MODULI SPACES OF SEMISTABLE SHEAVES ON SINGULAR GENUS 1 CURVES

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Abstract. We find some equivalences of the derived category of coherent sheaves on a Gorenstein genus one curve that preserve the (semi)-stability of pure dimensional sheaves. Using them we establish new identifications between certain Simpson moduli spaces of semistable sheaves on the curve. For rank zero, the moduli spaces are symmetric powers of the curve whilst for positive rank there are only a finite number of non-isomorphic spaces. We prove similar results for the relative semistable moduli spaces on an arbitrary genus one fibration with no conditions either on the base or on the total space. For a cycle E_N of projective lines, we show that the unique degree 0 stable sheaves are the line bundles having degree 0 on every irreducible component and the sheaves $\mathcal{O}(-1)$ supported on one irreducible component. We also prove that the connected component of the moduli space that contains vector bundles of rank r is isomorphic to the r-th symmetric product of the rational curve with one node.

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Introduction

Elliptic fibrations have been used in string theory, notably in connection with mirror symmetry on Calabi-Yau manifolds and D-branes. The study of relative moduli spaces of semistable sheaves on elliptic fibrations, aside from its mathematical importance, provides a geometric background to string theory. In the case of integral elliptic fibrations, a complete description is already known and among the papers considering the problem we can cite [3, 7, 8, 21].

A study of these relative spaces for a more general class of genus one fibrations (for instance, with non-irreducible fibers and even singular total spaces) turns out to be an interesting problem.

On the one hand, for sheaves of rank 1 a fairly complete study of a class of these moduli spaces (compactified relative Jacobians), including those associated to relatively minimal elliptic surfaces, was carried out by one of the authors in [23, 24] (see also [11, 12]). On the other hand, nowadays it is well understood the efficient key idea of the "spectral cover construction" discovered for the first time by Friedman-Morgan-Witten in [15] and widely used later by many authors. The method shows how useful is the theory of integral functors and Fourier-Mukai transforms in the problem. The study developed by two of the authors in [19] and [20] on relative integral functors for singular fibrations gives a new insight in this direction.

From the results in that paper one gets new information about moduli spaces of relative semistable sheaves of higher rank for a *genus one fibration* $p\colon S \to B$, that is, a projective Gorenstein morphism whose fibers are curves of arithmetic genus one and trivial dualizing sheaf but without further assumptions on S or B.

The fiber of the relative moduli space over a point $b \in B$ is just the absolute moduli space of semistable sheaves on S_b, so that in order to start with the relative problem one has to know in advance the structure of the absolute moduli spaces for the possible degenerations of an elliptic curve. There are some cases where the structure of the singular fibers is known. For smooth elliptic surfaces over the complex numbers, the classification was given by Kodaria [22] and for smooth elliptic threefolds over a base field of characteristic different from 2 and 3, they were classified by Miranda [25]. In both cases, the possible singular fibers are plane curves of the same type, the so-called Kodaira fibers. Nevertheless, in a genus one fibration non-plane curves can appear as degenerated fibers. So that our genus one fibrations may have singular fibers other than the Kodaira fibers. The study of the moduli spaces of vector bundles on smooth elliptic dates back to Atiyah [1] and Tu [37], who proved that for an elliptic curve X there is an isomorphism $\mathcal{M}(r,d) \equiv Sym^m X$, where $m = \gcd(r,d)$, between the moduli space of semistable sheaves of rank r and degree d and the symmetric product of the curve. A very simple way to prove this isomorphism is by using Fourier-Mukai transforms (cf. [29, 18]). This method has been generalized to irreducible elliptic curves (i.e., rational curves with a simple node or cusp) in [2, Chapter 6]) obtaining that $\mathcal{M}(r,d) \equiv Sym^m X$, where $m = \gcd(r,d)$ also in this case.

In the case of singular curves, the moduli spaces of semistable torsion free sheaves were first constructed and studied by Seshadri [31]; his construction can now be seen as a particular case of the general construction of the moduli spaces of semistable pure sheaves due to Simpson [34]. The properties of these moduli spaces and their degeneration properties have been studied by many authors (see, for instance, [32, 26, 27, 11, 12, 23, 24]).

The paper is divided in two parts. In the first part, we consider X an arbitrary Gorenstein genus one curve with trivial dualizing sheaf. The group of all integral functors that are exact autoequivalences of $D^b_c(X)$ is still unknown and a criterion characterizing those Fourier- Mukai transforms that preserve semistability for a non-irreducible curve of arithmetic genus 1 seems to be a difficult problem. Here we find some equivalences of its derived category $D^b_c(X)$ of coherent sheaves that preserve the (semi)-stability of pure dimensional sheaves. One is given by the ideal of the diagonal and the other is provided by twisting by an ample line bundle (see Theorem 1.20). Our proof follows the ideas in [7, 29] where the result was proved for a smooth elliptic curve and in [2] for an irreducible singular elliptic curve. The results of this section allow to ensure that for rank zero, the moduli spaces are the symmetric powers of the curve whilst for positive rank there are only a finite number of non-isomorphic moduli spaces (see Corollary 1.25). Unlike the case of a smooth curve, moduli spaces of semistable sheaves on a curve with many irreducible components are not normal (even in the case where the rank is 1). Its structure depends very strongly on the particular configuration of every single curve. The difficulty in determining the stability conditions for a sheaf in this case points out the relevance of the identifications of Corollary 1.25. In fact, for a curve with two irreducible components endowed with a polarization of minimal degree, they reduce the study either to the case of rank 0 or degree 0. Coming back to the relative case, the section finishes with Corollary 1.28 which establishes new identifications between certain relative Simpson moduli spaces of (semi)stable sheaves for a genus one fibration.

In the second part, we focus our study in a curve of type E_N and in the case of degree 0 which is particularly interesting as in this case semistability does not depend on the polarization. Proposition 2.3 computes the Grothendiek group of coherent sheaves for any reduced connected and projective curve whose irreducible components are isomorphic to $\mathbb{P}^1$. The discrete invariants corresponding to the Grothendieck group behave well with respect to Fourier-Mukai transforms and are important tools for the analysis of the moduli spaces. Although a description of all torsion-free sheaves on a cycle of projective lines E_N is known, as we mentioned above, it is by no means a trivial problem to find out which of them are semistable. For instance, contrary to what happens for an elliptic curve, semistability is not guaranteed by the simplicity of the sheaf. For E_1, that is, a rational curve with one node, this was done in [9] for the degree zero case and in [10] otherwise. Using the description of indecomposable torsion-free sheaves on E_N given in [5] and the study of semistable torsion-free sheaves on E_N and on tree-like curves of [23], Theorem 2.11 proves that the only degree 0 stable sheaves that exist are the line bundles having degree 0 on every irreducible component of X and $\mathcal{O}_{C_i}(-1)$ for some irreducible component C_i. Then Corollary 2.13 gives the possible Jordan-Holder factors of any degree 0 semistable sheaf. In the integral case, if the sheaf is indecomposable all Jordan-Hölder factors are isomorphic to each other. This is no longer the case for cycles of projective lines. Proposition 2.16 computes the graded object of any indecomposable semistable sheaf of degree 0. The structure of the connected component of the moduli space that contains vector bundles of rank r is given in Theorem 2.20. Namely, it is isomorphic to the r-th symmetric product $Sym^r E_1$ of the rational curve with one node. Having studied the case of degree zero, the results of the first part of the paper allow to cover other cases (see Remark 2.21). In particular, the connected component of the moduli space that contains vector bundles of rank r and degree rh, where h is the degree of the polarization, is also isomorphic to $Sym^r E_1$.

In this paper, scheme means separated scheme of finite type over an algebraically closed field k of characteristic zero. By a Gorenstein or a Cohen-Macaulay morphism, we understand a *flat* morphism of schemes whose fibers are respectively Gorenstein or Cohen-Macaulay. For any scheme X we denote by $D(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. This is the essential image of the derived category of quasi-coherent sheaves in the derived category $D(\mathfrak{Mod}(X))$ of all $\mathcal{O}_X$-modules [6, Corollary 5.5]. Analogously $D^+(X)$, $D^-(X)$ and $D^b(X)$ denote the derived categories of complexes which are respectively bounded below, bounded above and bounded on both sides, and have quasi-coherent cohomology sheaves. The subscript c will refer to the corresponding subcategories of complexes with coherent cohomology sheaves. By a point we always mean a closed point. As it is usual, if $x \in X$ is a point, $\mathcal{O}_x$ denotes the skyscraper sheaf of length 1 at x, that is, the structure sheaf of x as a closed subscheme of X, while the stalk of $\mathcal{O}_X$ at x is denoted $\mathcal{O}_{X,x}$.

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1. Fourier-Mukai transforms preserving stability

1.1. **A non-trivial Fourier-Mukai transform on genus one curves.** Let X and Y be proper schemes. We denote the two projections of the direct product $X \times Y$ to X and Y by π_X and π_Y.

Let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times Y)$. The integral functor of kernel $\mathcal{K}^\bullet$ is the functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y} \colon D(X) \to D(Y)$ defined as

$$\Phi^{\mathcal{K}^\bullet}_{X \to Y}(\mathcal{F}^\bullet) = \mathbf{R}\pi_{Y*}(\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)$$

and it maps $D^-(X)$ to $D^-(Y)$.

In order to determine whether an integral functor maps bounded complexes to bounded complexes, the following notion was introduced in [19].

Definition 1.1. Let $f \colon Z \to T$ be a morphism of schemes. An object $\mathcal{E}^\bullet$ in $D^b_c(Z)$ is said to be of *finite homological dimension over* T if $\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet$ is bounded for any $\mathcal{G}^\bullet$ in $D^b_c(T)$.

The proof of the following lemma can also be found in [20, Proposition 2.7].

Lemma 1.2. *Assume that X is a projective scheme and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times Y)$. The functor $\Phi^{\mathcal{K}^\bullet}_{X \to Y}$ maps $D^b_c(X)$ to $D^b_c(Y)$ if and only if $\mathcal{K}^\bullet$ has finite homological dimension over X.* □

Let us suppose that X is a projective Gorenstein curve with arithmetic genus $\dim H^1(X, \mathcal{O}_X) = 1$ such that its dualizing sheaf is trivial. This includes all the so-called Kodaria fibers, that is, all singular fibers of a smooth elliptic surface over the complex numbers (classified by Kodaira in [22]) and of a smooth elliptic threefold over a base field of characteristic different from 2 and 3 (classified by Miranda in [25]). In these two cases, all fibers are plane curves. Here, we do not need to assume that our curve X is a plane curve. Notice also that an irreducible curve of arithmetic genus one has always trivial dualizing sheaf, but this is no longer true for reducible curves. Therefore in [19] we defined a genus one fibration as a projective Gorenstein morphism $p\colon S \to B$ whose fibers have arithmetic genus one and trivial dualizing sheaf.

