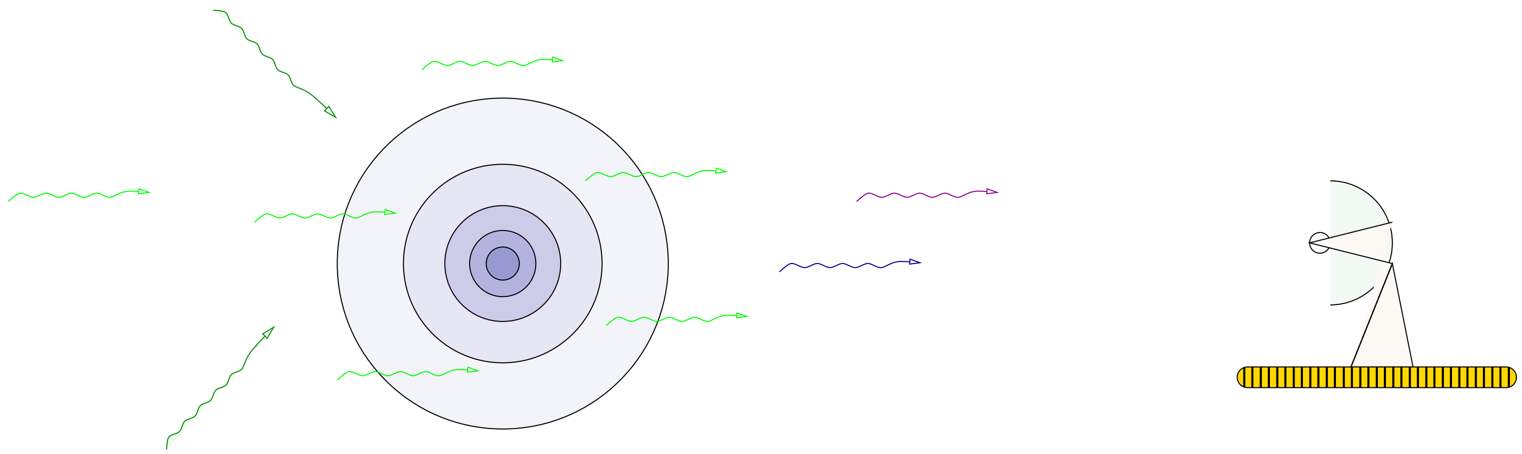
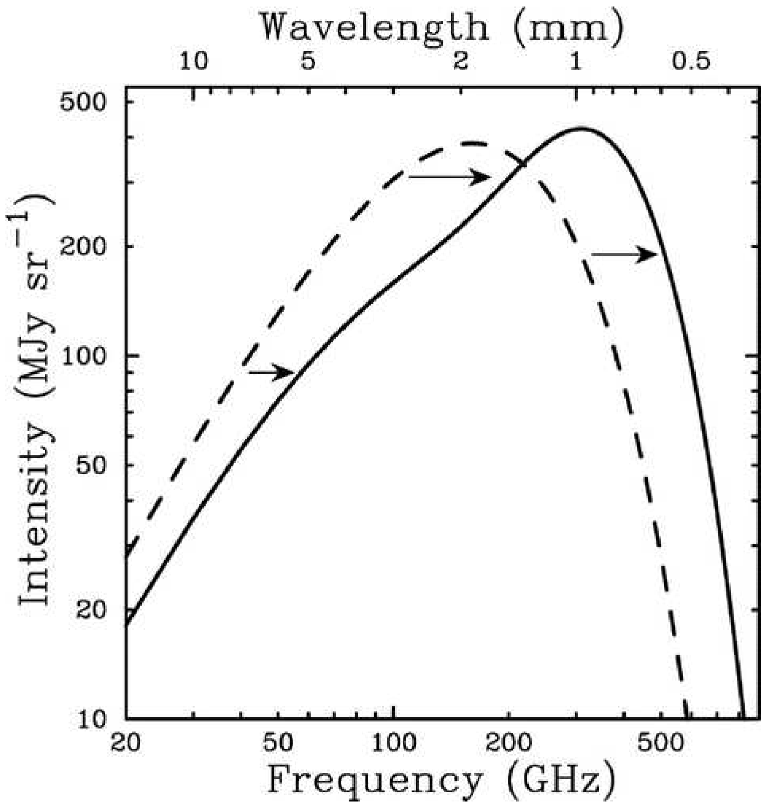
The Sunyaev-Zel'dovich Effect.

Effect of Clusters of Galaxies on the CMB.



