{U}_t(c,\mu) \equiv \int_{\tau(\Sigma)} U_t(c,\tau)d\mu.$$

We note the following result about $\overline{U}_t(c,\cdot)$ and skip its proof.

Proposition 1. *The utility functions $\overline{U}_t(c,\cdot):\mathscr{M}\to\mathbb{R}$ are continuous and linear in $\mathscr{M}$.*

Remark 1. Recall that the utility functions $U_t(c,\cdot)$ are continuous but, typically, *not* linear in $\tau(\Sigma)$.

Remark 2. To summarize, we have the following mathematical description. We can think of actions as elements that belong to the set of continuous functions $C[\tau(\Sigma)]$ that map $\tau(\Sigma)$ to $\mathbb{R}$ (i.e. the expected utility function associated with that action). Letting $C[\tau(\Sigma)]^*$ represent the dual space of $C[\tau(\Sigma)]$, the Riesz Representation Theorem tells us that $\mathscr{M}$ is the set of elements in $C[\tau(\Sigma)]^*$ whose norm is equal to one. The topology that we imposed on $\mathscr{M}$ is the one induced by the weak* topology on $C[\tau(\Sigma)]^*$ (that is, the coarsest topology such that every member of $C[\tau(\Sigma)]$, understood as an element of $C[\tau(\Sigma)]^{**}$, is a continuous function that maps $C[\tau(\Sigma)]^*$ to $\mathbb{R}$).

Given any $\mu\in\mathscr{M}$ we write $\overline{\mathrm{PBR}}_t(\mu)$ for the set of actions $c\in C_t$ that maximize $\overline{U}_t(c,\mu)$. As usual, we also define the set of mixed actions $\overline{\mathrm{BR}}_t(\mu)\equiv\Delta(\overline{\mathrm{PBR}}_t(\mu))$. The sets $\overline{\mathrm{PBR}}(\mu)$ and $\overline{\mathrm{BR}}(\mu)$ are defined accordingly. The correspondence $\overline{\mathrm{BR}}$ is upper semicontinuous and convex valued.

The following result follows directly from the definitions.

Proposition 2. *The strategy function σ is a Nash equilibrium of the Poisson game Γ if and only if $\sigma\in\overline{\mathrm{BR}}(\delta(\sigma))$.*

It is convenient to recast the definition of dominated actions using the extension of the utility functions to $\mathscr{M}$. We do so in the next proposition and state it without proof.

Proposition 3. *Action α is dominated by β for players of type t if and only if $\overline{U}_t(\alpha,\mu)\le\overline{U}_t(\beta,\mu)$ for every $\mu\in\mathscr{M}$ and $\overline{U}_t(\alpha,\mu')<\overline{U}_t(\beta,\mu')$ for some $\mu'\in\mathscr{M}$.*

Moreover, an action α is strictly dominated by β if the strict inequality holds for every $\mu\in\mathscr{M}$.

We are now in a position to characterize the set of dominated actions for a given type. The next theorem is reminiscent of classical results that hold in finite normal form games (see Gale and Sherman [12], Bohnenblust et al. [4], Pearce [33]).

Theorem 1. *An action $\alpha \in \Delta(C_t)$ is undominated for a player of type t if and only if there is a $\mu^\circ \in \mathscr{M}^\circ$ such that $\alpha \in \overline{\mathrm{BR}}_t(\mu^\circ)$.*

Proof. If there is a measure μ° that assigns positive probability to every open set in $\tau(\Sigma)$ and $\alpha \in \overline{\mathrm{BR}}_t(\mu^\circ)$ then action α cannot be dominated.

Suppose now that $\alpha \notin \overline{\mathrm{BR}}_t(\mu^\circ)$ for every $\mu^\circ \in \mathscr{M}^\circ$. Fix some $\rho^\circ \in \mathscr{M}^\circ$, some $0 < \varepsilon < 1$, and construct the infinite two-player zero-sum game $\Gamma(t,\alpha,\rho^\circ,\varepsilon) \equiv (\Delta(C_t), \mathscr{M}, V^\varepsilon_{\alpha,\rho^\circ})$ where, for any $\beta \in \Delta(C_t)$ and $\mu \in \mathscr{M}$, player one's payoff function $V^\varepsilon_{\alpha,\rho^\circ}$ is given by:

$$V^\varepsilon_{\alpha,\rho^\circ}(\beta,\mu) \equiv \overline{U}_t(\beta, \varepsilon\rho^\circ + (1-\varepsilon)\mu) - \overline{U}_t(\alpha, \varepsilon\rho^\circ + (1-\varepsilon)\mu).$$

Let $(\beta^\varepsilon, \mu^\varepsilon)$ be a Nash equilibrium of $\Gamma(t,\alpha,\rho^\circ,\varepsilon)$.[4] We have

$$0 = V^\varepsilon_{\alpha,\rho^\circ}(\alpha,\mu^\varepsilon) < V^\varepsilon_{\alpha,\rho^\circ}(\beta^\varepsilon,\mu^\varepsilon) \leq V^\varepsilon_{\alpha,\rho^\circ}(\beta^\varepsilon,\mu) \text{ for every } \mu \in \mathscr{M}. \tag{4.1}$$