Using the theory of spherical objects by Seidel and Thomas [30], we have the following

Proposition 1.3. *Let X be a projective Gorenstein curve with arithmetic genus 1 and trivial dualizing sheaf. Let $\mathcal{I}_\Delta$ be the ideal sheaf of the diagonal immersion $\delta\colon X \hookrightarrow X \times X$. One has:*

1. *The ideal sheaf $\mathcal{I}_\Delta$ is an object in $D^b_c(X \times X)$ of finite homological dimension over both factors.*
2. *The functor $\Phi = \Phi^{\mathcal{I}_\Delta}_{X \to X}\colon D^b_c(X) \to D^b_c(X)$ is an equivalence of categories.*
3. *The integral functor $\widehat{\Phi} = \Phi^{\mathcal{I}^*_\Delta}_{X \to X}\colon D^b_c(X) \to D^b_c(X)$ where $\mathcal{I}^*_\Delta$ is the dual sheaf is a shift of the quasi-inverse of Φ with $\widehat{\Phi} \circ \Phi \simeq [-1]$ and $\Phi \circ \widehat{\Phi} \simeq [-1]$.*

Proof. (1) Denote by $\pi_i\colon X \times X \to X$ with $i = 1, 2$ the two projections. By the symmetry, to see that $\mathcal{I}_\Delta$ is of finite homological dimension over both factors it is enough to prove it over the first one. Using the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X \times X} \to \delta_* \mathcal{O}_X \to 0\,,$$

it suffices to see that $\delta_*\mathcal{O}_X$ has finite homological dimension over the first factor. We have then to prove that for any bounded complex $\mathcal{F}^\bullet$ on X, the complex $\delta_*\mathcal{O}_X \otimes \pi_1^*\mathcal{F}^\bullet$ is also a bounded complex and this follows from the projection formula for δ.

(2) Since X is a projective Gorenstein curve of genus one and trivial dualizing sheaf, $\mathcal{O}_X$ is a spherical object of $D^b_c(X)$. By [30] the twisted functor $T_{\mathcal{O}_X}$, along the object $\mathcal{O}_X$, is an equivalence of categories. Since $\Phi \simeq T_{\mathcal{O}_X}[-1]$, the statement follows.

(3) By [20, Proposition 2.9], the functor $\Phi^{\mathcal{I}^\vee_\Delta[1]}_{X \to X}$ is the right adjoint to Φ where $\mathcal{I}^\vee_\Delta = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X \times X}}(\mathcal{I}_\Delta, \mathcal{O}_{X \times X})$ is the dual in the derived category, then it is enough to prove that $\mathcal{I}^\vee_\Delta$ is isomorphic to $\mathcal{I}^*_\Delta$, the ordinary dual. Indeed, one has to check that $\mathcal{E}xt^i_{\mathcal{O}_{X \times X}}(\mathcal{I}_\Delta, \mathcal{O}_{X \times X}) = 0$ for $i \geq 1$. Let us consider the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X \times X} \to \delta_* \mathcal{O}_X \to 0\,.$$

Taking local homomorphisms in $\mathcal{O}_{X \times X}$, we get an exact sequence

$$0 \to \mathcal{O}_{X \times X} \to \mathcal{I}^*_\Delta \to \mathcal{E}xt^1_{\mathcal{O}_{X \times X}}(\delta_*\mathcal{O}_X, \mathcal{O}_{X \times X}) \to 0$$

and isomorphisms

$$\mathcal{E}xt^{i-1}_{\mathcal{O}_{X \times X}}(\mathcal{I}_\Delta, \mathcal{O}_{X \times X}) \simeq \mathcal{E}xt^i_{\mathcal{O}_{X \times X}}(\delta_*\mathcal{O}_X, \mathcal{O}_{X \times X}) \qquad \text{for all } i > 1,$$

which proves our claim because X is Gorenstein. □

We shall use the following notation: for an integral functor $\Phi\colon D^b_c(X) \to D^b_c(X)$, Φ^j denotes the j-th cohomology sheaf of Φ, unless confusion can arise. Remember that a sheaf $\mathcal{E}$ on X is said to be WIT_i-Φ if $\Phi(\mathcal{E}) \simeq \Phi^i(\mathcal{E})[-i]$. In this case, we denote the unique non-zero cohomology sheaf $\Phi^i(\mathcal{E})$ by $\widehat{\mathcal{E}}$.

Note that in our particular situation, since $\mathcal{I}_\Delta$ is flat over the first factor and the fibers of π_2 are of dimension one, for any sheaf $\mathcal{E}$ on X one has $\Phi^j(\mathcal{E}) = 0$ unless $0 \leq j \leq 1$.

We now collect some well-known properties about WIT sheaves.

Proposition 1.4. *The following results hold:*

(1) *There exists a Mukai spectral sequence*

$$E_2^{p,q} = \widehat{\Phi}^p(\Phi^q(\mathcal{E})) \implies \begin{cases} \mathcal{E} & \text{if } p+q=1 \\ 0 & \text{otherwise}. \end{cases}$$

(2) *Let $\mathcal{E}$ be a WIT_i-Φ sheaf on X. Then $\widehat{\mathcal{E}}$ is a WIT_{1-i}-$\widehat{\Phi}$ sheaf on X and $\widehat{\widehat{\mathcal{E}}} = \mathcal{E}$.*

(3) *For every sheaf $\mathcal{E}$ on X, the sheaf $\Phi^0(\mathcal{E})$ is WIT_1-$\widehat{\Phi}$, while the sheaf $\Phi^1(\mathcal{E})$ is WIT_0-$\widehat{\Phi}$.*

(4) *There exists a short exact sequence*

$$0 \longrightarrow \widehat{\Phi}^1(\Phi^0(\mathcal{E})) \longrightarrow \mathcal{E} \longrightarrow \widehat{\Phi}^0(\Phi^1(\mathcal{E})) \longrightarrow 0\,.$$

Proof. (1) and (2) follow from [2, Eq. 2.35 and Prop. 2.34]. (3) is a direct consequence of (1) and (4) is the exact sequence of lower terms of the Mukai spectral sequence. □

1.2. **Preservation of the absolute stability for some equivalences.**

1.2.1. *Pure sheaves and Simpson stability.* A notion of stability and semistability for pure sheaves on a projective scheme with respect to an ample divisor was given by Simpson in [34]. He also proved the existence of the corresponding moduli spaces.

Let X be a projective scheme of dimension n over an algebraically closed field k of characteristic zero and fix H a polarization, that is, an ample divisor on X. For any coherent sheaf $\mathcal{E}$ on X, denote $\mathcal{E}(sH) = \mathcal{E} \otimes \mathcal{O}_X(sH)$.

The *Hilbert polynomial* of $\mathcal{E}$ with respect to H is defined to be the unique polynomial $P_{\mathcal{E}}(s) \in \mathbb{Q}[s]$ given by

$$P_{\mathcal{E}}(s) = h^0(X, \mathcal{E}(sH)) \text{ for all } s \gg 0\,.$$

This polynomial has the form

$$P_{\mathcal{E}}(s) = \frac{r(\mathcal{E})}{m!} s^m + \frac{d(\mathcal{E})}{(m-1)!} s^{m-1} + \dots$$

where $r(\mathcal{E}) \geq 0$ and $d(\mathcal{E})$ are integer numbers and its degree $m \leq n$ is equal to the dimension of the support of $\mathcal{E}$.

Definition 1.5. A coherent sheaf $\mathcal{E}$ is *pure* of dimension m if the support of $\mathcal{E}$ has dimension m and the support of any nonzero subsheaf $0 \to \mathcal{F} \to \mathcal{E}$ has dimension m as well.

When X is integral, pure sheaves of dimension n are precisely torsion-free sheaves. We can then adopt the following definition.

Definition 1.6. A coherent sheaf $\mathcal{E}$ on X is *torsion-free* if it is pure of dimension $n = \dim X$, and it is a *torsion* sheaf if the dimension of its support is $m < n$.

When X is a projective curve with a fixed polarization H, the Hilbert polynomial of a coherent sheaf $\mathcal{E}$ on X is then

$$P_{\mathcal{E}}(s) = r(\mathcal{E})s + d(\mathcal{E}) \in \mathbb{Z}[s],$$

a polynomial with integer coefficients and at most of degree one. It is constant precisely for torsion sheaves.

The (Simpson) *slope* of $\mathcal{E}$ is defined as

$$\mu_S(\mathcal{E}) = \frac{d(\mathcal{E})}{r(\mathcal{E})}\,.$$

This is a rational number if $r(\mathcal{E}) \neq 0$ and it is equal to infinity for torsion sheaves. It allows us to define (Simpson) μ-stability and μ-semistability for pure sheaves as usual.

Definition 1.7. A sheaf $\mathcal{E}$ on X is (Simpson) μ_S-stable (resp. μ_S-semistable) with respect to H, if it is pure and for every proper subsheaf $\mathcal{F} \hookrightarrow \mathcal{E}$ one has $\mu_S(\mathcal{F}) < \mu_S(\mathcal{E})$ (resp. $\mu_S(\mathcal{F}) \leq \mu_S(\mathcal{E})$).

With these definitions any torsion sheaf on X is μ_S-semistable and it is μ_S-stable if and only if it has no proper subsheaves, that is, it is isomorphic to $\mathcal{O}_x$, the structure sheaf of a point $x \in X$. As a particular case of Simpson's work [34], we have the following existence result. Fixing a polynomial $P(s) = rs+d \in \mathbb{Z}[s]$, and a polarization H on X, if the class of μ_S-semistable sheaves on X, with respect to H, with Hilbert polynomial equal to P is non-empty, then it has a coarse moduli space $\mathcal{M}_X(r,d)$ which is a projective scheme over k. Rational points of $\mathcal{M}_X(r,d)$ correspond to S-equivalence classes of μ_S-semistable sheaves with Hilbert polynomial $P(t) = rt + d$.

Remark 1.8. When X is an integral curve and $\mathcal{E}$ is a coherent sheaf on it, one has classical notions of rank of $\mathcal{E}$, as the rank at the generic point of X, and degree of $\mathcal{E}$, as $\chi(\mathcal{E}) - \operatorname{rk}(\mathcal{E})\chi(\mathcal{O}_X)$. The Riemann-Roch theorem gives us what is the relation between the coefficients of the Hilbert polynomial and the usual rank and degree of $\mathcal{E}$, namely

$$\begin{aligned} r(\mathcal{E}) &= \deg(X) \cdot \operatorname{rk}(\mathcal{E}) \\ d(\mathcal{E}) &= \deg(\mathcal{E}) + \operatorname{rk}(\mathcal{E}) \cdot \chi(\mathcal{O}_X) \end{aligned}$$

where $\deg(X)$ is the degree of X defined in terms of the polarization H. In this case, the Simpson notions of μ_S-stability and μ_S-semistability are equivalent to the usual ones for torsion-free sheaves. Thus, for integral curves μ_S-semistability does not depend on the polarization. This is no longer true for non-integral curves (see [23] for more details). △

1.2.2. *Invariants of the transforms and the WIT condition.* In the rest of this section we will assume that X is a projective Gorenstein curve of arithmetic genus 1 with trivial dualizing sheaf and H is a fixed polarization on it of degree h.

