Evolution of Density Perturbations.

Newtonian Cosmology.

⊙ The evolution of matter and radiation density perturbations requires a full relativistic treatment. Approximate solutions in specific cases can be found using a newtonian treatment.

⊙ The newtonian evolution of a self gravitating perfect fluid is described by:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) &= 0 \qquad (continuity) \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} &= -\nabla\phi \qquad (Euler) \\
\nabla^2\phi &= 4\pi G\rho \qquad (Poisson)
\end{aligned}$$

These equations have no physical solution and require an approximation known as 'Jeans swindle'.

In an expanding Universe is very useful to introduce comoving coordinates:

$$\underbrace{\vec{r}}_{Physical\ Coordinates} = a(t) \underbrace{\vec{x}}_{Comoving\ Coordinates}$$

and

$$\vec{v} = \frac{d\vec{r}}{dt} = \underbrace{\dot{a}\vec{x}}_{Hubble\ Expansion} + \underbrace{a\dot{\vec{x}}}_{Peculiar\ Velocity} = H\vec{r} + a\vec{u}$$

where $H = \dot{a}/a$ is the Hubble constant and $\vec{\omega} = a\vec{u}$ is the peculiar velocity.

We shall redefine physical magnitudes in terms of their mean and 'contrast' from the mean:

$$\rho(\vec{x}, t) = \bar{\rho}(t)[1 + \delta(\vec{x}, t)]$$

In comoving coordinates, newtonian equations become:

$$
\begin{aligned}
\frac{\partial \delta}{\partial t} + \nabla_{\vec{x}} \cdot \vec{u} + \underline{\nabla_{\vec{x}} \cdot (\vec{u}\delta)} &= 0 \qquad (Continuity) \\
\frac{\partial \vec{u}}{\partial t} + 2H\vec{u} + \underline{(\vec{u} \cdot \nabla_{\vec{x}})\vec{u}} &= -a^{-2}\nabla_{\vec{x}}\phi \qquad (Euler) \\
\nabla^2_{\vec{x}}\phi &= 4\pi G\bar{\rho}\delta a^2 \qquad (Poisson)
\end{aligned}
$$

In Linear Perturbation Theory the terms $\boxed{\nabla_{\vec{x}} \cdot (\vec{u}\delta)}$ and $\boxed{(\vec{u} \cdot \nabla_{\vec{x}})\vec{u}}$ are assumed to be negligible.

Velocity field: Solenoidal and Irrotational components.

Let us decompose the peculiar velocity field on irrotational and solenoidal components: $\vec{\omega} = \vec{\omega}_i + \vec{\omega}_s$ defined by:

$$\nabla \times \vec{\omega}_i = 0; \qquad \nabla \cdot \vec{\omega}_s = 0$$

From Euler equation we can deduce:

$$\frac{\partial a\vec{\omega}}{\partial t} = \nabla_x \phi \qquad \Longrightarrow \qquad \frac{\partial a\vec{\omega}_i}{\partial t} = \nabla_x \phi, \qquad \frac{\partial a\vec{\omega}_s}{\partial t} = 0$$

Therefore, the solenoidal part of the velocity field decays with the expansion:

$$\frac{\partial a\vec{\omega}_s}{\partial t} = 0 \qquad \Longrightarrow \qquad a\vec{\omega}_s = const \qquad \vec{\omega}_s = a(t_{in})\vec{\omega}_s(t_{in})/a(t)$$

This result is consequence of the conservation of angular momentum.

Fourier transforms of the equations of evolution.