The weak inequality follows from player two's Nash equilibrium conditions and the strict inequality follows because α is never a best response against any element in $\mathscr{M}^\circ$. Hence, β^ε dominates α in the zero-sum game $\Gamma(t,\alpha,\rho^\circ,\varepsilon)$. Passing to a subsequence if necessary, consider the limit β^* of $\{\beta^\varepsilon\}$ as ε goes to zero. Define the function V_α as follows:

$$V_\alpha(\beta,\mu) \equiv \overline{U}_t(\beta,\mu) - \overline{U}_t(\alpha,\mu).$$

From (4.1) we know that $V^\varepsilon_{\alpha,\rho^\circ}(\beta^\varepsilon,\mu) > 0$ for every $\mu \in \mathscr{M}$. Hence, by continuity, $V_\alpha(\beta^*,\mu) \geq 0$ for every $\mu \in \mathscr{M}$. Now we only need to find $\mu' \in \mathscr{M}$ such that $V_\alpha(\beta^*,\mu') > 0$.

For ε small enough the carrier $\mathscr{C}(\beta^*)$ is a subset of the carrier $\mathscr{C}(\beta^\varepsilon)$, therefore, for such small values of ε we also have $V^\varepsilon_{\alpha,\rho^\circ}(\beta^*,\mu) > 0$ for every $\mu \in \mathscr{M}$. Since, by definition, $V_\alpha(\beta^*, \varepsilon\rho^\circ + (1-\varepsilon)\mu) = V^\varepsilon_{\alpha,\rho^\circ}(\beta^*,\mu)$, we can conclude that α is dominated by β^* in the original Poisson game. $\square$

This result implies that a definition of perfection that guarantees that players do not play dominated actions needs to be based on elements of the set $\mathscr{M}^\circ$. Hence, we define a *perturbation* as a pair $(\varepsilon,\mu^\circ) \in (0,1) \times \mathscr{M}^\circ$. The interpretation is that with vanishing probability ε, the average behavior of the population is perturbed towards the completely mixed measure μ°. Thus, a Nash equilibrium of such a perturbed game is a strategy function σ that satisfies $\sigma \in \overline{\mathrm{BR}}((1-\varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$. Moreover, a strategy function satisfies this property

[4] There is always a Nash equilibrium. In particular, if d_p is the Prokhorov metric on $\mathscr{M}$, Prokhorov's Theorem implies that $(\mathscr{M}, d_p)$ is a compact metric space because $(\tau(\Sigma), d)$ is also a compact metric space (Billingsley [3, pg. 37]). The set $\mathscr{M}$ is also a nonempty and convex subset of a normed vector space. Furthermore, the payoff function $V^\varepsilon_{\alpha,\rho^\circ}$ is linear in both arguments, making the associated best response correspondence convex valued (and upper semicontinuous). Existence of Nash equilibrium follows from the Fan-Glicksberg fixed point theorem.

if and only if it is a Nash equilibrium of a suitably defined utility-perturbed Poisson game.

Given a Poisson game $\Gamma = (n, \mathscr{T}, r, C, (C_t)_{t\in\mathscr{T}}, u)$ and a perturbation (ε, μ°) we define the *perturbed Poisson game* $\Gamma_{\varepsilon,\mu^\circ} \equiv \big(n, \mathscr{T}, r, C, (C_t)_{t\in\mathscr{T}}, u(\cdot \mid \varepsilon, \mu^\circ)\big)$ where the utility functions are given, for every type $t \in \mathscr{T}$ and every action $c \in C_t$, by

$$u_t(c, x \mid \varepsilon, \mu^\circ) \equiv (1-\varepsilon)u_t(c,x) + \varepsilon \int_{\tau(\Sigma)} U_t(c,\tau)d\mu^\circ. \tag{4.2}$$

Proposition 4. *Given a perturbation (ε, μ°), the strategy function σ is a Nash equilibrium of $\Gamma_{\varepsilon,\mu^\circ}$ if and only if $\sigma \in \overline{\mathrm{BR}}((1-\varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$.*

Proof. Just notice that for every $t \in \mathscr{T}$ and every $c \in C_t$,

$$\begin{aligned}
\overline{U}_t(c, (1-\varepsilon)\delta(\sigma) + \varepsilon\mu^\circ) &= (1-\varepsilon)\overline{U}_t(c, \delta(\sigma)) + \varepsilon\overline{U}_t(c, \mu^\circ)\\
&= (1-\varepsilon)U_t(c, \tau(\sigma)) + \varepsilon \int_{\tau(\Sigma)} U_t(c,\tau)d\mu^\circ\\
&= (1-\varepsilon) \sum_{x\in Z(C)} \mathbf{P}(x \mid \tau(\sigma))u_t(c,x) + \varepsilon \int_{\tau(\Sigma)} U_t(c,\tau)d\mu^\circ\\
&= \sum_{x\in Z(C)} \mathbf{P}(x \mid \tau(\sigma)) \left[(1-\varepsilon)u_t(c,x) + \varepsilon \int_{\tau(\Sigma)} U_t(c,\tau)d\mu^\circ\right]\\
&= \sum_{x\in Z(C)} \mathbf{P}(x \mid \tau(\sigma))u_t(c, x \mid \varepsilon, \mu^\circ) = U_t(c, \tau(\sigma) \mid \varepsilon, \mu^\circ).
\end{aligned}$$