Since the curve X may be a singular curve, we will work with the Hilbert polynomial of a sheaf instead of its Chern characters that might not be defined. The following proposition computes the Hilbert polynomial of the transform of $\mathcal{E}$ by the equivalences Φ and $\widehat{\Phi}$ of the previous subsection and by $\Psi = \Phi_{X\to X}^{\delta_*\mathcal{O}_X(H)}$ and $\widehat{\Psi} = \Phi_{X\to X}^{\delta_*\mathcal{O}_X(-H)}$. Remember that for a bounded complex $\mathcal{F}^\bullet$, the Euler characteristic is defined to be the alternate sum

$$\chi(\mathcal{F}^\bullet) = \sum_i (-1)^i \chi(\mathcal{H}^i(\mathcal{F}^\bullet))\,.$$

and the Hilbert polynomial is by definition $P_{\mathcal{F}^\bullet}(s) = \chi(\mathcal{F}^\bullet(sH))$.

Proposition 1.9. *Let $\mathcal{E}$ be a sheaf on X with Hilbert polynomial $P_{\mathcal{E}}(s) = rs + d$. Then*

(1) *The Hilbert polynomial of the complex $\Phi(\mathcal{E})$ (resp. $\widehat{\Phi}(\mathcal{E})$) is equal to $(dh-r)s-d$ (resp. $(dh+r)s+d$).*

(2) *The Hilbert polynomial of the sheaf $\Psi(\mathcal{E})$ (resp. $\widehat{\Psi}(\mathcal{E})$) is equal to $rs+d+r$ (resp. $rs+d-r$).*

Proof. (1) Denote $\mathcal{O} = \mathcal{O}_{X\times X}$ and consider the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O} \to \delta_* \mathcal{O}_X \to 0\,.$$

In the derived category $D^b_c(X)$, this induces an exact triangle

$$\Phi(\mathcal{E}) \to \Phi^{\mathcal{O}}_{X\to X}(\mathcal{E}) \to \mathcal{E} \to \Phi(\mathcal{E})[1] \tag{1.1}$$

for any sheaf $\mathcal{E}$ on X. Since the Euler characteristic is additive for exact triangles in the derived category, the Hilbert polynomial of the complex $\Phi(\mathcal{E})$ is equal to

$$\chi(\Phi^{\mathcal{O}}_{X\to X}(\mathcal{E})(sH)) - (rs+d)\,.$$

If $p\colon X \to \operatorname{Spec} k$ is the projection of X onto a point and $\pi_i\colon X\times X \to X$ are the natural projections, the base-change formula for the diagram

$$\begin{tikzcd} X\times X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"'] & \operatorname{Spec} k \end{tikzcd}$$

shows that

$$\Phi^{\mathcal{O}}_{X\to X}(\mathcal{E}) = \mathbf{R}\pi_{2*}(\pi_1^*\mathcal{E}) \simeq p^*\mathbf{R}p_*(\mathcal{E}) \simeq p^*\mathbf{R}\Gamma(X,\mathcal{E}) = \mathbf{R}\Gamma(X,\mathcal{E}) \otimes_k \mathcal{O}_X\,. \tag{1.2}$$

Then $\chi(\Phi^{\mathcal{O}}_{X\to X}(\mathcal{E})(sH)) = \chi(\mathbf{R}\Gamma(X,\mathcal{E})\otimes_k \mathcal{O}_X(sH)) = \chi(\mathcal{E})\chi(\mathcal{O}_X(sH)) = d(sh)$, and the result follows. The Hilbert polynomial of $\widehat{\Phi}(\mathcal{E})$ is computed using that $\Phi\circ\widehat{\Phi} \simeq [-1]$.

(2) Since the equivalence Ψ (resp. $\widehat{\Psi}$) is given by twisting by the line bundle $\mathcal{O}_X(H)$ (resp. $\mathcal{O}_X(-H)$), this part is immediate. □

Remark 1.10. Notice that if $\mathcal{L}$ is an arbitrary line bundle on X, the second coefficient of the Hilbert polynomial of the transform $\Phi^{\delta_*\mathcal{L}}_{X\to X}(\mathcal{E})$ is not in general a linear function of the coefficients r and d (see Example 2.2). △

Any sheaf $\mathcal{E}$ on X is WIT_0-Ψ and WIT_0-$\widehat{\Psi}$. In order to prove the preservation of stability under the equivalence of $D^b_c(X)$ defined by the ideal of the diagonal $\mathcal{I}_\Delta$, we shall need a description of semistable sheaves WIT_i with respect to the Fourier-Mukai transforms Φ and $\widehat{\Phi}$.

Corollary 1.11. *Let $\mathcal{E}$ be a non-zero sheaf on X.*

(1) *If $\mathcal{E}$ is WIT_0-Φ, then $\mu_S(\mathcal{E}) > 1/h$.*
(2) *If $\mathcal{E}$ is WIT_1-Φ, then $\mu_S(\mathcal{E}) \leq 1/h$.*
(3) *If $\mathcal{E}$ is WIT_0-$\widehat{\Phi}$, then $\mu_S(\mathcal{E}) > -1/h$.*
(4) *If $\mathcal{E}$ is WIT_1-$\widehat{\Phi}$, then $\mu_S(\mathcal{E}) \leq -1/h$*

Proof. Let $P_{\mathcal{E}}(s) = rs+d$ the Hilbert polynomial of $\mathcal{E}$. By Proposition 1.9, the Hilbert polynomial of $\Phi(\mathcal{E})$ is $P_{\Phi(\mathcal{E})}(s) = (dh-r)s-d$. Suppose that $\mathcal{E}$ is WIT_0-Φ. Then $\widehat{\mathcal{E}} = \Phi(\mathcal{E})$ and $dh \geq r$. If $dh = r$, the transform $\widehat{\mathcal{E}}$ is a torsion sheaf of length

$-d = -r/h \leq 0$ and thus it is equal to zero. But this is absurd because $\mathcal{E}$ is non-zero. The second statement follows straightforwardly and the proof for $\widehat{\Phi}$ is similar. □

Remark 1.12. The following easy properties will be used in the rest of the section:

(1) Torsion sheaves on X are WIT_0 with respect to both equivalences Φ and $\widehat{\Phi}$.
(2) If a sheaf $\mathcal{E}$ is WIT_1 with Hilbert polynomial $P_{\mathcal{E}}(s) = rs + d$, then $r \neq 0$.

△

Proposition 1.13. *If $\mathcal{E}$ is a μ_S-semistable sheaf on X, then*

(1) *$\mathcal{E}$ is WIT_0-Φ if and only if $\mu_S(\mathcal{E}) > 1/h$.*
(2) *$\mathcal{E}$ is WIT_1-Φ if and only if $\mu_S(\mathcal{E}) \leq 1/h$.*

Proof. If $\mathcal{E}$ is a torsion sheaf, the result follows from Remark 1.12. Suppose then that $\mathcal{E}$ is torsion-free and consider the short exact sequence

$$0 \longrightarrow \widehat{\Phi}^1(\Phi^0(\mathcal{E})) \longrightarrow \mathcal{E} \longrightarrow \widehat{\Phi}^0(\Phi^1(\mathcal{E})) \longrightarrow 0\,.$$

(1) The direct implication is given by Corollary 1.11. Let us prove the converse. If $\mu_S(\mathcal{E}) > 1/h$ and $\mathcal{E}$ is not WIT_0-Φ, by the above exact sequence $\widehat{\Phi}^0(\Phi^1(\mathcal{E}))$ is a non-zero quotient of $\mathcal{E}$ and WIT_1-Φ. By Corollary 1.11, its slope is $\mu_S(\widehat{\Phi}^0(\Phi^1(\mathcal{E}))) \leq 1/h$ and consequently, $\mu_S(\widehat{\Phi}^1(\Phi^0(\mathcal{E}))) > \mu_S(\mathcal{E})$. This contradicts the semistability of $\mathcal{E}$. Thus, $\mathcal{E}$ is WIT_0-Φ.

(2) The direct implication follows again from Corollary 1.11. For the converse we proceed as before. If $\mu_S(\mathcal{E}) \leq 1/h$ and $\mathcal{E}$ is not WIT_1-Φ, by the exact sequence $\widehat{\Phi}^1(\Phi^0(\mathcal{E}))$ is a non-zero subsheaf of $\mathcal{E}$ and WIT_0-Φ. By Corollary 1.11, its slope is $\mu_S(\widehat{\Phi}^1(\Phi^0(\mathcal{E}))) > 1/h$ and this contradicts the semistability of $\mathcal{E}$. Thus, $\mathcal{E}$ is WIT_1-Φ. □

There exists a similar result for $\widehat{\Phi}$ whose proof is analogous.

Proposition 1.14. *If $\mathcal{E}$ is a μ_S-semistable sheaf on X, then*

(1) *$\mathcal{E}$ is WIT_0-$\widehat{\Phi}$ if and only if $\mu_S(\mathcal{E}) > -1/h$.*
(2) *$\mathcal{E}$ is WIT_1-$\widehat{\Phi}$ if and only if $\mu_S(\mathcal{E}) \leq -1/h$.*

□

The following simple but useful result is known as the Parseval theorem (see for instance [2] for a proof).

Proposition 1.15. *Let $\Phi\colon D(X) \to D(Y)$ be an exact fully faithful functor, $\mathcal{F}$ a WIT_i-Φ sheaf and $\mathcal{G}$ a WIT_j-Φ sheaf on X. Then for all k, one has*

$$\operatorname{Ext}^k_X(\mathcal{F},\mathcal{G}) \simeq \operatorname{Ext}^{k+i-j}_Y(\widehat{\mathcal{F}},\widehat{\mathcal{G}})$$

In particular if $\mathcal{F}$ is a simple WIT-Φ sheaf, then the transform $\widehat{\mathcal{F}}$ is also simple. □

Proposition 1.16. *Let $\mathcal{E}$ be a simple (resp. indecomposable) semistable sheaf on X. Then the transform $\widehat{\mathcal{E}}$ with respect to both Φ and $\widehat{\Phi}$ is also a simple (resp. indecomposable) sheaf.*

Proof. By Propositions 1.13 and 1.14, $\mathcal{E}$ is WIT with respect to both Φ and $\widehat{\Phi}$. Then $\widehat{\mathcal{E}}$ is indecomposable when $\mathcal{E}$ is so. Moreover, if $\mathcal{E}$ is simple, $\widehat{\mathcal{E}}$ is simple by the Parseval formula (Proposition 1.15). □

1.2.3. *Preservation of (semi)stability.* If X is an irreducible curve of arithmetic genus one, the group of exact auto-equivalences of its derived category $D^b_c(X)$ is described in [10]. As it happens for the smooth case, this group is generated by the trivial equivalences (twists by line bundles on X, automorphisms of X and the shift functor [1]) together with the Fourier-Mukai transform Φ whose kernel is the ideal of the diagonal. Then, taking into account that on integral curves tensoring by line bundles preserves trivially the (semi)stability of sheaves, the fact that the non-trivial Fourier-Mukai functor Φ transforms (semi)stable sheaves into (semi)stable sheaves (up to shift) and stable sheaves into stable ones (also up to shift) (cf. [2]) ensures that any auto-equivalence of the derived category $D^b_c(X)$ preserves stability.