The equations of motion are more easily studied (and solved!!!) in Fourier space. Let us define:

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3k\delta(k,t)e^{i\vec{k}\vec{x}}$$

When necessary, will further assume that $\delta(k,t) = \delta(k)D(t)$. The equations of evolution become:

$$\begin{aligned}
\dot{\delta}_{\mathbf{k}} + i\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}} + \underline{\frac{i}{(2\pi)^3}\int d^3k'\delta_{\mathbf{k}}(\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}-\mathbf{k}'})} &= 0 \\
\dot{\mathbf{u}}_{\mathbf{k}} + 2H\mathbf{u}_{\mathbf{k}} + \underline{\frac{1}{(2\pi)^3}\int d^3k' i[\mathbf{u}_{\mathbf{k}'}(\mathbf{k}-\mathbf{k}')]\mathbf{u}_{\mathbf{k}-\mathbf{k}'}} &= ia^{-2}\mathbf{k}\phi_{\mathbf{k}} \\
|\mathbf{k}|^2\phi_{\mathbf{k}} &= -4\pi G\bar{\rho}\delta_{\mathbf{k}}a^2
\end{aligned}$$

Evolution of density perturbations: I-Matter Domination.

Neglecting non-linear terms, the coupling of Continuity, Euler and Poisson equation yields:

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} - 4\pi G\bar{\rho}\delta_{\mathbf{k}} = 0 \qquad (1)$$

The scale factor $a(t)$ is fixed by the geometry. For a $\Omega_m = 1$, $\Omega_\Lambda = \Omega_k = 0$, $a(t) \sim t^{2/3}$ and eq. (1) becomes:

$$\ddot{\delta}_{\mathbf{k}} + \frac{4}{3t}\dot{\delta}_{\mathbf{k}} - \frac{2}{3t^2}\delta_{\mathbf{k}} = 0 \qquad (2)$$

This second order diferential equation has two solutions: 'growing' and 'decaying' modes

$$\delta_{\mathbf{k}} = A_{\mathbf{k}}t^{2/3} + B_{\mathbf{k}}t^{-1}; \qquad \mathbf{u}_{\mathbf{k}} = \frac{2}{3}\frac{i\mathbf{k}}{|\mathbf{k}|^2}A_{\mathbf{k}}t^{-1/3}$$

Evolution of density perturbations: II-Radiation Domination.

During radiation domination: $\Omega_\gamma + \Omega_m \simeq 1$. The exact treatment requires a GR approach. We shall use the newtonian approximation assuming that:

1. Relativistic particles are smooth on scales $\lambda \ll ct \Longrightarrow$ The only gravitational term that drives the evolution of density perturbations is due to the matter.
2. The relativistic component alters the background expansion rate.

$$\left(\frac{\dot{a}}{a}\right) = \frac{8\pi G}{3}(\bar{\rho}_\gamma + \bar{\rho}_m)$$

If we know chhange variables from t to $\eta = \rho_m/\rho_\gamma$ then eq. (1) becomes

$$\frac{d^2\delta_{\mathbf{k}}}{d\eta^2} + \frac{2+3\eta}{2\eta(1+\eta)}\frac{d\delta_{\mathbf{k}}}{d\eta} = \frac{3}{2}\frac{\delta_{\mathbf{k}}}{\eta(1+\eta)}$$

with solutions

growing mode: $\delta \propto 1 + \frac{3}{2}\eta$

decaying mode: $\delta \propto (1 + \frac{3}{2}\eta) \ln \left[\frac{(1+\eta)^{1/2}+1}{(1+\eta)^{1/2}-1} \right] - 3(1+\eta)^{1/2}$

Fluctuations in the matter component can not grow till the Universe becomes matter dominated.

♠ A physical reason to understand the lack of growth of matter density perturbations during the radiation dominated regime is that matter density perturbations do not have enough time to grow since the free-fall time $t_{ff} \sim (G\rho_m)^{-1/2}$ is much larger than the characteristic expansion time:

$$H = \left(\frac{4\pi G \rho_\gamma}{3} \right)^{1/2} \qquad \Rightarrow \qquad t_{exp} = H^{-1} \sim (G\rho_\gamma)^{-1/2}$$

and since $\rho_\gamma \gg \rho_m$ then $t_{exp} \ll t_{ff}$.

Evolution of Superhorizon-Sized Perturbations.