□

Note that, given a perturbation (ε, μ°), we can first normalize utility functions in $\Gamma_{\varepsilon,\mu^\circ}$ by dividing them by $(1-\varepsilon)$ and think of the perturbation as adding, for each type $t \in \mathscr{T}$ and each action $c \in C_t$, the constant value $\frac{\varepsilon}{1-\varepsilon}\int_{\tau(\Sigma)} U_t(c,\tau)d\mu^\circ$ to the function $U_t(c,\cdot)$. In Example 1, for instance, for any perturbation (ε, μ°) the value that is added to $U_t(a,\cdot)$ by the perturbation in the corresponding perturbed Poisson game is always strictly larger than the value added to $U_t(b,\cdot)$ (see Figure 1). This "lifts" the expected utility function $U_t(a,\cdot)$ more than $U_t(b,\cdot)$ and makes action b strictly dominated in the perturbed Poisson game. Note as well that, given a Poisson game Γ, the set of all perturbed Poisson games (as defined above) is a strict subset of the set of Poisson games that can be generated by perturbing the utility functions in Γ.

Taking the perturbations to zero, we introduce a new definition of perfection for Poisson games.

Definition 5 (Outer-perfection)**.** The strategy function σ is an *outer-perfect equilibrium* if there is a sequence of perturbations $\{(\varepsilon^k, \mu^k)\}_k$ and a sequence of strategy functions $\{\sigma^k\}_k$ such that $\{\varepsilon^k\}_k$ converges to zero, $\{\sigma^k\}_k$ converges to σ, and σ^k is a Nash equilibrium of $\Gamma_{\varepsilon^k,\mu^k}$ for every k.

Every perturbed Poisson game has a Nash equilibrium. For any sequence of Poisson games we can construct an associated sequence of Nash equilibria. Such a sequence is contained in the compact set Σ so it has a subsequence that converges. Hence, every Poisson game has an outer-perfect equilibrium. (Furthermore, it can also be proved that if the sequence of strategies $\{\sigma^k\}_k$ supports an outer-perfect equilibrium σ given the sequence of perturbations $\{(\varepsilon^k, \mu^k)\}_k$ then the sequence of perturbed equilibria $\{(1-\varepsilon^k)\delta(\sigma^k) + \varepsilon^k \mu^k\}_k$ converges weakly to $\delta(\sigma)$.) The major difference between this concept and the usual implementation of perfection in finite or infinite normal form games (Selten [36], Simon and Stinchcombe [37]; also, inner-perfect equilibria in the present paper) is that the set of perturbations *is not* a subset of the set of mixed strategies in the game. Thus, a perturbed Poisson game *cannot* be interpreted as a game where players make mistakes when implementing their intended actions. We adhere to the view that, if the possibility of mistakes is real then it should be properly modeled in the game (Kohlberg and Mertens [19, p. 1005, footnote 3]). Perturbations are just technical devices used to obtain desirable game theoretical properties. We complete this argument presenting in Appendix B an example that illustrates the inadequacy of a concept of strategic stability based on inner-perfect equilibria (which is the concept that, in the current context, does admit a motivation based on players' mistakes when playing the game). We also show at the end of this section that outer-perfection neither implies nor is implied by inner-perfection.

The following two corollaries follow from Theorem 1.

Corollary 1. *Every outer-perfect equilibrium is a Nash equilibrium in undominated strategies.*

Proof. Theorem 1 implies that, for any perturbation (ε, μ°), every dominated action in the Poisson game Γ becomes strictly dominated in $\Gamma_{\varepsilon,\mu^\circ}$. Hence, it is used with probability zero in every Nash equilibrium of $\Gamma_{\varepsilon,\mu^\circ}$. Since an outer-perfect equilibrium is the limit point of a sequence of Nash equilibria of perturbed Poisson games, such a dominated action is also used with probability zero in any outer-perfect equilibrium of Γ. □

Corollary 2. *If* $\#\mathscr{T} = 1$ *then every undominated equilibrium is outer-perfect.*

Proof. Let σ be an undominated equilibrium. Because σ is an equilibrium, $\sigma \in \overline{\mathrm{BR}}(\delta(\sigma))$. Because σ is undominated, there is a measure $\mu^\circ \in \mathscr{M}^\circ$ such that $\sigma \in \overline{\mathrm{BR}}(\mu^\circ)$. Therefore, Proposition 1 implies that $\sigma \in \overline{\mathrm{BR}}((1-\varepsilon)\delta(\sigma) + \varepsilon\mu^\circ)$. Taking ε to zero proves the result. □

In the next example we show that Corollary 2 does not generalize to Poisson games with more than two types.

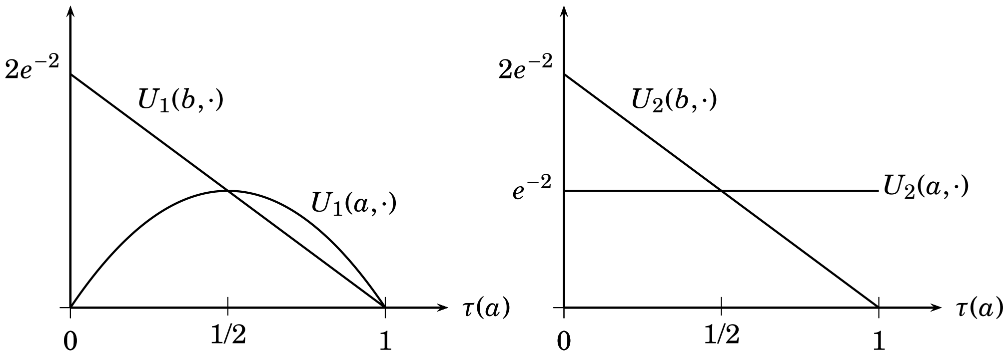

FIGURE 2. Utility functions in Example 2.