However, this is no longer true for non-irreducible curves. Actually, if X is a non-irreducible curve, there are examples of equivalences of $D^b_c(X)$ that do not preserve semistability, and we can find examples of such equivalences among those of the most simple type, namely, among the equivalences $\Phi^{\delta_*\mathcal{L}}_{X\to X}$ consisting of twisting by a line bundle $\mathcal{L}$.

Consider, for instance, a curve X of type E_2, that is, two rational curves meeting transversally at two points (cf. Figure 1).

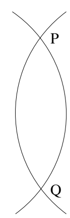

FIGURE 1. The curve E_2

Take a line bundle $\mathcal{L}$ on X which has degree 2 in one irreducible component and degree -2 in the other one. From Propositions 6.2 and 6.3 in [23], $\mathcal{O}_X$ is a stable sheaf but $\mathcal{L}$ is not even semistable. Then twisting by $\mathcal{L}$ is a Fourier-Mukai transform which does not preserve semistability.

Remark 1.17. A straightforward computation shows that both $H^0(X,\mathcal{L})$ and $H^1(X,\mathcal{L})$ are one dimensional vector spaces. Using Equations (1.1) and (1.2) we deduce that $\Phi(\mathcal{E})$ is a complex with two nonzero cohomology sheaves. This proves that simple (unstable) sheaves on X may fail to be WIT. $\triangle$

Thus it is important to characterize the auto-equivalences of $D^b_c(X)$ which preserve stability on a non-irreducible curve X of arithmetic genus 1. This seems to be a difficult task, and here we just provide non-trivial instances of such equivalences.

To begin with, note that twisting by the ample sheaf $\mathcal{O}_X(H)$ trivially preserves stability, that is, the transform of a μ_S-(semi)stable sheaf by the equivalences Ψ or $\widehat{\Psi}$ is again μ_S-(semi)stable (cf. Proposition 1.9).

In this section we prove that the non-trivial Fourier-Mukai functors Φ and $\widehat{\Phi}$ preserve semistability as well.

Lemma 1.18. *Let $\mathcal{E}$ be a sheaf on X with Hilbert polynomial $P_\mathcal{E}(s) = dhs+d$ and $d > 0$. Then $\mathcal{E}$ is WIT_1-Φ if and only if $\mathcal{E}$ is a torsion free μ_S-semistable sheaf. Analogously if $P_\mathcal{E}(s) = dhs - d$ with $d > 0$, then $\mathcal{E}$ is WIT_1-$\widehat{\Phi}$ if and only if $\mathcal{E}$ is a torsion free μ_S-semistable sheaf.*

Proof. If $\mathcal{E}$ is WIT_1-Φ, then any subsheaf is WIT_1-Φ as well. Since by Remark 1.12 torsion sheaves are WIT_0-Φ, this proves that $\mathcal{E}$ is torsion-free. Moreover, if $\mathcal{F} \hookrightarrow \mathcal{E}$ is a subsheaf, by Corollary 1.11, $\mu_S(\mathcal{F}) \leq 1/h = \mu_S(\mathcal{E})$, so that $\mathcal{E}$ is μ_S-semistable. The converse is part of Proposition 1.13. The proof for $\widehat{\Phi}$ is similar. $\square$

Proposition 1.19. *Let $\mathcal{T}$ a non-zero torsion sheaf on X. Then the transform $\widehat{\mathcal{T}}$ with respect to both equivalences Φ and $\widehat{\Phi}$ is a torsion-free μ_S-semistable sheaf.*

Proof. Since $\mathcal{T}$ is a torsion sheaf, $\mathcal{T}$ is WIT_0-Φ and its Hilbert polynomial is $P_{\mathcal{T}}(s) = d$ with $d > 0$. The transform $\widehat{\mathcal{T}}$ is WIT_1-$\widehat{\Phi}$ and, by Proposition 1.9, its Hilbert polynomial is $P_{\widehat{\mathcal{T}}}(s) = dhs - d$. We conclude by Lemma 1.18. The proof for $\widehat{\Phi}$ is the same. $\square$

We state now the result that ensures the preservation of semistability under the Fourier-Mukai transforms Φ and $\widehat{\Phi}$.

Theorem 1.20. *Let X be a projective Gorenstein curve of arithmetic genus one and trivial dualizing sheaf. Fix a polarization H on X. Let $\mathcal{E}$ be a pure dimensional sheaf on X. If $\mathcal{E}$ is μ_S-semistable with respect to H, then its transform $\widehat{\mathcal{E}}$ with respect to both equivalences Φ and $\widehat{\Phi}$ is also μ_S-semistable with respect to H.*

Proof. Let $P_{\mathcal{E}}(s) = rs + d$ be the Hilbert polynomial of $\mathcal{E}$ with respect to the fixed polarization H and, as before, denote by h the degree of H. If $r = 0$, the result is proved in Proposition 1.19, so that we can assume that $\mathcal{E}$ is a torsion-free sheaf. Let us distinguish the following cases:

(1) If $dh = r$, $\widehat{\mathcal{E}}$ is a torsion sheaf and thus semistable.

(2) Suppose now that $dh > r$. By Proposition 1.13, $\mathcal{E}$ is WIT_0-Φ, so that $\widehat{\mathcal{E}}$ is WIT_1-$\widehat{\Phi}$. The same argument as in Lemma 1.18 shows that $\widehat{\mathcal{E}}$ is torsion-free. If $\widehat{\mathcal{E}}$ is not μ_S-semistable, there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \widehat{\mathcal{E}} \longrightarrow \mathcal{G} \longrightarrow 0\,,$$

with $\mu_S(\mathcal{G}) < \mu_S(\widehat{\mathcal{E}}) < \mu_S(\mathcal{F})$. Moreover, the existence of Harder-Narasimhan filtrations allows us to assume that $\mathcal{G}$ is a torsion-free μ_S-semistable sheaf. Since $\widehat{\mathcal{E}}$ is torsion-free and WIT_1-$\widehat{\Phi}$, by Proposition 1.13 $\mu_S(\widehat{\mathcal{E}}) < -1/h$ where the last inequality is strict because $\mathcal{E}$ is torsion-free. Then $\mathcal{G}$ is a μ_S-semistable sheaf with $\mu_S(\mathcal{G}) < -1/h$, so that $\mathcal{G}$ is WIT_1-$\widehat{\Phi}$ by Proposition 1.14. By applying the inverse Fourier-Mukai transform one obtains that $\mathcal{F}$ is WIT_1-$\widehat{\Phi}$ as well and that there is an exact sequence

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow \mathcal{E} \longrightarrow \widehat{\mathcal{G}} \longrightarrow 0\,.$$

Since $\mathcal{E}$ is torsion-free, one has $r(\widehat{\mathcal{F}}) \neq 0$ and $1/h < \mu_S(\widehat{\mathcal{F}}) \leq \mu_S(\mathcal{E})$ where the first inequality is due to the fact that $\widehat{\mathcal{F}}$ is WIT_0-Φ and the second is by the μ_S-semistability of $\mathcal{E}$. Since

$$\mu_S(\widehat{\mathcal{E}}) = \frac{\mu_S(\mathcal{E})}{1 - h\mu_S(\mathcal{E})} \quad \text{and} \quad \mu_S(\mathcal{F}) = \frac{\mu_S(\widehat{\mathcal{F}})}{1 - h\mu_S(\widehat{\mathcal{F}})}\,,$$

one obtains $\mu_S(\mathcal{F}) \leq \mu_S(\widehat{\mathcal{E}})$; this contradicts $\mu_S(\mathcal{F}) > \mu_S(\widehat{\mathcal{E}})$.

(3) Suppose finally that $dh < r$. Let us prove that $\widehat{\mathcal{E}}$ is a torsion-free sheaf. Indeed, a torsion subsheaf $\mathcal{T} \neq 0$ of $\widehat{\mathcal{E}}$ should necessary be WIT_0-$\widehat{\Phi}$ and $\mu_S(\widehat{\mathcal{T}}) = 1/h$. By

applying $\widehat{\Phi}$ we obtain $\widehat{\mathcal{T}} \hookrightarrow \mathcal{E}$ which contradicts the μ_S-semistability of $\mathcal{E}$. If $\widehat{\mathcal{E}}$ is not μ_S-semistable, as in the previous case we can assume the existence of an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \widehat{\mathcal{E}} \longrightarrow \mathcal{G} \longrightarrow 0$$

with $\mathcal{F}$ torsion-free, $\mathcal{G}$ WIT_0-$\widehat{\Phi}$ and $\mu_S(\mathcal{F}) > \mu_S(\widehat{\mathcal{E}}) > \mu_S(\mathcal{G})$. Moreover $\widehat{\mathcal{E}}$ is WIT_0-$\widehat{\Phi}$, so that $\mu_S(\mathcal{F}) > \mu_S(\widehat{\mathcal{E}}) > -1/h$. Then $\mathcal{F}$ is WIT_0-$\widehat{\Phi}$, by Proposition 1.14. Applying the inverse Fourier-Mukai transform we get the exact sequence

$$0 \longrightarrow \widehat{\mathcal{F}} \longrightarrow \mathcal{E} \longrightarrow \widehat{\mathcal{G}} \longrightarrow 0\,.$$

Since $\mu_S(\mathcal{F}) > \mu_S(\widehat{\mathcal{E}}) > -1/h$ we obtain

$$\mu_S(\widehat{\mathcal{F}}) = \frac{\mu_S(\mathcal{F})}{1 + h\mu_S(\mathcal{F})} > \frac{\mu_S(\widehat{\mathcal{E}})}{1 + h\mu_S(\widehat{\mathcal{E}})} = \mu_S(\mathcal{E})\,,$$

but this contradicts the fact that $\mathcal{E}$ is μ_S-semistable.

An analogous discussion proves the result for the Fourier-Mukai transform $\widehat{\Phi}$. □

The proof of the theorem implies the following consequences

Corollary 1.21. *Let $\widehat{\mathcal{E}}$ be the transform of a μ_S-semistable sheaf $\mathcal{E}$ with respect to Φ (resp. $\widehat{\Phi}$). The following holds*

(1) *If $\mathcal{E}$ is a torsion-free sheaf and $\mu_S(\mathcal{E}) \neq 1/h$ (resp. $-1/h$), then $\widehat{\mathcal{E}}$ is also torsion-free.*
(2) *If $\mu_S(\mathcal{E}) \neq 1/h$ (resp. $-1/h$) and $\mathcal{E}$ is μ_S-stable, then $\widehat{\mathcal{E}}$ is μ_S-stable as well.*
(3) *If $\mu_S(\mathcal{E}) = 1/h$ (resp. $-1/h$), then $\widehat{\mathcal{E}}$ is μ_S-stable if and only if $d = 1$.*

□

Remark 1.22. Note that if $\mathcal{E}$ is semistable with $\mu_S(\mathcal{E}) = 1/h$, $\widehat{\mathcal{E}}$ is a torsion sheaf and, even when the sheaf $\mathcal{E}$ is stable, we can only ensure the stability of $\widehat{\mathcal{E}}$ for $d = 1$. When $d > 1$, if $\mathcal{E}$ is indecomposable (for instance, if it is stable), the transform $\widehat{\mathcal{E}}$ is a torsion sheaf and it is indecomposable by Proposition 1.16; thus it is supported at a single point $x \in X$. If x is a smooth point, then $\widehat{\mathcal{E}} \simeq \mathcal{O}_{X,x}/\mathfrak{m}_x^d$. The structure of torsion sheaves supported at a singular point is much more complicated (see [9] for more details). Nevertheless, if $\widehat{\mathcal{E}}$ is supported at a possibly singular point x, one can see by induction on the length d that $\widehat{\mathcal{E}}$ is always S-equivalent to $\oplus_d \mathcal{O}_x$. △

Since the transform $\widehat{\mathcal{O}}_x = \Phi(\mathcal{O}_x)$ is the ideal sheaf $\mathfrak{m}_x$ of the point x, and $\widehat{\Phi}(\mathcal{O}_x) = \mathfrak{m}_x^*$, we deduce the following stability result.