♠ Superhorizon sized perturbations require a full relativistic treatment. Since the growth of perturbations with wavelength $\lambda > d_H$ can not be affected by processes like viscosity, free streaming, the evolution of such perturbations can be analyzed with the following argument:

♠ Consider a spherical region containing matter with a mean density ρ_1, embedded in a $k = 0$ Friedmann Universe of density ρ_o, with $\delta\rho = \rho_1 - \rho_o$ small and positive. Since the evolution of a spherical symmetric region is not affected by the matter outside, this regionn evolves as a Friedmann model with $k = 1$. Therefore:

$$H_1^2 + \frac{1}{a_1^2} = \frac{8\pi G}{3}\rho_1; \qquad\qquad H_o^2 = \frac{8\pi G}{3}\rho_o$$

where $H_o = (\dot{a}_o/a_o)$ and $H_1 = (\dot{a}_1/a_1)$. We will compare the evolution of the perturbed universe and the background universe when their expansion rates are equal, i.e., when $\boxed{H_o = H_1}$.

From the previous equations

$$\delta = \frac{\rho_1 - \rho_o}{\rho} = \frac{3}{8\pi G(\rho_o a_1^2}$$

In general, if $H_o = H_1$ at aime time t_i, then $a_o \neq a_1$, but if δ is small, then their difference will be small and we can set $a_1 \approx a_o$. Since ρ_o is the background density, we obtain:

$$\delta \propto \begin{cases} a^2 & (RD) \\ a & (MD) \end{cases}$$

Perturbations outside the horizon always grow in the MD and RD regimes.

The Matter Power Spectrum.

Characterizing Matter Density Perturbations.

The properties of the density field are not known in detail but on average. The statistical properties of the density field are given in terms of its moments:

2-point: $\langle \delta(\mathbf{x})\delta(\mathbf{x}')\rangle$

3-point: $\langle \delta(\mathbf{x})\delta(\mathbf{x}'\delta(\mathbf{x}''))\rangle$

4-point: $\langle \delta(\mathbf{x})\delta(\mathbf{x}'\delta(\mathbf{x}''\delta(\mathbf{x}'''))\rangle$

$\cdots$

where the average is taken over Universes with the same statistical properties. Each of those Universes is called a REALIZATION of the underlying density field.

The ergodic theorem is assumed to apply:

$$\langle A \rangle = \sum_{Universes} A(\mathbf{x}) = \frac{1}{V}\int_V d^3x' A(\mathbf{x}-\mathbf{x}')$$

The 2nd order moment is called 2-point correlation function. Because of homogeneity and isotropy, it follows that:

$$\langle \delta(\mathbf{x})\delta(\mathbf{x}')\rangle \equiv \xi(\mathbf{x},\mathbf{x}') = \xi(|\mathbf{x}-\mathbf{x}'|)$$

The power spectrum is the average amplitude of the Fourier components of the density field:

$$P(k) = \langle |\delta_{\mathbf{k}}|^2\rangle \equiv A^2(k) = A_S k^{n_S}$$

It is assume to behave like a power law.

Theorem 1. *The correlation function is the Fourier transform of the power spectrum.*

$$
\begin{aligned}
\xi(\mathbf{x},\mathbf{x}') &= \frac{1}{V}\int_V d^3y\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{x}'-\mathbf{y}) \\
&= \frac{1}{V}\frac{1}{(2\pi)^6}\int_V d^3yd^3kd^3k'\delta(k)e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\delta(k')e^{i\mathbf{k}'(\mathbf{x}'-\mathbf{y})} \\
&= \frac{1}{(2\pi)^3}\int d^3kd^3k'\delta(k)\delta(k')e^{i\mathbf{k}\mathbf{x}+\mathbf{k}'\mathbf{x}'}\frac{1}{V}\frac{1}{(2\pi)^3}\int d^3ye^{-i(\mathbf{k}+\mathbf{k}')\mathbf{y}} \\
&= \frac{1}{(2\pi)^3}\int d^3kd^3k'\delta(k)\delta(k')e^{i(\mathbf{k}\mathbf{x}+\mathbf{k}'\mathbf{x}')}\delta_{Dirac}(\vec{k}+\vec{k'}) \\
&= \frac{1}{(2\pi)^3}\int d^3k|\delta(k)|^2e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \equiv \frac{1}{(2\pi)^3}\int d^3kP(\mathbf{k})|^2e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}
\end{aligned}
$$

The Power Spectrum.