Example 2. Take a Poisson game with expected number of players $n = 2$, set of types $\mathscr{T} = \{1,2\}$, and set of actions $C = C_1 = C_2 = \{a,b\}$. The probability of each type is $r_1 = 2/3$ and $r_2 = 1/3$. Utility functions are as follows:[5]

$$u_1(a,x) = \begin{cases} 1 & \text{if } x(a) = x(b) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad u_2(a,x) = e^{-2} \quad \text{for every } x,$$

$$u_1(b,x) = (2 - x(a))e^{-2}, \qquad u_2(b,x) = (2 - x(a))e^{-2}.$$

The corresponding expected utility functions U_1 and U_2 are plotted in Figure 2. As we can see, no type has a dominated action. It is easy to see that the strategy function $\sigma = (\frac{3}{4}a + \frac{1}{4}b, b)$ is an undominated Nash equilibrium such that $\tau(\sigma) = \frac{1}{2}a + \frac{1}{2}b$. However, it is not outer-perfect. Given that $U_1(a,\cdot)$ is always below $U_2(a,\cdot)$ and that $U_1(b,\cdot) = U_2(a,\cdot)$, for any $\mu \in \mathscr{M}^\circ$ such that type 1 players are indifferent between a and b, necessarily, players of type 2 strictly prefer a to b. Hence, we cannot construct a sequence of perturbed Poisson games whose associated sequence of Nash equilibria converges to σ. (In turn, the Nash equilibrium $(\frac{1}{4}a + \frac{3}{4}b, a)$ is indeed outer-perfect.)

We now explore further the relationship between inner-perfect and outer-perfect equilibria. We have already seen above that an inner-perfect equilibrium can be a dominated strategy. Therefore, not every inner-perfect equilibrium is outer-perfect. We can also easily illustrate this last fact here with the strategy function $\sigma = (\frac{3}{4}a + \frac{1}{4}b, b)$ in Example 2. Indeed, the sequence of completely mixed strategies $\sigma^\varepsilon = \big((\frac{3-2\varepsilon}{4})a + (\frac{1+2\varepsilon}{4})b, \varepsilon a + (1-\varepsilon)b\big)$ converges to σ. Given that $\tau(\sigma^\varepsilon) = \tau(\sigma)$ for every small enough ε, the strategy function σ is a best response against every element in such a sequence. Thus, σ is an inner-perfect equilibrium (in undominated strategies).

[5] Note, however, that the utility functions u_i are *not* bounded, contrary to our assumption when we defined Poisson games. We chose unbounded utility functions only for the sake of simplicity in the exposition of the result.

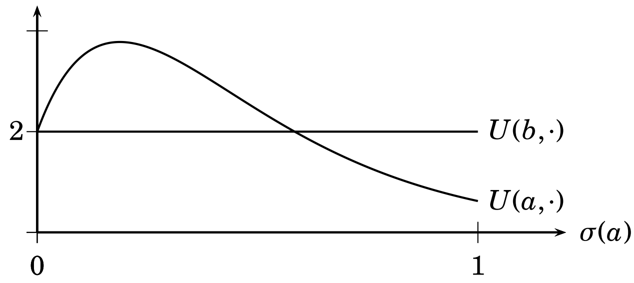

FIGURE 3. Utility functions in Example 3.

On the other hand, as we show in the next example, not every outer-perfect equilibrium is inner-perfect.

Example 3. Let the Poisson game Γ have expected number of players equal to $n = 4$, only one type, set of actions $C = \{a, b\}$ and utility function:

$$u(a,x) = \begin{cases} 2 & \text{if } x(a) = 0, \\ 8 & \text{if } x(a) = 1, \\ 0 & \text{otherwise,} \end{cases}$$
$$u(b,x) = 2 \quad \text{for every } x.$$

We represent the corresponding utility functions $U(a,\cdot)$ and $U(b,\cdot)$ in Figure 3.[6] Strategies a and b are both undominated. Furthermore, (b) is a Nash equilibrium of the game and, since $\#\mathcal{T} = 1$, Corollary 2 implies that it is also an outer-perfect equilibrium. However, it is not inner-perfect as for any completely mixed strategy close to strategy (b) action a is strictly preferred to action b.

We summarize these observations in the next proposition.

Proposition 5. *A Nash equilibrium in undominated actions is not necessarily outer-perfect even if it is also an inner-perfect equilibrium. Moreover, an outer-perfect equilibrium is not necessarily inner-perfect.*

5. STABILITY

In the following example, we show that the process of iterated deletion of dominated actions can lead to different solutions depending on the order of elimination.