Corollary 1.23. *If X is a polarized Gorenstein curve of arithmetic genus one and trivial dualizing sheaf, then the maximal ideal $\mathfrak{m}_x$ for any point $x \in X$ and its dual $\mathfrak{m}_x^*$ are stable sheaves.*

Remark 1.24. When X is irreducible, this is a trivial fact. For Gorenstein reducible curves of arithmetic genus bigger or equal than 2, the semistability of $\mathfrak{m}_x$ for an arbitrary point x of X has been recently proved in [13] using different techniques. △

The equivalences Φ and Ψ define scheme isomorphisms between the corresponding moduli spaces:

Corollary 1.25. *Let (r, d) be a pair of integers with $r \geq 0$.*

(1) *The Fourier-Mukai functors* Φ *and* Ψ *induce scheme isomorphisms of moduli spaces*

$$\begin{aligned}
&\mathcal{M}_X(r,d)\simeq\mathcal{M}_X(dh-r,-d) \quad \textit{for } d/r>1/h,\\
&\mathcal{M}_X(r,d)\simeq\mathcal{M}_X(-dh+r,d) \quad \textit{for } d/r\leq 1/h, \textit{ and}\\
&\mathcal{M}_X(r,d)\simeq\mathcal{M}_X(r,d+r)\,.
\end{aligned}$$

(2) *The moduli space* $\mathcal{M}_X(r,d)$ *is isomorphic either to* $\mathcal{M}_X(0,d_0)\simeq Sym^{d_0}(X)$ *with* $d_0>0$*, or to* $\mathcal{M}_X(r_0,0)$ *with* $r_0>0$ *or to* $\mathcal{M}_X(r_0,d_0)$ *with* $2r_0/h\leq d_0<r_0$.

Proof. For the first part see for instance [2, Corollary 2.65]. Let us prove the second one. By Remark 1.22 and arguing as in the proof of [2, Corollary 3.33], one proves that $\mathcal{M}_X(0,d)\simeq Sym^d(X)$. Consider the family B of all pairs of integers (r',d') with $r'\geq 0$ that are related with (r,d) by the isomorphisms in (1). Take (r_0,d_0) in B such that $r_0\geq 0$ is the minimum r' among the pairs in B. If $r_0=0$, then $d_0>0$ and $\mathcal{M}_X(r,d)\simeq\mathcal{M}_X(0,d_0)$. Assume then that $r_0>0$. Since by applying the equivalence Ψ it is possible to increase d and by applying $\widehat{\Psi}$ to decrease it, if one considers now C as the family of all pairs in B with $r'=r_0$, one can choose d_0 as the minimum d' among pairs in C such that $0\leq d_0<r_0$. If $d_0=0$, then $\mathcal{M}_X(r,d)\equiv\mathcal{M}_X(r_0,0)$. If $d_0>0$, we claim that all the sheaves in $\mathcal{M}_X(r_0,d_0)$ are WIT_0-Φ. Indeed, otherwise $r_0\leq r_0-d_0h$ by the choice of r_0 and then $d_0<0$ which contradicts our choice of d_0. Then, it has to be $r_0\leq d_0h-r_0$ so that $2r_0/h\leq d_0<r_0$ and the proof is complete. □

Remark 1.26. Notice that if X has only two irreducible components, as it happens for the Kodaira fibers E_2 (cf. Figure 1) and III, and the polarization H has degree $h=2$, Corollary 1.25 reduces the study of the moduli spaces $\mathcal{M}_X(r,d)$ just to the case $d=0$. Some results in this case can be found in the next section. △

Remark 1.27. If X is irreducible, we can take $h=1$, so that r and d are the usual rank and degree. In this situation, the last case in (2) of Corollary 1.25 does not occur; moreover, there is an isomorphism $\mathcal{M}(r_0,0)\simeq\mathcal{M}_X(0,r_0)$. We get then that $\mathcal{M}(r,d)\simeq Sym^{r_0}(X)$. Using the transforms Φ and Ψ and the Euclid algorithm, one can see that $r_0=\gcd(r,d)$, as proven in [2, Chapter 6] by generalizing an argument described for smooth elliptic curves by Bridgeland [7] and Polishchuk [29]. A consequence is that there are no stable sheaves on X if $\gcd(r,d)>1$. As already mentioned in the introduction, a complete description of these moduli spaces can be found in [2, Chapter 6]. △

1.3. **Relative moduli spaces.** Let $p\colon S\to B$ a genus one fibration, that is, a projective Gorenstein morphism whose fibers are curves of arithmetic genus one and trivial dualizing sheaf but without further assumptions on S, B or the fibers; in particular, non-reduced fibers are allowed. Consider the relative integral functor

$$\Phi=\Phi^{\mathcal{I}_\Delta}_{S\to S}\colon D^b_c(S)\to D^b_c(S)\,,$$

with kernel the ideal sheaf $\mathcal{I}_\Delta$ of the relative diagonal immersion $\delta\colon S\hookrightarrow S\times_B S$. By [20, Proposition 2.16], it is an equivalence of categories.

Fix a relative polarization $\mathcal{H}$ on the fibers of p and denote $\hat{p}\colon\mathcal{M}_{S/B}(r,d)\to B$ the relative coarse moduli space of μ_S-semistable sheaves on the fibers (with respect to the induced polarization) that have Hilbert polynomial $P(s)=rs+d$. Closed points of the fiber $\hat{p}^{-1}(b)=\mathcal{M}_{S_b}(r,d)$ represent S-equivalence classes of μ_S-semistable sheaves on the fiber S_b with Hilbert polynomial $P(s) = rs + d$. Denote also Ψ the relative auto-equivalence of $D^b_c(S)$ given by twisting by the line bundle $\mathcal{O}_{S/B}(\mathcal{H})$.

Taking into account [2, Corollary 6.3] and Corollary 1.25, we get:

Corollary 1.28. *Let (r, d) be a pair of integers with $r \geq 0$.*

(1) *The Fourier-Mukai functors Φ and Ψ induce scheme isomorphisms of moduli spaces*

$$\begin{aligned}
&\mathcal{M}_{S/B}(r,d) \simeq \mathcal{M}_{S/B}(dh-r,-d) \quad \text{for } d/r > 1/h,\\
&\mathcal{M}_{S/B}(r,d) \simeq \mathcal{M}_{S/B}(-dh+r,d) \quad \text{for } d/r \leq 1/h, \text{ and}\\
&\mathcal{M}_{S/B}(r,d) \simeq \mathcal{M}_{S/B}(r,d+r)\,.
\end{aligned}$$

(2) *The moduli space $\mathcal{M}_{S/B}(r,d)$ is isomorphic either to $\mathcal{M}_{S/B}(0,d_0) \simeq Sym^{d_0}(S/B)$ with $d_0 > 0$, or to $\mathcal{M}_{S/B}(r_0,0)$ with $r_0 > 0$ or to $\mathcal{M}_{S/B}(r_0,d_0)$ with $2r_0/h \leq d_0 < r_0$.*

□

2. Moduli spaces of degree zero sheaves for E_N

In this section, we give a description of the connected component of the moduli space $\mathcal{M}_X(r,0)$ with $r > 0$ containing vector bundles when X is a curve of type E_N, that is, a cycle of N projective lines (cf. Figure 2). The description is achieved by combining two different ingredients; the first one is the description of indecomposable torsion-free sheaves on cycles E_N given in [14, 5], and the second one is the description of (semi)stable line bundles on tree-like curves and cycles carried out by one of the authors in [24, 23].

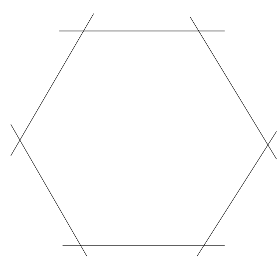

Figure 2. The curve E_6

2.1. **Coherent sheaves on reducible curves.** We collect some results about coherent sheaves on reducible curves. Let X be any projective connected and reduced curve over an algebraically closed field k. Denote by $C_1, \dots, C_N$ the irreducible components of X and by $x_1, \dots, x_k$ the intersection points of $C_1, \dots, C_N$. Let $\mathcal{E}$ a coherent sheaf on X and denote $\mathcal{E}_{C_i} = (\mathcal{E} \otimes \mathcal{O}_{C_i})/\text{torsion}$ its restriction to C_i modulo torsion. Let $r_i = r_i(\mathcal{E})$ and $d_i = d_i(\mathcal{E})$ be the rank and the degree of $\mathcal{E}_{C_i}$.

Definition 2.1. The *multirank* and *multidegree* of a coherent sheaf $\mathcal{E}$ on X are the N-tuples $\underline{r}(\mathcal{E}) = (r_1, \dots, r_N)$ and $\underline{d}(\mathcal{E}) = (d_1, \dots, d_N)$.

Let H be a polarization on X of degree h and denote by h_i the degree of H on C_i. As in [31], for any pure dimension one sheaf $\mathcal{E}$ on X, there is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_{C_1} \oplus \cdots \oplus \mathcal{E}_{C_N} \to T \to 0\,, \tag{2.1}$$

where T is a torsion sheaf whose support is contained in the set $\{x_1, \dots, x_k\}$. The first arrow in the sequence (2.1) is the composition of the canonical morphisms $\mathcal{E} \to \pi_*(\pi^*\mathcal{E}) \to \pi_*(\pi^*\mathcal{E}/tors)$, where π is the normalization morphism of X.

Let $P_{\mathcal{E}}(s) = rs + d$ be the Hilbert polynomial of $\mathcal{E}$ with respect to H. Since the Hilbert polynomial is additive, from the above exact sequence, one obtains that

$$r = r_1 h_1 + \cdots + r_N h_N \,.$$

Hence, there is a natural decomposition

$$\mathcal{M}_X(r,d) = \coprod_{\underline{r}\in\mathbb{Z}^N} \mathcal{M}_X(\underline{r},d) \tag{2.2}$$

where $\mathcal{M}_X(\underline{r},d)$ is the moduli space of those semistable sheaves $\mathcal{E}$ on X of multirank $\underline{r}$ and the union runs over all N-tuples $\underline{r} = (r_1, \dots, r_N)$ of non-negative integers such that $r = r_1 h_1 + \cdots + r_N h_N$.