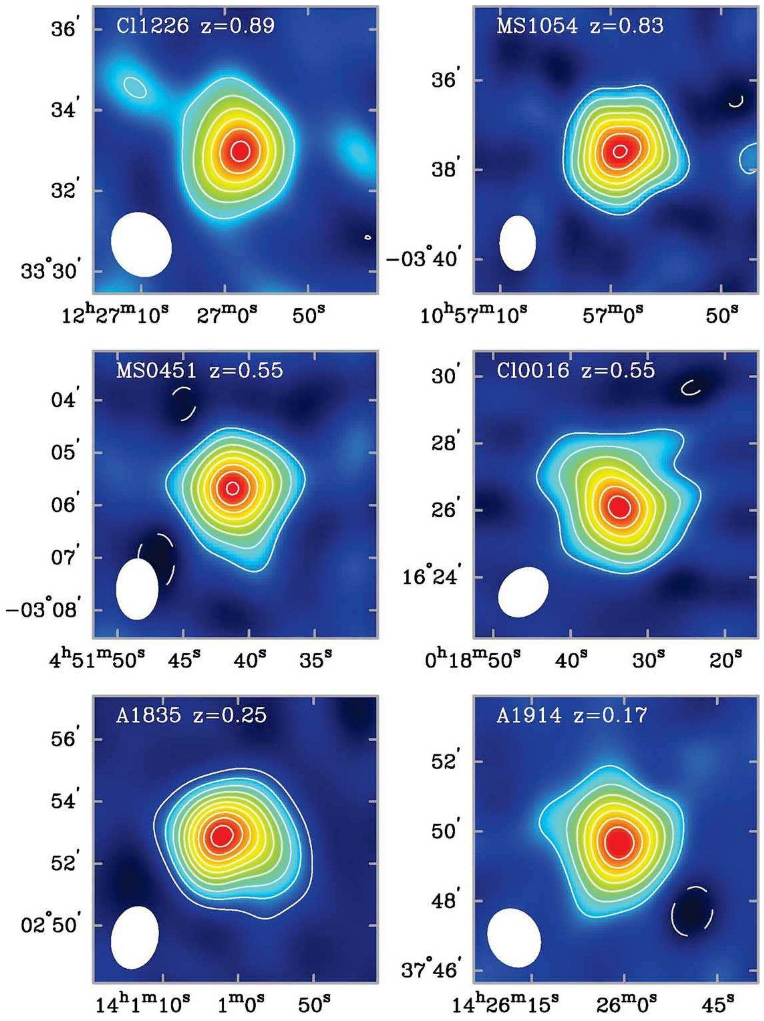
Electrons in the cluster potential well transfer energy to CMB photons.

The SZE.



The SZE is a distortion of the CMB spectrum caused by the free electrons locked in the potential wells of clusters of galaxies. Being a spectral distortion, **is INDEPENDENT of redshift**.

Undistorded CMB spectrum (dashed line) and SZ distortion (solid) produced by a fictional clusters 1000 times more massive than Coma.



This example shows that the SZ effect is independent of redshift. Clusters with the same X-ray luminosity but different redshifts have the same SZ brightness.

The Thermal SZE.

Induced by the thermal motion of Intra Cluster electrons. CMB photons are Inverse Compton scattered by the IC medium. Each scattered photon gains an energy:

$$\frac{\Delta E}{E} \approx \frac{k_B T_e}{m_e c^2}$$

The brightness and temperature distribution of photons changes:

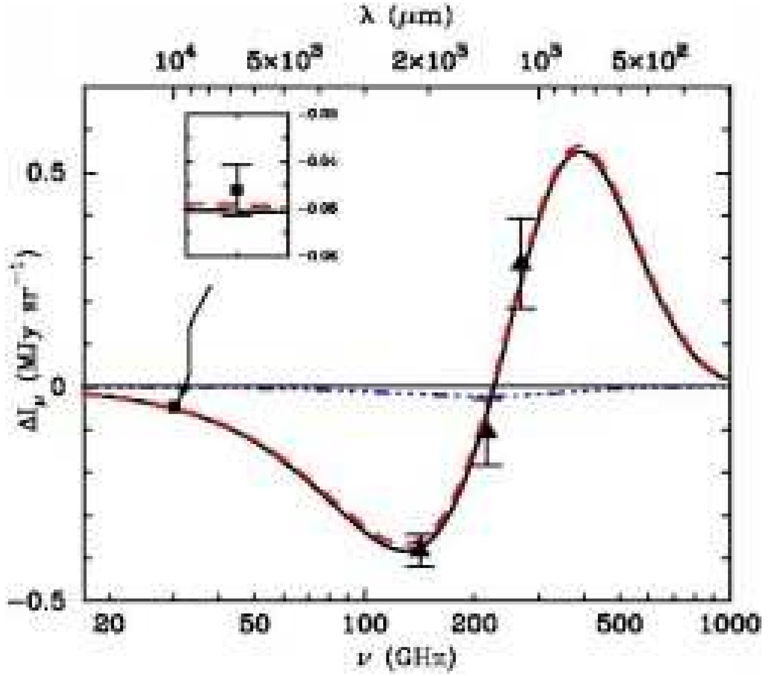
$$\frac{\Delta T_{TSZ}}{T_o} = g(\nu)\frac{k_B \sigma_T}{m_e c^2}\int n_e T_e dl, \qquad g(\nu) = \left(x\frac{e^x+1}{e^x-1} - 4\right)(1+\text{relativistic corrections})$$