Example 4 (Iterated dominance)**.** This example shows why iterated dominance and existence force us to use a set valued solution concept. Consider a Poisson

[6] An analogous picture can be obtained from a Poisson model of a congestion problem such as the Farol Bar game proposed by Arthur [2]. Each agent has two alternatives: drinking a beer at home (action b) or at a bar (action a). The utility of drinking in the bar alone is the same as the one from drinking at home. Furthermore, the utility of drinking in the bar is increasing in the company up to a point where the bar is too crowded and it starts to decline.

game with set of types $\mathscr{T} \equiv \{1,2\}$ with probabilities $r_1 = 1/4$ and $r_2 = 3/4$, and set of actions $C_1 = C_2 = C \equiv \{a,b,c,d\}$. Preferences are given by the following utility functions.

$$u_1(a,x) = \begin{cases} 2 \text{ if } x(a) \geq x(b), \\ 0 \text{ otherwise,} \end{cases} \qquad u_2(a,x) = \begin{cases} 1 \text{ if } x(c) > x(d), \\ 0 \text{ otherwise,} \end{cases}$$

$$u_1(b,x) = \begin{cases} 1 \text{ if } x(b) \geq x(a), \\ 0 \text{ otherwise,} \end{cases} \qquad u_2(b,x) = \begin{cases} 1 \text{ if } x(d) > x(c), \\ 0 \text{ otherwise,} \end{cases}$$

$$u_1(c,x) = 0 \text{ for all } x \in Z(C), \qquad u_2(c,x) = 0 \text{ for all } x \in Z(C),$$

$$u_1(d,x) = -1 \text{ for all } x \in Z(C). \qquad u_2(d,x) = -1 \text{ for all } x \in Z(C).$$

Actions c and d are dominated for both types. If we first eliminate d from C_1 and C_2 then b is dominated for type 2 players. Eliminating b from C_2 and c from C_1 and C_2, we see that at least 3/4 of the population choose a. Correspondingly, the best action for players of type 1 for every remaining strategy of the population is to also play a. We obtain that (a,a) survives the process of iterated deletion of dominated actions.

On the other hand, if we first eliminate c from C_1 and C_2 then a is dominated for type 2 players. We can eliminate a for type 2 players and d for every player in the game to conclude that at least 3/4 of the population choose b. Provided the expected number of players n is large enough, choosing b dominates choosing a for players of type 1. With this order of elimination of dominated actions, only the strategy function (b,b) survives. Note that the two equilibria that we obtain through the process of iterated deletion of dominated actions also induce different expected utility to the players in the game.

Thus, if we want to provide a definition of equilibrium that is robust against iterated deletion of dominated actions we are led to define a set-valued concept. In the previous example, e.g., such an equilibrium concept would have to include both (a,a) and (b,b).

Following Kohlberg and Mertens [19] we say that a set of equilibria is *stable* if it is minimal with respect to the following property:

Property (S). *$S \subset \Sigma$ is a closed set of Nash equilibria of Γ satisfying: for any $\varepsilon > 0$ there is a $\bar{\eta} > 0$ such that for any perturbation (η, μ°) with $0 < \eta < \bar{\eta}$ we can find a σ that is ε-close to S and satisfies $\sigma \in \overline{\mathrm{BR}}((1-\eta)\delta(\sigma) + \eta\mu^\circ)$.*

Remark 3. Property **(S)** in Kohlberg and Mertens [19] requires that every close by strategy perturbed game have a Nash equilibrium close to S. Taking into account the space of perturbations used here (see page 9), we instead directly perturb the best response correspondence by perturbing the aggregate behavior of the population. As a result, Property **(S)** here requires that every close by perturbed correspondence have a fixed point close to S. Of course, Proposition 4 implies that such a fixed point is a Nash equilibrium of a close by (payoff) perturbed Poisson game.

Proposition 6. *Every Poisson game has a stable set.*

Existence of stable sets in Poisson games is a particular case of a more general existence result. Stable sets in Poisson games are an example of *Q-robust* sets of fixed points (McLennan [26, Definition 8.3.5]). Loosely speaking, a set of fixed points X of a correspondence F is *essential* if every correspondence "close" to F has a fixed point close to X. This means that X is stable against every small perturbation of F. For instance, we can show that the set of *all* Nash equilibria of a (Poisson) game is essential. We can weaken this concept and restrict the set of allowed perturbations to those belonging to some class Q. (In our case, the characterization given in Theorem 1 indicates that we only consider perturbations caused by altering the average behavior of the population towards some $\mu^\circ \in \mathscr{M}^\circ$.) Then we say that a set of fixed points X of a correspondence F is *Q-robust* if every correspondence "close" to F that can be obtained through a perturbation in Q has a fixed point close to X. McLennan [26] shows that, if F is an upper semicountinuous and closed valued correspondence, every Q-robust set contains a minimal Q-robust set and that every connected Q-robust set contains a minimal connected Q-robust set (Theorem 8.3.8). However, not every stable set is necessarily connected.

Indeed, let us modify the utility functions in Example 3 so that the expected utility functions are those depicted in Figure 4. (Utility functions $u(a,\cdot)$ and $u(b,\cdot)$ can be found that generate such $U(a,\cdot)$ and $U(b,\cdot)$.) In this new game, $U(a,\cdot)$ and $U(b,\cdot)$ coincide in three isolated points. Furthermore, there are 2 stable sets, $\{\sigma^*\}$ and $\{(a),(b)\}$. Not only is the latter stable set disconnected but also its members belong to different connected components of Nash equilibria. Hence, $\{\sigma^*\}$ is the unique connected stable set.

We now prove that stable sets satisfy admissibility.