♠ A particular case of the previous theorem is:

$$\xi(0) = \left(\frac{\delta\rho}{\rho}\right)^2 = \int \frac{k^3 P(k)}{2\pi^2}\frac{dk}{k}$$

For isotropy we shall assume that the power spectrum depends only on $k = |\vec{k}|$.

The fluctuation power per logarithmic interval, or variance, appears very often in discussions, and is usually denoted by:

$$\Delta^2(k) = \frac{k^3 P(k)}{2\pi^2}$$

Normalization of the Matter Power Spectrum.

♠ From the density contrast δ we can define the density contrast on a given mass scale $(\delta M/M)$: is the mass excess on a volume V. The volume could be a sphere of a given size or could have smooth edges. We define the window function $W(r)$ as an adequate statistical weight when considering volume averages:

$$V = 4\pi \int_o^{\infty} r^2 W(r) dr$$

Usually, the window functions are normalized to have unit volume. Two commonly used window functions and their Fourier transforms are:

$$W(r) = \begin{cases} 1 & r \leq R \\ 0 & r > R \end{cases} \quad V = \frac{4\pi R^3}{3} \quad W(k) = 3V\left[\frac{\sin kR}{(kR)^3} - \frac{\cos kR}{(kR)^2}\right]$$

$$W(r) = \exp\left(-\frac{r^2}{2R^2}\right) \quad V = (2\pi)^{3/2} R^3 \quad W(k) = V \exp\left(-\frac{1}{2}k^2 R^2\right)$$

♠ The mass excess is:

$$\left(\frac{\delta M}{M}\right)^2 = \frac{1}{V^2}\langle(\int \delta(\mathbf{x}+\mathbf{r})W(\mathbf{r})d^3r)^2\rangle = \int \Delta^2(k)|W(k)|^2 d\ln k$$

♠ The matter power spectrum can be normalized measuring the amplitude of the matter power spectrum from galaxy catalogs. For the CfA redshift survey, Davis & Peebles (1973) found that

$$\sigma_8 = \left(\frac{\delta M}{M}\right)(R = 8h^{-1}Mpc) = 1$$

WARNING: Galaxy catalogs do not give the matter distribution but the distribution of luminous matter.

Statistical Properties of the Density Field.

An important class of density fields are those whose phases are random: the Fourier coefficients $\delta_{\mathbf{k}}$ for any given realization on a volume V have their amplitude drawn from the power spectrum with random angular phases.

$$P(k) = A^2(k) \qquad \Longrightarrow \qquad \delta(\mathbf{k}) = A(k)e^{i\theta_{\mathbf{k}}}; \qquad 0 < \theta_{\mathbf{k}} \leq 2\pi$$

The random variable $\theta_{\mathbf{k}}$ follows a uniform distribution in the interval $[0, 2\pi]$.

The central limit theorem guarantees that the density fluctuation δ at any point obeys Gaussian statistics; the probability distribution of $\delta(\vec{x})$ at each point $\vec{x}$ is:

$$P(\delta(\vec{x}))d\delta = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2(\vec{x})}{2\sigma^2}\right) d\delta$$

where σ is the variance of the density field.

Problems.

- The variance of the matter density field is

$$\sigma_M^2(R) = \langle \left(\frac{\delta M}{M}\right)^2_R \rangle = \int_o^\infty \frac{dk}{k} W_k^2 \frac{k^3}{2\pi^2} P(k)$$

For a power law spectrum $P(k) = Ak^n$, the integral can be evaluated analytically. Show that $\sigma_M(R) \propto R^{-(n+3)/2} \propto M^{-(n+3)/6}$.

- WMAP normalization of the power spectrum requires $\sigma_M(R = 8h^{-1}Mpc) = \sigma_8 = 0.8$. The 'Great Attractor' is a region of $\sim 50h^{-1}Mpc$ with an overdensity $(\delta M/M) \sim 0.7$. What is the probability of finding a 'Great Attractor' region if the power spectrum is $n = 0, -1, -2$?. How many 'Great Attractor' regions will we find within the current horizon?.