This decomposition becomes necessary in the analysis of some moduli spaces $\mathcal{M}_X(r,d)$ because, as it was mentioned in Remark 1.10, if Γ is an equivalence of $D^b_c(X)$ and $\mathcal{E}$ is a WIT-Γ sheaf, it is not true in general that the coefficients of the Hilbert polynomial of $\Gamma(\mathcal{E})$ are linear functions of the coefficients of the Hilbert polynomial of $\mathcal{E}$. This means that the action of an arbitrary equivalence of $D^b_c(X)$ does not send all connected components of the moduli space $\mathcal{M}_X(r,d)$ into the same moduli space $\mathcal{M}_X(r',d')$. Here we have an example of this fact.

Example 2.2. Let X be a curve of type E_2, that is, two rational curves C_1 and C_2 meeting transversally at two points with a polarization H such that $h_1 = h_2 = 1$ (Figure 1). Let $\mathcal{L}$ be a line bundle on X with multidegree $\underline{d}(\mathcal{L}) = (d_1, d_2)$ with $d_1 \neq d_2$ and take $\Gamma = \Phi^{\delta_*\mathcal{L}}_{X\to X}$ the equivalence defined by twisting by $\mathcal{L}$. If $\mathcal{E}$ is a coherent sheaf on X with Hilbert polynomial $P_{\mathcal{E}}(s) = rs+d$ and multirank $\underline{r}(\mathcal{E}) = (r_1, r_2)$, the Hilbert polynomial of $\Gamma(\mathcal{E})$ is equal to $rs + d + (r_1 d_1 + r_2 d_2)$. Thus, the connected component $\mathcal{M}_X((r,0),d) \subseteq \mathcal{M}_X(r,d)$ is sent by Γ into the moduli space $\mathcal{M}_X(r, d + r d_1)$ while the component $\mathcal{M}_X((0,r),d)$ is mapped into $\mathcal{M}(r, d + r d_2)$.

The following proposition provides better invariants. If $K(D^b_c(X))$ denotes the Grothendieck group of the triangulated category $D^b_c(X)$ (cf. [17]), one has $K(D^b_c(X)) \simeq K(\mathrm{Coh}(X))$ and this group is usually denoted $K_\bullet(X)$.

Proposition 2.3. *Let X be any reduced connected and projective curve. If every irreducible component of X is isomorphic to $\mathbb{P}^1$ the projective line, then there is an isomorphism $K_\bullet(X) \simeq \mathbb{Z}^{N+1}$ where N is the number of irreducible components of X. Moreover the above isomorphism is defined by sending the class of any coherent sheaf $[\mathcal{E}]$ in $K_\bullet(X)$ to $(\underline{r}(\mathcal{E}), \chi(\mathcal{E})) \in \mathbb{Z}^{N+1}$.*

Proof. Let us denote by $C_1, \dots, C_N$ the irreducible components of the curve X. Since the curve is connected and its irreducible components are isomorphic to $\mathbb{P}^1$ we conclude that any two points are rationally equivalent, that is $A_0(X) = \mathbb{Z}[x]$, where $[x]$ is the class of a point of X. On the other hand it is well known [16, Example 1.3.2] that the n-th Chow group of an n-dimensional scheme is the free abelian group on its n-dimensional irreducible components, therefore $A_1 = \mathbb{Z}[C_1] \oplus \cdots \oplus \mathbb{Z}[C_N]$.

The normalization $\pi\colon \widetilde{X} \to X$ is a Chow envelope of X, thus $\pi_*\colon K_\bullet(\widetilde{X}) \to K_\bullet(X)$ is surjective [16, Lemma 18.3]. Since $\widetilde{X} = \widetilde{C}_1 \coprod \cdots \coprod \widetilde{C}_N$ one has

$$K_\bullet(\widetilde{X}) \simeq K_\bullet(\widetilde{C}_1) \oplus \cdots \oplus K_\bullet(\widetilde{C}_N).$$

Taking into account that $\widetilde{C}_i \simeq \mathbb{P}^1$ we get $K_\bullet(\widetilde{X}) = \bigoplus_{i=1}^N \mathbb{Z}[\mathcal{O}_{\widetilde{C}_i}] \oplus \mathbb{Z}[\mathcal{O}_{\tilde{x}_i}]$, where $\tilde{x}_i \in \widetilde{C}_i$. Given any two points $\tilde{x}_i$, $\tilde{y}_i$ in $\widetilde{C}_i \simeq \mathbb{P}^1$ we know that $[\mathcal{O}_{\tilde{x}_i}] = [\mathcal{O}_{\tilde{y}_i}]$. Therefore, since X is connected, the surjectivity of π_* implies that $K_\bullet(X)$ is generated by

$$[\mathcal{O}_{C_1}], \dots, [\mathcal{O}_{C_N}], [\mathcal{O}_x].$$

By the Riemann-Roch theorem for algebraic schemes [16, Theorem 18.3] there is a homomorphism $\tau_X \colon K_\bullet(X) \to A_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the following properties:

(1) For any k-dimensional closed subvariety Y of X, one has $\tau_X([\mathcal{O}_Y]) = [Y] +$ terms of dimension $< k$,
(2) $(\tau_X)_{\mathbb{Q}} \colon K_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism.

Using the first property one easily sees that $\{\tau_X([\mathcal{O}_{C_1}]), \dots, \tau_X([\mathcal{O}_{C_N}]), \tau_X([\mathcal{O}_x])\}$ is a basis of $A_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Taking into account that $\{[\mathcal{O}_{C_1}], \dots, [\mathcal{O}_{C_N}], [\mathcal{O}_x]\}$ is a system of generators of $K_\bullet(X)$, the second property of τ_X implies that it is also a basis.

The final statement follows now straightforwardly by applying the integer valued mapping $(\underline{r}(-), \chi(-))$ to the basis of $K_\bullet(X)$ above constructed. □

All the functions defining the isomorphism of Proposition 2.3, that is, the ranks r_i and the Euler characteristic χ, are additive on exact triangles of $D^b_c(X)$. Hence, any equivalence Γ of $D^b_c(X)$ induces a group automorphism γ of $K_\bullet(X)$, such that there is a commutative square

$$\begin{tikzcd} D^b_c(X) \arrow[r, "\Gamma"] \arrow[d] & D^b_c(X) \arrow[d] \\ K_\bullet(X) \arrow[r, "\gamma"] & K_\bullet(X) \end{tikzcd}$$

where the vertical arrows are the natural ones.

Note that if $\mathcal{E}$ is a vector bundle on X, then $\sum_{i=1}^N d_i = d$. Thus, the category of vector bundles on X of rank r and degree d (whose Hilbert polynomial is $P_{\mathcal{E}}(s) = rhs + d$) decomposes as

$$\mathrm{VB}_X(r,d) \simeq \coprod_{\underline{d} \in \mathbb{Z}^N} \mathrm{VB}_X(r, \underline{d}) \,,$$

where now the union runs over all $\underline{d} = (d_1, \dots, d_N) \in \mathbb{Z}^N$ such that $\sum_{i=1}^N d_i = d$. However, since for non-locally free sheaves it is not true that $\sum_{i=1}^N d_i = d$, there is not a similar decomposition for the moduli space $\mathcal{M}_X(r,d)$.

To finish this subsection, let us show how μ_S-semistability behaves under direct and inverse images by Galois coverings of reducible curves. For non-singular projective and irreducible varieties, similar results were proved by Takemoto in [36].

Lemma 2.4. *Let X be a projective connected and reduced curve whose irreducible components are smooth. Let H be a polarization on X and let $f \colon Y \to X$ be an ètale Galois covering of degree n where Y is also connected.*

(1) *If $\mathcal{E}$ is a torsion-free sheaf on X such that $f^*\mathcal{E}$ is μ_S-semistable with respect to f^*H, then $\mathcal{E}$ is μ_S-semistable with respect to H.*
(2) *A torsion-free sheaf $\mathcal{F}$ on Y is μ_S-semistable with respect to f^*H if and only if $f_*(\mathcal{F})$ is μ_S-semistable with respect to H.*

Proof. (1) By (2.1), if $C_1, \dots, C_N$ are the irreducible components of X, there is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_{C_1} \oplus \cdots \oplus \mathcal{E}_{C_N} \to T \to 0\,,$$

where $\mathcal{E}_{C_i}$ is a vector bundle on C_i for all i and T is a torsion sheaf. Because C_i is smooth, one has that $P_{f^*\mathcal{E}_{C_i}}(s) = nP_{\mathcal{E}_{C_i}}(s)$. Moreover, since f is flat,

$$0 \to f^*\mathcal{E} \to f^*\mathcal{E}_{C_1} \oplus \cdots \oplus f^*\mathcal{E}_{C_r} \to f^*T \to 0$$

is also an exact sequence. Then the additivity of the Hilbert polynomial allow us to conclude that $P_{f^*\mathcal{E}}(s) = nP_{\mathcal{E}}(s)$ where both polynomials are computed with the corresponding polarizations. Hence

$$\mu_S(f^*\mathcal{E}) = \mu_S(\mathcal{E}) \tag{2.3}$$

and the result follows.

(2) By (1), to prove the direct implication it suffices to prove that $f^*f_*(\mathcal{F})$ is semistable with respect to f^*H. Since any Galois ètale covering with X and Y connected is trivialized by itself, $f^*f_*(\mathcal{F}) = \oplus\rho^*\mathcal{F}$ where ρ runs over the Galois group of f. Then the result follows because any extension of two μ_S-semistable sheaves with the same slope is μ_S-semistable as well. The converse is true for any finite morphism. □

It is important to remark that Equation (2.3) is true because we are considering the Simpson's slope and not the usual slope, defined for a torsion-free sheaf as the quotient obtained by dividing the degree by the rank. For smooth varieties, it is a well-known fact (cf. [36]) that if f is an unramified covering then the relation between the usual slopes of $\mathcal{E}$ and of $f^*\mathcal{E}$ is given instead by $\mu(f^*\mathcal{E}) = \deg f \cdot \mu(\mathcal{E})$.

2.2. **Indecomposable torsion free sheaves on E_N.** The purpose of this subsection is just to state some known results about the classification of indecomposable vector bundles and torsion free sheaves on cycles E_N of projective lines that we shall use later. This classification was obtained for the first time by Drozd and Greuel in [14] for arbitrary base fields. Nevertheless, for algebraically closed base fields, one can find a geometric description of indecomposable torsion free sheaves on E_N in [5] that follows the classical description of vector bundles on elliptic curves given by Oda in [28] (see Theorem 2.6 below) and allows to study which of these sheaves are semistable.

Following the same argument that Atiyah used for smooth elliptic curves and taking into account that $\operatorname{Ext}^1(\mathcal{O}_{E_N}, \mathcal{O}_{E_N}) = k$, it is possible to inductively prove the following result.