Proposition 7. *Every point of a stable set is an outer-perfect, hence, undominated, equilibrium.*

Proof. Let S be a stable set and let $\sigma \in S$ be a strategy function that is not an outer-perfect equilibrium. Therefore, there is some $\bar{\varepsilon} > 0$ and some $\bar{\eta} > 0$ such that for every $\eta < \bar{\eta}$ and every μ° we have $\varsigma \notin \overline{\mathrm{BR}}((1-\eta)\delta(\varsigma)+\eta\mu^\circ)$ whenever $d(\sigma,\varsigma) < \bar{\varepsilon}$. This implies that no strategy in the open ball $B(\sigma,\bar{\varepsilon}/2) \equiv \{\varsigma : d(\sigma,\varsigma) < \bar{\varepsilon}/2\}$ is an outer-perfect equilibrium either. It follows that $S\setminus\{B(\sigma,\bar{\varepsilon}/2)\}$

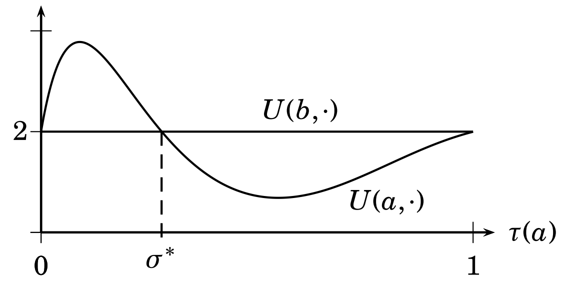

FIGURE 4. Utility functions in a game with a disconnected stable set.

satisfies Property **(S)** so that either it is a stable set or it contains one. By minimality, S is not a stable set. □

Furthermore, stable sets are robust to elimination of dominated actions in the following sense:

Proposition 8. *A stable set contains a stable set of any game obtained by deletion of a dominated pure action or a pure action that is an inferior response to any strategy function in the stable set.*

Proof. Take a stable set S of the Poisson game Γ. Let $c \in C_t$ be an action that is either dominated for players of type t or satisfies $c \notin \mathrm{BR}_t(\tau(\sigma))$ for every $\sigma \in S$. Let $\tilde{\Gamma}$ be the reduced game obtained from Γ by deleting c from C_t. We know that $\sigma \in S$ implies $\sigma_t(c) = 0$. Therefore, every strategy function in S can be considered as a strategy function in the smaller game $\tilde{\Gamma}$. Let $\tilde{\Sigma} \subset \Sigma$ be the resulting space of mixed strategies, so that $\tau(\tilde{\Sigma}) \subset \tau(\Sigma)$. Furthermore, let $\tilde{\mathscr{M}}$ be the set of Borel measures on $\tau(\tilde{\Sigma})$.

Fix ε and choose an η as in Property **(S)**. Consider the measures $\mu \in \mathscr{M}^\circ$ and $\tilde{\mu} \in \tilde{\mathscr{M}}^\circ$. For any $0 < \kappa < 1$, we have $\mu^\kappa \equiv \kappa\mu + (1-\kappa)\tilde{\mu} \in \mathscr{M}^\circ$. Hence, there is a σ^κ that is ε-close to S such that $\sigma^\kappa \in \overline{\mathrm{BR}}((1-\eta)\delta(\sigma^\kappa) + \eta\mu^\kappa)$. Taking the limit as κ approaches zero gives us, by continuity, a strategy function $\tilde{\sigma}$ that is ε-close to S and satisfies $\tilde{\sigma} \in \overline{\mathrm{BR}}((1-\eta)\delta(\tilde{\sigma}) + \eta\tilde{\mu})$. We conclude that S satisfies Property **(S)** in $\tilde{\Gamma}$. Thus, either S is a stable set of $\tilde{\Gamma}$ or it contains one. □

Remark 4. Robustness against elimination of inferior responses has been used to formalize *forward induction* (Kohlberg and Mertens [19], Mertens [27]). Once we fix a solution of the game, players should consider as "certain not to be employed" those behaviors of the opponents in which some players use an inferior response against every member of the solution. If we accept that, we can ask that the solution be robust against the deletion of such inferior responses (Kohlberg [18, p. 13]; Hillas and Kohlberg [15, p. 1645]). A similar reasoning may be used in Poisson games. Therefore, in this sense, we can say that stable sets in a Poisson game satisfy forward induction.

6. Examples

We now compute the stable sets of the past examples. In Example 1 the only undominated action is a so, by admissibility, $\{(a)\}$ is the unique stable set.

In Example 2 the strict equilibrium (b,b) is, of course, a singleton stable set. The strategy function (a,a) is a Nash equilibrium such that, for every small perturbation, the corresponding perturbed Poisson game has a Nash equilibrium close to (a,a). To see this note that action a is a strict best response for type 2 players. For players of type 1, if a perturbation "lifts" $U(a,\cdot)$ more than $U(b,\cdot)$ then a is a strict best response in the perturbed game. (See the discussion following Equation (4.2).) On the other hand, if a perturbation "lifts" $U(b,\cdot)$ more than $U(a,\cdot)$ then both functions cross at some point close to $\tau(a) = 1$. Hence, $\{(a,a)\}$ is also a stable set. Finally, we can also see that the strategy function $(\frac{1}{4}a + \frac{3}{4}b, a)$ is *strictly outer-perfect* and, consequently, also a singleton stable set.[7]

The Poisson game in Example 3 (see also the game described in footnote 6) has two Nash equilibria that are also outer-perfect, the pure strategy (b) and a mixed strategy σ^* that satisfies $U(a,\sigma^*) = U(b,\sigma^*)$. The set $\{(b)\}$ is not stable because for those perturbations that, in the resulting perturbed Poisson game, "lift" $U(a,\cdot)$ more than $U(b,\cdot)$ there is no Nash equilibrium close to (b). In turn, $\{\sigma^*\}$ is clearly stable.