The Harrison-Zel'dovich Power Spectrum.

Notation and earlier results.

t_{in}: moment when a perturbation comes into the horizon.

t_{eq}: moment of matter radiation equality; $\rho_\gamma(t_{eq}) = \rho_m(t_{eq})$.

t_o: present time.

k, λ: Comoving wavenumber and wavelength; $k = 2\pi/\lambda$.

$\Delta(k,t)$: amplitude of a perturbation of given wavenumber at time t:

$$\Delta^2(k,t) = D_+(t)\frac{k^3}{2\pi^3}P(k)$$

$D_+(t)$: growth factor.

$$D_+(t) = \begin{cases} D_+(t_{in})(t/t_{in})^{2/3} & if \quad t_{in} > t_{eq} \\ D_+(t_{in}) = const. & if \quad t_{in} \leq t_{eq} \end{cases}$$

♣ We define the moment a perturbation comes within the horizon when its physical length coincides with the size of the horizon:

$$\lambda(t_{in})a(t_{in}) = d_H(t_{in})$$

Since $d_H(t) \sim t$ and during MATTER DOMINATION $a(t) \sim t^{2/3}$, then $\lambda(t_{in}) \sim t_{in}^{1/3}$. Therefore:

$$k_{in}^3 t_{in} = const = k_{eq}^3 t_{eq}$$

♣ Harrison (1970) & Zeldovich (1972) prescription.

ALL PERTURBATIONS THAT COME WITHIN THE HORIZON HAVE THE SAME AMPLITUDE

$$D_+(t_{in})\Delta(k_{in}) = const = D_+(t_{eq})\Delta(k_{eq}) \qquad \forall t_{in}$$

This prescription gives the amplitude of the matter power spectrum at different moments in time.

Matter power spectrum at any given time.

♠ Let us assume that that the perturbation comes into the horizon during RD: $t_{in} < t_{eq}$; taking into account

$$D_+(t_{in}) = D_+(t_{eq}) \qquad \text{and} \qquad D_+(t_{in})\Delta(k_{in}) = D_+(t_{eq})\Delta(k_{eq})$$

then

$$D_+(t_{eq})\Delta(k_{in}) = D_+(t_{eq})\Delta(k_{eq}) \qquad \Longrightarrow \qquad D_+(t_o)\Delta(k_{in}) = D_+(t_o)\Delta(k_{eq})$$

If a perturbation comes into the horizon before Matter–Radiation equality, its amplitude today is related to the amplitude at MR eq as:

$$P(k_{in}) = P(k_{eq}) \left(\frac{k_{in}}{k_{eq}}\right)^{-3}$$

♠ Let us assume that a perturbation comes into the horizon at a time $t_{in} > t_{eq}$; i.e, during MD.

$$D_+(t_o) = D_+(t_{in}) \left(\frac{t_o}{t_{in}}\right)^{2/3} = D_+(t_{in}) \left(\frac{t_o}{t_{eq}}\right)^{2/3} \left(\frac{t_{eq}}{t_{in}}\right)^{2/3}$$

therefore

$$\begin{aligned} D_+(t_o)\Delta(k_{in}) &= D_+(t_{in})\Delta(k_{in}) \left(\frac{t_o}{t_{eq}}\right)^{2/3} \left(\frac{t_{eq}}{t_{in}}\right)^{2/3} = D_+(t_{eq})\Delta(k_{eq}) \left(\frac{t_o}{t_{eq}}\right)^{2/3} \left(\frac{t_{eq}}{t_{in}}\right)^{2/3} \\ &= D_+(t_o)\Delta(k_{eq}) \left(\frac{t_{eq}}{t_{in}}\right)^{2/3} = D_+(t_o)\Delta(k_{eq}) \left(\frac{k_{eq}}{k_{in}}\right)^{2} \end{aligned}$$