with $x = h\nu/k_B T_{CMB}$. The integral is along the line of sight (l.o.s.). If clusters are isothermal, $T_e \sim const$ and we can define the optical opacity to the SZ effect as:

$$\tau_e = \sigma_T \int n_e dl$$

For the brightest clusters $\tau_e \sim 5 \times 10^{-3}$, $T_e \sim 10$KeV.

The Kinematic SZE.



Induced by the motion of the cluster as a whole:

$$\frac{\Delta T_{KSZ}}{T_o} = \tau_e \frac{\vec{v}\vec{n}}{c}$$

where $\vec{v}$ is the cluster peculiar velocity and $\vec{n}$ is the l.o.s. Unfortunately, it has the SAME frequency dependence than the CMB.

SZE spectrum of Abell 2163 at 30, 140, 218 and 270GHz. The best fit TSZ (dashed), KSZ (dotted) and TSZ-KSZ (solid) are shown.

Mesurement of the Hubble Constant.

Emission and absorption from a slab of hot gas have different dependencies with electron density

$$\left.\begin{array}{l} E \propto \int n_e^2 dl \\ A \propto \int n_e dl \end{array}\right\} \Longrightarrow$$

A^2/E is a density weighted measure of the size of the cluster.

If θ is the angle subtended by the cluster:

$$D_A \propto (A^2/E\theta)$$

Results from A2218, A665: $H_o = 55 \pm 17 \mathrm{kms}^{-1}$

The Integrated Sachs-Wolfe Effect.

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The integrated Sachs-Wolfe effect results from the late time decay from gravitational potential fluctuations.

$$\frac{\Delta T_{ISW}(\hat{n})}{T_o} = -\frac{2}{c^2}\int_0^{r_{rec}} dr \frac{\partial \Phi(r,\hat{n})}{\partial r}. \tag{3}$$

The integral is with respect to conformal distance (or look-back time) from the observer at redshift $z = 0$,

$$r(z) = \int_0^{r_{rec}} \frac{dz'}{H(z')} \qquad H^2(z) = H_o^2[\Omega_m(1+z)^3 + \Omega_\Lambda + (\Omega_k(1+z)^2)]$$

the integral is not exact so, in general

$$\int_0^{r_{rec}} dr\, \frac{\partial \Phi(r,\hat{n})}{\partial r} \neq \Phi(r_0) - \Phi(0)$$

Multipole Expansion I.

◇ To compute the Power Spectrum of CMB temperature anisotropies induced by the ISW effect we need to carry out the multipole expansion of eq. (3)

The multipole moments are defined as

$$a_{lm} = \int d\Omega_{\hat{n}} T(\hat{n}) Y^*_{lm}(\Omega_{\hat{n}}) = -\frac{2}{c^2} \int d\Omega_{\hat{n}} Y^*_{lm}(\Omega_{\hat{n}}) \int_0^{r_{rec}} dr \frac{\partial \Phi(r, \hat{n})}{\partial r}. \tag{4}$$

We shall take Fourier transforms of the gravitational field:

$$\Phi(r, \hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Phi(k) e^{-i\vec{k}\hat{n}r} \tag{5}$$

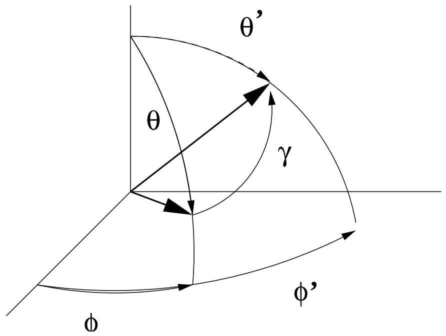
and since

$$\nabla^2\Phi = 4\pi G\rho_m(a)\delta(a)a^2 \qquad \Rightarrow \qquad \Phi(k) = -\frac{3}{2}\Omega_{m,o}\frac{\delta(k,0)}{k^2 H_o^{-2}}F(z)$$

where $F(z) = D(z)/(1+z)$, and $D(z)$ is the growth factor of matter density perturbations in a particular cosmological model.