Example 4 has a one dimensional and connected set of Nash equilibria that goes from (a,a) to (b,b). (In every point of this set type 2 players are indifferent between a and b and, in a subset of it, type 1 players are also indifferent between a and b). We already argued that a stable set must contain (a,a) and (b,b). In both equilibria type 1 players play a strict best response. Given that neither a nor b are dominated, there are close by perturbed games that "lift" $U_2(a,\cdot)$ more than $U_2(b,\cdot)$ as well as perturbed games that "lift" $U_2(b,\cdot)$ more than $U_2(a,\cdot)$. The strategy function (a,a) is a Nash equilibrium of every game in the first class of perturbed games while (b,b) is a Nash equilibrium of every game in the second class of perturbed games. Therefore, by minimality, the unique stable set is $\{(a,a),(b,b)\}$.

We conclude this section analyzing a variation of the referendum example proposed at the beginning of Section 3.

[7] A strictly outer-perfect equilibrium of a Poisson game Γ is a Nash equilibrium σ^* with the property that *every* perturbed Poisson game sufficiently close to Γ has a Nash equilibrium close to σ^*.

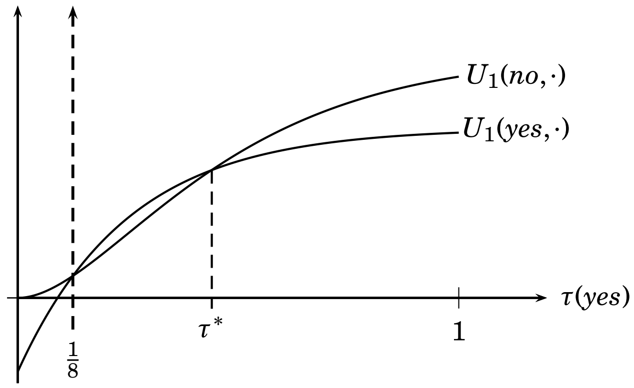

FIGURE 5. Utility functions for type 1 players in Example 5.

Example 5 (A voting example)**.** There is a referendum where voters have to vote either *yes* or *no* to some new policy and at least $K > 1$ voters should vote *yes* for the policy to be implemented. For concreteness, let us assume $K = 2$ and that the expected number of players is 4. Every voter prefers the new policy to the status quo. Let us fix players' payoff from the outcome of the election equal to 1 if the policy is implemented and equal to 0 if it is not. Suppose further that there are two types of voters. Type 1 voters incur a cost $c = \frac{1}{2}e^{-\frac{1}{2}}(\approx 0.3)$ if they vote *yes* whereas type 2 voters do not have any cost of voting. Let the probability that a player is of type 2 be equal to $\frac{1}{8}$.

Voting *yes* is a weakly dominant action for type 2 players. They are only indifferent between *yes* and *no* under the strategy function (no, no). The expected utility functions of type 1 players are depicted in Figure 5. Given the utility values chosen, $U_1(no,\cdot)$ represents the probability that *two or more* voters vote *yes* for each value of $\tau(yes)$. On the other hand, $U_1(yes,\cdot)$ is equal to the probability that *one or more* voters vote *yes* minus the cost c. The two functions cross in two isolated points, $\frac{1}{8}$ and $\tau^*(\approx 0.44)$. When $\tau(yes) < \frac{1}{8}$, few other voters are expected to vote *yes* and type 1 voters prefer to vote *no* because their probability of being pivotal is not enough to overcome their cost to voting *yes*. As the number of other voters who are expected to vote *yes* increases, the probability of being pivotal in the referendum increases and type 1 voters start preferring voting *yes* to voting *no*. When the number of other voters who are expected to vote *yes* grows even larger so that $\tau(yes) > \tau^*$, the probability that the other voters meet the threshold necessary for the policy to be implemented increases, making *yes* again an inferior response for type 1 voters.

Hence, this game has three isolated Nash equilibria. There is a dominated Nash equilibrium where every player votes *no*, a *low support* equilibrium where only type 2 players play *yes*, and a *high support* equilibrium where type 2 voters play *yes* and type 1 voters play *yes* with probability $\frac{8}{7}(\tau^* - \frac{1}{8})$. The last two Nash equilibria are both outer-perfect, however, the high support equilibrium is the unique stable set of the game. Indeed, consider the low support equilibrium and eliminate the dominated action *no* for type 2 players. In the reduced game at least $\frac{1}{8}$ of the population vote *yes*. Thus, if we only consider values of τ such that $\tau(yes) \geq \frac{1}{8}$, we obtain a picture similar to the one in Figure 3. In the same fashion as in that example, it can be seen that there are close by perturbed games that do not have a Nash equilibrium close to the low support equilibrium. From this we conclude that the unique stable set of the game consists only of the high support equilibrium.

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APPENDIX A. STABLE SETS IN GENERIC POISSON GAMES

We show that for generic Poisson games every Nash equilibrium is a singleton stable set. We do this in a similar fashion to Carbonell-Nicolau [8] who uses Fort's Theorem (Fort [11]) to show that, for some large families of infinite normal-form games, generic members are such that every Nash equilibrium is *essential*.[8] We point out, however, that the same caveat that is usually raised upon this type of genericity results applies here. The examples of Poisson games that we find in applications are nongeneric: there typically is a non-injective function mapping action profiles to events (in the case of voting games, e.g., pivotal events) where utilities are defined instead.