and finally

$$D_+(t_o)\Delta(k_{in})k_{in}^2 = const = D_+(t_o)\Delta(k_{eq})k_{eq}^2$$

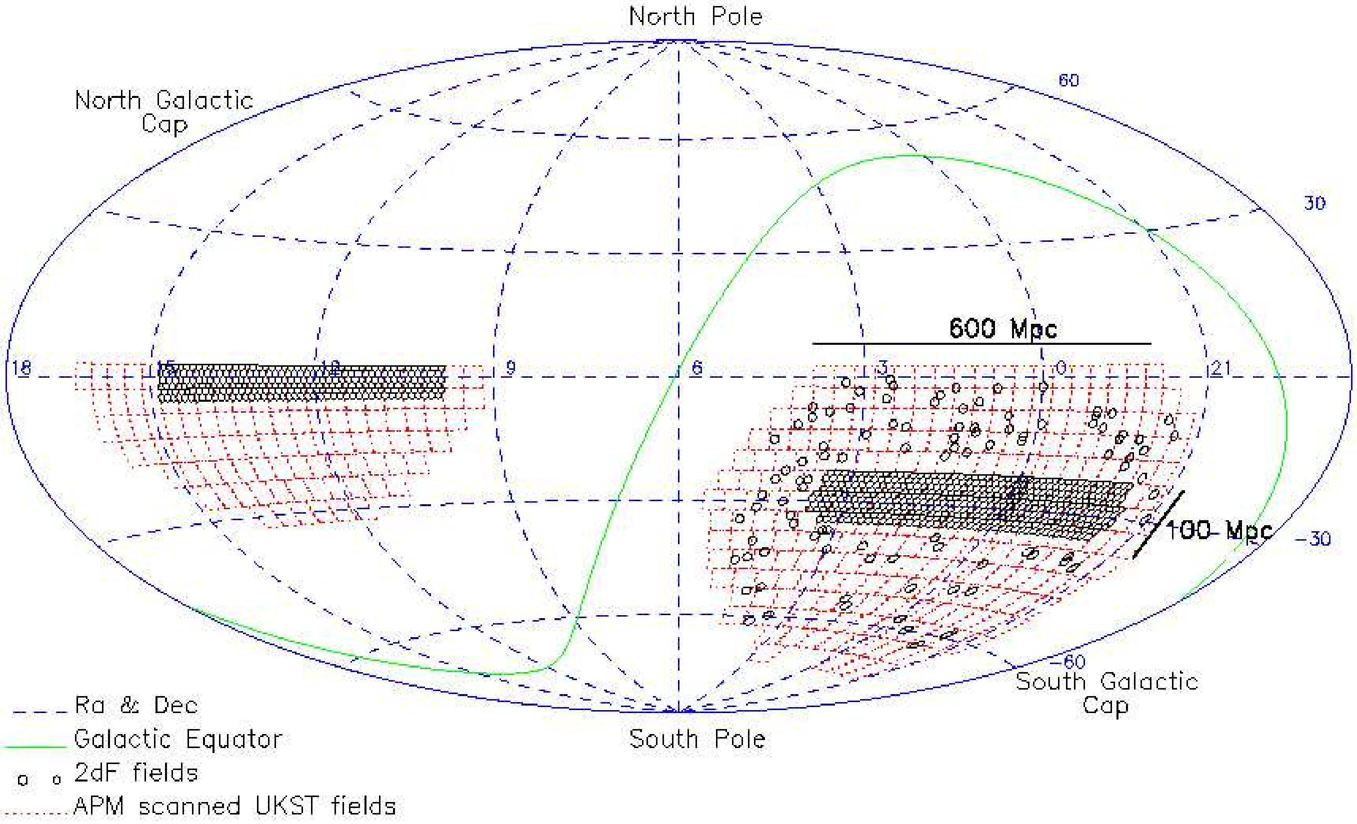
therefore, in this case if a perturbation comes into the horizon AFTER Matter–Radiation equality, its amplitude today is related to the amplitude at MR eq as:

$$P(k_{in}) = P(k_{eq}) \left(\frac{k_{in}}{k_{eq}}\right)^1$$

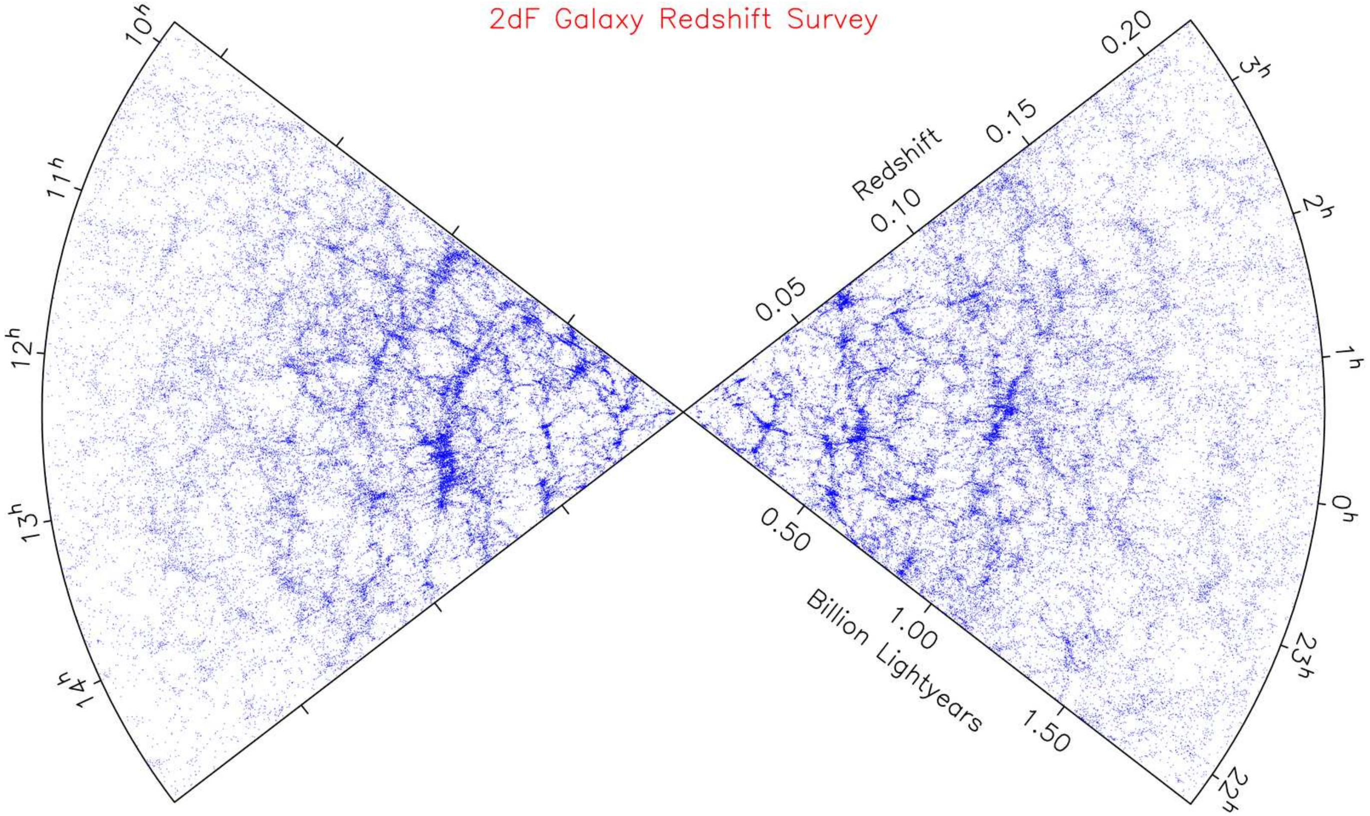
Observations of the Matter Power Spectrum.

♠ Matter Power Spectrum can be obtained from Galaxy Redshift Surveys. Recent Surveys: 2dFRGS and SDSS.

2dF Survey Map.



2dF Cone.



2dF Galaxy Redshift Survey
Redshift
0.05
0.10
0.15
0.20
Billion Lightyears
0.50
1.00
1.50
10h
11h
12h
13h
14h
3h
2h
1h
0h
23h
22h

2dF Matter Power Spectrum.