Lemma 2.5. *Let E_N be a cycle of N projective lines. For any integer $m \geq 1$ there is a unique indecomposable vector bundle $\mathcal{F}_m$ on E_N appearing in the exact sequence*

$$0 \to \mathcal{F}_{m-1} \to \mathcal{F}_m \to \mathcal{O}_{E_N} \to 0\,, \quad \mathcal{F}_1 = \mathcal{O}_{E_N}\,.$$

□

Theorem 2.6. [5, Theorem 19] *Let E_N be a cycle of N projective lines and I_k be a chain of k projective lines (cf. Figure 3). Let $\mathcal{E}$ be an indecomposable torsion free sheaf on E_N. The following holds:*

(1) *If $\mathcal{E}$ is a vector bundle, there is an étale covering $\pi_r \colon E_{rN} \to E_N$, a line bundle $\mathcal{L}$ on E_{rN} and a number $m \in \mathbb{N}$ such that*

$$\mathcal{E} \simeq \pi_{r*}(\mathcal{L} \otimes \mathcal{F}_m)\,.$$

The integers r, m are determined by $\mathcal{E}$. Moreover, when $r > 1$ the multidegree $\underline{d}(\mathcal{L})$ of the line bundle $\mathcal{L}$ is non-periodic.

(2) *If $\mathcal{E}$ is not locally free, then there exits a finite map $p_k\colon I_k \to E_N$ (defined as the composition of some π_r and some closed immersion $i\colon I_k \hookrightarrow E_{rN}$) and a line bundle $\mathcal{L}$ on I_k such that $\mathcal{E} \simeq p_{k*}(\mathcal{L})$.*

□

Remember that if $\underline{d} = (d_1, \dots, d_N, d_{N+1}, \dots, d_{2N}, \dots, d_{(r-1)N}, \dots, d_{rN})$, then the non-periodicity means that $\underline{d} \neq \underline{d}[t]$ for $t = 1, \dots, r-1$ where

$$\underline{d}[1] = (d_{N+1}, \dots, d_{2N}, \dots, d_{(r-1)N}, \dots, d_{rN}, d_1, \dots, d_N)$$

and $\underline{d}[t] = (\underline{d}[t-1])[1]$.

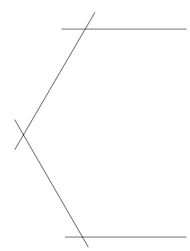

FIGURE 3. The curve I_4

2.3. **Locally free sheaves on cycles.**

Proposition 2.7. *Let $\mathcal{E}$ be a pure dimension one sheaf on E_N. Then*

$$\sum_{i=1}^{N} d_i(\mathcal{E}) - \chi(\mathcal{E}) \leq 0\,,$$

and $\sum_{i=1}^{N} d_i(\mathcal{E}) - \chi(\mathcal{E}) = 0$ if and only if $\mathcal{E}$ is locally free.

Proof. Since the function $\sum_{i=1}^{N} d_i(\mathcal{E}) - \chi(\mathcal{E})$ is additive over direct sums of sheaves, we can assume that $\mathcal{E}$ is indecomposable. We can then apply Theorem 2.6. If $\mathcal{E}$ is not locally free, then $\mathcal{E} \simeq p_{k*}(\mathcal{L})$, where $\mathcal{L}$ is a line bundle on a chain I_k of projective lines and $p_k\colon I_k \to E_N$ is a finite morphism. One has $\chi(\mathcal{E}) = \chi(\mathcal{L}) = 1 + \sum_D d(\mathcal{L}_D)$, where the sum runs over the irreducible components D of I_k. Moreover $\sum_D d(\mathcal{L}_D) = \sum_{i=1}^{N} d_i(\mathcal{E})$ and then $\sum_{i=1}^{N} d_i(\mathcal{E}) - \chi(\mathcal{E}) = -1$.

If $\mathcal{E}$ is locally free, then $\mathcal{E} \simeq \pi_{s*}(\mathcal{L} \otimes \mathcal{F}_m)$ for a finite morphism $\pi_s\colon E_{sN} \to E_N$ and a line bundle $\mathcal{L}$ on E_{sN}. Thus $\chi(\mathcal{E}) = \chi(\mathcal{L} \otimes \mathcal{F}_m) = m\chi(\mathcal{L}) = m\sum_{j=1}^{sN} d(\mathcal{L}_{D_j})$, where the sum runs over the irreducible components D_j of E_{sN}. Thus $\sum_{i=1}^{N} d_i(\mathcal{E}) - \chi(\mathcal{E}) = 0$. □

Remark 2.8. Proposition 2.7 is not true for the chain I_N. For any pure dimension one sheaf $\mathcal{E}$ on I_N one has $\sum_{i=1}^{N} d_i - \chi(\mathcal{E}) < 0$. △

2.4. **Stable sheaves of degree zero on E_N.**

Let $X = E_N$ be as above, a cycle of projective lines. Suppose now that the number of irreducible components of X is $N \geq 2$. Fix a polarization H on X of degree h and let $\mathcal{E}$ be a coherent sheaf on X. Since $\chi(\mathcal{O}_X) = 0$, the second coefficient of the Hilbert polynomial, that coincides with its Euler characteristic $\chi(\mathcal{E})$, can be seen in view of Remark 1.8 as its degree. Hence, from now on, if the Hilbert polynomial of $\mathcal{E}$ is $P_{\mathcal{E}}(s) = rs$, then we will refer to it as a degree 0 sheaf.

When $r = h$, the structure of the component $\mathcal{M}_X((1,\dots,1),0) \subset \mathcal{M}_X(h,0)$ was determined by one of the authors in [23] and [24]. There, one can find the following result.

Lemma 2.9. *Let $\mathcal{E}$ be a pure dimension one sheaf on X of degree 0 and multirank $\underline{r}(\mathcal{E}) = (1,\dots,1)$.*

(1) *The (semi)stability of $\mathcal{E}$ does not depend on the polarization.*
(2) *$\mathcal{E}$ is μ_S-stable if and only if it is a line bundle and its multidegree is $\underline{d}(\mathcal{E}) = (0,\dots,0)$.*
(3) *If $\mathcal{E}$ has multirank $\underline{r}(\mathcal{E}) = (1,\dots,1)$ and is strictly semistable, then its graded object is $Gr(\mathcal{E}) \simeq \oplus_{i=1}^N \mathcal{O}_{C_i}(-1)$, where $C_1,\dots,C_N$ are the irreducible components of X.*

Lemma 2.10. *Let $\mathcal{L}$ be a line bundle on X. For any integer $m \geq 1$, the vector bundle $\mathcal{F}_m \otimes \mathcal{L}$ is μ_S-semistable if and only if $\mathcal{L}$ is μ_S-semistable. For $m > 1$, $\mathcal{F}_m \otimes \mathcal{L}$ is never a μ_S-stable sheaf.*

Proof. Using the exact sequences that define $\mathcal{F}_m$ in Lemma 2.5 one gets that $P_{\mathcal{F}_m \otimes \mathcal{L}}(s) = mP_{\mathcal{L}}(s)$ so that $\mu_S(\mathcal{F}_m \otimes \mathcal{L}) = \mu_S(\mathcal{L})$ for any $m \geq 1$. Thus, the statement about the non-stability $\mathcal{F}_m \otimes \mathcal{L}$ for $m > 1$ is straightforward.

Assume that $\mathcal{F}_m \otimes \mathcal{L}$ is μ_S-semistable. If $\mathcal{L}$ is not μ_S-semistable, by Lemma 3.3 in [23], there exists a proper subcurve $Z \subset X$ such that $\mu_S(\mathcal{L}) > \mu_S(\mathcal{L}_Z)$ where $\mathcal{L}_Z$ is the restriction of $\mathcal{L}$ to Z. Since $P_{\mathcal{F}_m \otimes \mathcal{L}_Z}(s) = mP_{\mathcal{L}_Z}(s)$, the quotient $\mathcal{F}_m \otimes \mathcal{L}_Z$ contradicts the semistability of $\mathcal{F}_m \otimes \mathcal{L}$. The converse is proved by induction on m taking into account that the category of semistable sheaves of fixed slope is closed under extensions. □

In analogy to what happens for smooth elliptic curves and irreducible projective curves of arithmetic genus one, the following theorem proves that there are no stable degree 0 sheaves of higher rank.

Theorem 2.11. *Let X be a polarized curve of type E_N with $N \geq 2$. Let $\mathcal{E}$ be a sheaf of pure dimension 1 on X with Hilbert polynomial $P_{\mathcal{E}}(s) = rs$ and multirank $\underline{r}(\mathcal{E}) = (r_1,\dots,r_N)$. If $\mathcal{E}$ is μ_S-stable, then either it is isomorphic to $\mathcal{O}_{C_i}(-1)$ for some $i = 1,\dots,N$ or it is a locally free sheaf with multirank $\underline{r}(\mathcal{E}) = (1,\dots,1)$ and multidegree $\underline{d} = (0,\dots,0)$.*

Proof. Assume that $\mathcal{E}$ is not isomorphic to any of the sheaves $\mathcal{O}_{C_i}(-1)$, $i = 1,\dots,N$. By (2.1), we have an exact sequence

$$0 \to \mathcal{E} \to \mathcal{E}_{C_1} \oplus \cdots \oplus \mathcal{E}_{C_N} \to T \to 0$$

where $C_1,\dots,C_N$ denote the irreducible components of X. Let us consider a component C_i such that $\mathcal{E}_{C_i} \neq 0$. Then $\mathcal{E}_{C_i}$ is a vector bundle on $C_i \simeq \mathbb{P}^1$ of rank r_i and degree d_i, and the Grothendieck description of vector bundles on the projective line gives an isomorphism

$$\mathcal{E}_{C_i} \simeq \oplus_{j=1}^{r_i} \mathcal{O}_{\mathbb{P}^1}(\alpha_{i,j})$$

where the integers $\alpha_{i,j}$ satisfy $\sum_{j=1}^{r_i} \alpha_{i,j} = d_i$. Moreover, since we are assuming that $\mathcal{E}$ is not isomorphic to $\mathcal{O}_{C_i}(-1)$, the sheaf $\mathcal{O}_{\mathbb{P}^1}(\alpha_{i,j})$ is a strict quotient of $\mathcal{E}$ for every

$j = 1, \dots, r_i$, and the stability of $\mathcal{E}$ imposes that $\alpha_{i,j} \geq 0$ so that $d_i \geq 0$. By Proposition 2.7, $\mathcal{E}$ is a locally free sheaf.