Once we fix n, $\mathscr{T}$, r, C and $(C_t)_{t\in\mathscr{T}}$, a Poisson game is given by a function $u : \mathscr{T} \times C \times Z(C) \to \mathbb{R}$. Since $\mathscr{T}$ and C are finite and $Z(C)$ is countable, we can see such a function u as a point in the space of all bounded sequences ℓ^∞. Thus, the Nash equilibrium correspondence NE can be thought of as $\mathrm{NE} : \ell^\infty \to \Sigma$. Such a correspondence is upper semicontinuous and compact valued.

Recall that a G_δ set is a countable intersection of open sets. A topological space is called a *Baire space* if the union of any countable collection of closed

[8] A Nash equilibrium σ of a game Γ is essential (Wen-Tsün and Jia-He [38]) if every game close to Γ has a Nash equilibrium close to σ. Of course, an essential Nash equilibrium is a singleton stable set.

sets with empty interior has empty interior. Since $(\ell^\infty, \|\cdot\|_\infty)$ is a Banach space,[9] the Baire Category Theorem implies that it is also a Baire space.

Theorem 2 (Fort [11])**.** *If $F : X \to Y$ is an upper semicontinuous and compact valued correspondence from a Baire space to a metric space then F is both upper and lower semicontinuous at every point of a dense G_δ subset of X.*

At a lower semicontinuity point u of the Nash equilibrium correspondence, for every Nash equilibrium σ of u and every sequence of Poisson games $\{u^k\}_k$ converging to u, there is an associated sequence $\{\sigma^k\}_k$ converging to σ such that σ^k is a Nash equilibrium of u^k for every k. This is, in particular, true for every sequence of perturbed Poisson games converging to u. We thus conclude:

Corollary 3. *For every game in a dense G_δ set of Poisson games every Nash equilibrium is a singleton stable set.*

Appendix B. On the inadequacy of inner-perfection

In the following example we illustrate why a definition of stability based on inner-perfect equilibrium perturbations is not adequate even if it is accompanied by a restriction that only allows to select undominated actions.

Example 6. Consider the Poisson game Γ with $n = 2$, only one type, set of actions $C = \{a, b, c\}$, and utility function

$$\begin{aligned} u(a,x) &= \begin{cases} x(c)+1 & \text{if } x(a)+x(c)=1, \\ x(c) & \text{otherwise,} \end{cases} \\ u(b,x) &= \quad e^{-1} \qquad\qquad\quad \text{for all } x \in Z(C), \\ u(c,x) &= -1 \qquad\qquad\qquad \text{for all } x \in Z(C). \end{aligned}$$

Action $\gamma = \frac{1}{2}a + \frac{1}{2}b$ is not dominated: it does better than action a against the strategy (b) and it does better than action b against the strategy $(\frac{1}{2}b + \frac{1}{2}c)$. For any strategy-perturbation $(\varepsilon, \sigma^\circ)$ such that ε is close enough to zero, consider the strategy-perturbed game $\Gamma_{\varepsilon,\sigma^\circ}$ and the strategy

$$\varsigma_{\varepsilon,\sigma^\circ} \equiv \frac{1}{1-\varepsilon}\left(\left(\frac{1-z(\varepsilon\sigma^\circ(c))}{2} - \varepsilon\sigma^\circ(c) - \varepsilon\sigma^\circ(a)\right)a + \left(\frac{1+z(\varepsilon\sigma^\circ(c))}{2} - \varepsilon\sigma^\circ(b)\right)b\right),$$

where the correcting factor $z(\varepsilon\sigma^\circ(c)) > 0$ is chosen so that $P(1 \mid 1 - z(\varepsilon\sigma^\circ(c))) = e^{-1} - 2\varepsilon\sigma^\circ(c)$. The strategy $\varsigma_{\varepsilon,\sigma^\circ}$ is an undominated Nash equilibrium of the strategy-perturbed game $\Gamma_{\varepsilon,\sigma^\circ}$ which is close to $\sigma = (\frac{1}{2}a + \frac{1}{2}b)$. (Note that under the corresponding perturbed strategy $\sigma_{\varepsilon,\sigma^\circ} \equiv (1-\varepsilon)\varsigma_{\varepsilon,\sigma^\circ} + \varepsilon\sigma^\circ$ the expected value of $x(c)$ is $2\varepsilon\sigma^\circ(c)$ and, therefore, $U(a, \sigma_{\varepsilon,\sigma^\circ}) = e^{-1}$.)

Hence, $\{(\frac{1}{2}a + \frac{1}{2}b)\}$ would be a stable set of the Poisson game according to a definition of stability based on strategy-perturbations (i.e. inner-perfection).

[9] In our context, we can write $\|u\|_\infty = \max_{t\in\mathcal{T}} \max_{c\in C_t} \sup_{x\in Z(C)} |u_t(c,x)|$.

However, after eliminating the strictly dominated action c, action a and, consequently, action $\gamma = \frac{1}{2}a + \frac{1}{2}b$ become weakly dominated.

On the other hand, one can show that in this game the only stable set (according to our definition) is the set made of the strict equilibrium (b).

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