Remark: In Einstein-de Sitter models $D(z) = (1+z)$, so F(z) is a constant. Since the ISW effect depends on dF/dz, this model will not have this type of anisotropy.

Mathematical Formulae.



Rayleigh expansion of a plane wave:

$$e^{-i\vec{k}\hat{n}r} = 4\pi \sum_{lm} i^l j_l(kr) Y^*_{lm}(\Omega_{\hat{k}}) Y_{lm}(\Omega_{\hat{n}}),$$

orthogonality:

$$\int d\Omega_{\hat{n}} Y^*_{l'm'}(\Omega_{\hat{n}}) Y_{lm}(\Omega_{\hat{n}}) = \delta_{l'l}\delta_{m'm}$$

addition theorem of spherical harmonics:

$$\begin{aligned} \cos\gamma &= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \\ P_l(\cos\gamma) &= \frac{4\pi}{2l+1}\sum_m Y^*_{lm}(\theta,\phi)Y_{lm}(\theta',\phi') \end{aligned}$$

Multipole Expansion II.

Using previous formulae we obtain:

$$a_{lm} = -i^l \frac{2}{c^2} \int \frac{d^3k}{2\pi^2} Y^*_{lm}(\Omega_{\hat{k}}) \int_0^{rrec} dr j_l(kr) \frac{\partial \Phi(r, \hat{k})}{\partial r}.$$

After substituting the gravitational potential by the density field, we obtain:

$$a_{lm} = i^l \frac{3}{c^2} \Omega_{m,o} \int \frac{d^3k}{2\pi^2} Y^*_{lm}(\Omega_{\hat{k}}) \int_0^{zrec} dz j_l(kr(z)) \frac{dF}{dz} \left[\frac{\delta(k,0)}{k^2 H_o^{-2}} \right]$$

Statistical Averages.

⊙ Statistical averages at one point are the average of an ensemble of realizations Ω of the random variable δ at that point. If $P(\bar{\delta},\vec{k})$ is the probability of the variable δ taken a value $\bar{\delta}$ at a point $\vec{k}$, then

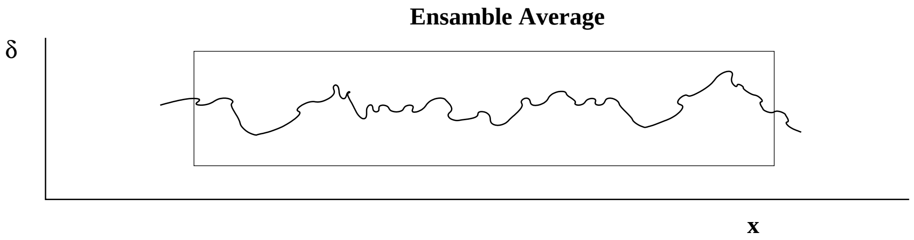
$$< \delta(\vec{k}) >= \int_{\Omega} P(\bar{\delta},\vec{k})\bar{\delta}d\Omega$$

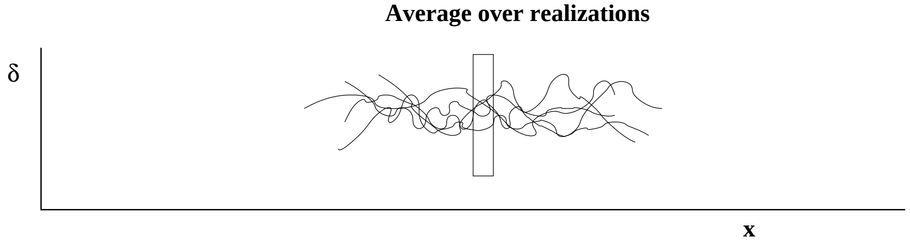
We have assumed $\int_{\Omega} P(\bar{\delta},\vec{k})d\Omega = 1$.

⊙ We assume the Ergodic Theorem holds: the statistical average over an ensamble of realizations is equivalent to an average over the VOLUME where the variable is defined:

$$< \delta(\vec{k}) >= \int_{V} \delta(\vec{k}+\vec{x})d^3x$$

where $\int_V d^3x = V$.





In a similar manner we can further define higher order moments. If $P(\bar{\delta}, \vec{k}|\bar{\delta}', \vec{k}')$ is the joint probability of variable δ taken a value $\bar{\delta}$ at position $\vec{k}$ when at position $\vec{k}'$ takes value $\bar{\delta}'$, then the realization average is

$$< \delta^*(\vec{k})\delta(\vec{k}') >= \int_\Omega P(\bar{\delta}, \vec{k}|\bar{\delta}', \vec{k}')\bar{\delta}(\vec{k}) \cdot \bar{\delta}(\vec{k}')d\Omega$$

and the ensamble average

$$< \delta^*(\vec{k})\delta(\vec{k}') >= \int_V \delta(\vec{k} + \vec{x})\delta(\vec{k}' + \vec{x})d^3x$$

Power Spectrum

The radiation power spectrum due to the ISW effect is:

$$\begin{aligned} < a^*_{lm} a_{l'm'} > &= \left(\frac{3\Omega_{m,o}}{c^2}\right)^2 i^{l+l'} \int \frac{d^3k}{2\pi^2}\frac{d^3k'}{2\pi^2} Y^*_{lm}(\Omega_{\hat{k}}) Y_{l'm'}(\Omega_{\hat{k}'}) < \delta^*(\vec{k})\delta(\vec{k}') > \\ &\cdot \int_0^{r_{rec}} dr j_l(kr)\frac{dF}{dr}\left[\frac{H_o}{k^2}\right] \int_0^{r_{rec}} dr' j_l(k'r')\frac{dF}{dr'}\left[\frac{H_o}{k'^2}\right] \end{aligned}$$

Taking into account the definition of power spectrum:

$$< \delta^*(\vec{k})\delta(\vec{k}') >= (2\pi)^3 \delta^D(\vec{k} + \vec{k}')P(k)$$

we arrive at

$$< a^*_{lm} a_{l'm'} >= \frac{8}{\pi c^4} \int d^3k Y^*_{lm}(\Omega_{\hat{k}}) Y_{l'm'}(\Omega_{\hat{k}}) I_l(k) I_{l'}(k) P(k)$$

In spherical coordinates: $d^3k = k^2 dk d\Omega_{\hat{k}}$. We can carry out the integration of the angular part using the orthogonality relation of spherical harmonics. Finally we arrive at:

$$C_l^{ISW} =< |a_{lm}|^2 >= \frac{2}{\pi}\int_0^\infty k^2 dk P(k) I_l^2(k)$$

where

$$I_l(k) = 3\Omega_{m,o}\frac{H_o^2}{c^2k^2}\int_0^{r_{rec}} dr j_l(kr)\frac{dF}{dr}$$

Order of Magnitude Estimate

If we denote $R_H = cH_o^{-1}$ and we assume that F' is small and constant, then

$$I_l(k) = 3\Omega_{m,o}\frac{1}{(kR_H)^2}\frac{dF}{dr}\int_o^{ro} j_l(kr)dr = 3\Omega_{m,o}\frac{F'}{(kR_H)^2}\frac{1}{k}\int_o^{uo} j_l(u)du$$

Taking into account that

$$\frac{dF}{dr} = R_H^{-1}\frac{dF}{dz}, \qquad \int_o^{uo} j_l(u)du \sim \Theta(k - l/R_H)$$

then $I_l(k) = (3\Omega_{m,o}/(kR_H)^3)(dF/dz)\Theta(k - l/R_H)$ and if we assume the matter power spectrum to be of Harrison-Zeldovich type:

$$P(k) = P(k_o)\left(\frac{k}{k_o}\right) \Longrightarrow C_l^{ISW} = \frac{2}{\pi}\frac{(3\Omega_{m,o})^2}{k_o R_H^6}\left(\frac{dF}{dz}\right)^2\int_o^{l/R_H} P(k_o)\frac{dk}{k^3}$$

and finally:

$$C_l^{ISW} = \frac{2}{\pi}(3\Omega_{m,o})^2\left(\frac{dF}{dz}\right)^2 P(k_o)\frac{1}{R_H^3 l^3}$$

As $R_H = 3 \times 10^3$, for $l = 2$ we have $k_o = l/R_H \sim 10^{-3}$; from the literature $P(k_o) \sim 310^{-3}$. If we assume that $dF/dz \sim 0.1$, corresponding to $\Delta F = 0.3$ in a redshift interval $\Delta z = 3$, then we obtain:

$$C_2^{ISW} = 1000(\mu K)^2 \Longrightarrow C_l^{ISW} = 1000(\mu K)^2\left(\frac{2}{l}\right)^3.$$

while the correct result, depending on the model, is a factor 2-3 smaller, but the order of magnitude is correct.