It remains to show that $r_i = 1$ for every $i = 1, \dots, N$. Suppose that the vector bundle $\mathcal{E}$ has higher rank. Since any stable vector bundle is indecomposable, we can apply Theorem 2.6. Notice that we can exclude the case $\mathcal{E} = \mathcal{L} \otimes \mathcal{F}_r$ because the sheaves $\mathcal{L} \otimes \mathcal{F}_r$ are strictly semistable by Lemma 2.10. Then, there is an étale covering $\pi_s \colon E_{sN} \to E_N$, a line bundle $\mathcal{L}$ on E_{sN} whose multidegree $\underline{d}(\mathcal{L})$ is non-periodic, and a number $m \in \mathbb{N}$, such that

$$\mathcal{E} \simeq \pi_{s*}(\mathcal{L} \otimes \mathcal{F}_m)\,.$$

If $m > 1$, the sheaf $\pi_{s*}(\mathcal{L} \otimes \mathcal{F}_m)$ is not stable because its subsheaf $\pi_{s*}(\mathcal{L} \otimes \mathcal{F}_{m-1})$ has the same slope. Hence, $m = 1$ and $\mathcal{E} \simeq \pi_{s*}(\mathcal{L})$. Since π_s is a finite morphism, $\chi(\mathcal{L}) = \chi(\mathcal{E}) = 0$, and the stability of $\mathcal{E}$ implies that $\mathcal{L}$ has to be a stable line bundle. By Lemma 2.9, the multidegree of $\mathcal{L}$ is $\underline{d} = (0, \dots, 0)$ which contradicts the non-periodicity. Then, $\mathcal{E}$ is a line bundle and we conclude the proof. □

This is related to the following result due to L. Bodnarchuck, presented by her in VBAC-2007: If $\mathcal{E}$ is a simple vector bundle of rank r, multidegree $\underline{d} = (d_1, \dots, d_N)$ and degree d on E_N, one has $(r, d) = (r, d_1, \dots, d_N) = 1$. Moreover, if these conditions are satisfied, the determinant gives an equivalence between the category of simple vector bundles of rank r and multidegree $(d_1, \dots, d_N)$ on E_N and $\mathrm{Pic}^{\underline{d}}(E_N)$. See [4, Thm. 1.2.2, Rem. 1.2.3] where the result is proved for Kodaira curves of type II, III and IV and stated for the cycles E_N, $N \leq 3$.

Using the results on the moduli space $\mathcal{M}^s_X((1, \dots, 1), 0)$ given in [24, Theorem 4.1] for $r = h$, we can summarize the structure of the open set of stable degree 0 sheaves as follows.

Corollary 2.12. *Let X be a curve of type E_N with $N \geq 2$ and H a polarization on it of degree h. Let $\mathcal{M}^s_X(r, 0)$ be the open subset of stable sheaves with Hilbert polynomial $P(s) = rs$. The following holds:*

(1) *If $r = h$, all the components of $\mathcal{M}^s_X(r, 0)$ given by (2.2) are empty except $\mathcal{M}^s_X((1, \dots, 1), 0)$ which is isomorphic to the multiplicative group $\boldsymbol{\mu} = k^*$. Moreover, the compactification of the component $\mathcal{M}^s_X((1, \dots, 1), 0)$ is isomorphic to a rational curve with one node.*
(2) *If $r = h_i$ for some $i = 1, \dots, N$, then $\mathcal{M}^s_X(r, 0)$ is a single point.*
(3) *Otherwise, $\mathcal{M}^s_X(r, 0)$ is empty.*

Corollary 2.13. *Let $\mathcal{E}$ be a semistable sheaf on X with Hilbert polynomial $P(\mathcal{E}) = rs$. If $\mathcal{F}$ is a Jordan-Hölder factor of $\mathcal{E}$, then $\mathcal{F}$ is isomorphic either to one of the sheaves $\mathcal{O}_{C_i}(-1)$, where C_i are the irreducible components of X, or to a line bundle $\mathcal{L}$ on X of multidegree $\underline{d}(\mathcal{L}) = (0, \dots, 0)$.*

Following the notions introduced in [10], this means that the shadows of any decomposable semistable sheaves of degree 0 contain at most 2 points.

Corollary 2.14. *Let $\underline{r} = (r_1, \dots, r_N) \in \mathbb{Z}^N$ such that $r = r_1 h_1 + \dots + r_N h_N$. The dimension of the connected component $\mathcal{M}((r_1, \dots, r_N), 0)$ is equal to the minimum of the r_i with $i = 1, \dots, N$.*

Proof. Suppose that r_1 is the minimum of the r_i's. Any sheaf of the form

$$(\oplus_{i=1}^{u} \mathcal{L}_i) \oplus (\oplus_{j=1}^{N} \mathcal{O}_{C_j}(-1)^{\oplus v_j})\,, \tag{2.4}$$

where u is at most r_1, $\mathcal{L}_i$ are line bundles on X all of them of multidegree $(0,\dots,0)$ and $u+v_j=r_j$ for all j, is a semistable sheaf that defines a point in $\mathcal{M}((r_1,\dots,r_N),0)$. By Corollary 2.13, if $\mathcal{E}$ is a semistable sheaf of degree 0 and multirank $\underline{r}(\mathcal{E})=(r_1,\dots,r_N)$, its graded object is of the form given by Equation 2.4. Since the group of stable line bundles $\operatorname{Pic}^0_s(X)$ of degree 0 on X is determined by the exact sequence

$$1\to\kappa^*\to\operatorname{Pic}^0_s(X)\to\prod_{i=1}^N\operatorname{Pic}^0(C_i)\to 1\,,$$

one gets that the dimension of this component is equal to r_1. □

In the case of a rational curve with one node or one cusp, as happens also for smooth elliptic curves, it is known [10] that all the Jordan-Hölder factors of any indecomposable sheaf are isomorphic to each other. This is no longer true for cycles $X=E_N$ of projective lines, as we will now prove.

Lemma 2.15. *For $m>1$, the graded object of the Atiyah indecomposable vector bundle $\mathcal{F}_m$ is $Gr(\mathcal{F}_m)=\oplus_{i=1}^m\mathcal{O}_X$*

Proof. This follows from the exact sequences that define $\mathcal{F}_m$ in Lemma 2.5 and from the fact that the structural sheaf $\mathcal{O}_X$ of any cycle X is stable by Lemma 2.9. □

Proposition 2.16. *Let $\mathcal{E}$ be a strictly semistable indecomposable sheaf on X with Hilbert polynomial $P(s)=rs$. If $\mathcal{E}$ is not locally free, its graded object is $Gr(\mathcal{E})\simeq\oplus_{i\in A}\mathcal{O}_{C_i}(-1)$ for a subset $A\subseteq\{1,\dots,N\}$. If $\mathcal{E}$ is locally free of rank r, then $Gr(\mathcal{E})$ is isomorphic either to $\mathcal{L}^{\oplus r}$ for a line bundle $\mathcal{L}$ of multidegree $\underline{d}(\mathcal{L})=(0,\dots,0)$, or to $\oplus_{i=1}^N\mathcal{O}_{C_i}(-1)^{\oplus r}$.*

Proof. If $\mathcal{E}$ is not locally free, by Theorem 2.6, $\mathcal{E}\simeq p_{k*}(\mathcal{L})$ where $\mathcal{L}$ is a line bundle of degree -1 on I_k and k, p_k and $\mathcal{L}$ are determined by $\mathcal{E}$. Since $p_k=\pi_r\circ i$ for some Galois covering π_r and some closed immersion i, using Lemma 2.4 above and [23, Lemma 3.2], one has that $\mathcal{E}$ is H-semistable if and only if $\mathcal{L}$ is $p_k^*(H)$-semistable. By [23, Theorem 4.5], there are not stable line bundles of degree -1 on I_k and there is exactly one strictly semistable line bundle whose graded object is $Gr(\mathcal{L})\simeq\oplus_D\mathcal{O}_D(-1)$ where the sum runs over all the irreducible components D of I_k. Since p_k is a finite morphism, p_{k*} is an exact functor, and then the direct image by p_k of a Jordan-Hölder filtration for $\mathcal{L}$ is a filtration for $\mathcal{E}$. Hence the graded object is also $Gr(\mathcal{E})\simeq\oplus_{i\in A}\mathcal{O}_{C_i}(-1)$ for a subset $A\subseteq\{1,\dots,N\}$.

Suppose now that $\mathcal{E}$ is a vector bundle. Assume first that $\mathcal{E}=\mathcal{L}\otimes\mathcal{F}_r$ for a line bundle $\mathcal{L}$ of multidegree $(0,\dots,0)$. By Lemma 2.10, $\mathcal{L}$ is semistable; if it is stable, then $Gr(\mathcal{E})=\mathcal{L}^{\oplus r}$ by Lemma 2.15; if $\mathcal{L}$ is strictly semistable, then its Jordan-Hölder factors are $\mathcal{O}_{C_1}(-1),\dots,\mathcal{O}_{C_N}(-1)$ by Corollary 2.13, and again by Lemma 2.15, one has that $Gr(\mathcal{E})\simeq\oplus_{i=1}^N\mathcal{O}_{C_i}(-1)^{\oplus r}$.

By Theorem 2.6, the only remaining case is when $\mathcal{E}\simeq\pi_{s*}(\mathcal{L}\otimes\mathcal{F}_m)$ for an étale covering $\pi_s\colon E_{sN}\to X=E_N$, a line bundle $\mathcal{L}$ on E_{sN} with degree zero and non-periodic multidegree, and a number $m\in\mathbb{N}$. By Lemmas 2.4 and 2.10, $\mathcal{E}$ is H-semistable if and only if $\mathcal{L}$ is π_r^*H-semistable. Using Lemma 2.9, we see that the line bundle $\mathcal{L}$ is not stable because otherwise one would have $\underline{d}(\mathcal{L})=(0,\dots,0)$ and this contradicts the non-periodicity of $\underline{d}(\mathcal{L})$. Proceeding as above one sees that the graded object of $\mathcal{L}\otimes\mathcal{F}_m$ is $\oplus\mathcal{O}_C(-1)^{\oplus m}$, where the sum runs over all the irreducible components C of E_{sN}. Since π_s is a finite morphism, one sees that $Gr(\mathcal{E})\simeq\oplus_{i=1}^N\mathcal{O}_{C_i}(-1)^{\oplus r}$. □

This shows that in this case there exist indecomposable vector bundles whose Jordan-Hölder factors are non-perfect.

2.5. **The biggest component of the moduli space.** In this subsection we describe completely the component $\mathcal{M}_X((\bar{r},\dots,\bar{r}),0)\subset\mathcal{M}_X(r,0)$ ($r=\bar{r}h$) of semistable sheaves of multirank $(\bar{r},\dots,\bar{r})$ and degree zero for the curve E_N (cf. Figure 2) with respect to an arbitrary polarization H.

For any smooth elliptic curve or a rational curve with one node or one cusp, it is well known that the moduli space $\mathcal{M}^s((1),0)$ is isomorphic to the curve. This is no longer true for reducible fibers. In the particular case of E_N, the moduli space $\mathcal{M}((1,\dots,1),0)$ is isomorphic to a rational curve with a node E_1. This was proved in [12] if $N=2$ and in [24] for any $N\geq 2$.

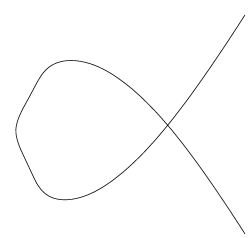

FIGURE 4. The rational curve with a node E_1

Let us describe the isomorphism $\mathcal{M}((1,\dots,1),0)\xrightarrow{\sim}E_1$, following [24, Proposition 3.2], and the inverse isomorphism. Let $C_1,\dots,C_N$ be the rational components of $X=E_N$ ordered cyclically and let us denote by $\gamma\colon X\to E_1$ the morphism which contracts $C_2,\dots,C_N$ and gives an isomorphism $C_1-\{x_1,x_N\}\xrightarrow{\sim}E_1-\{\bar{z}\}$, where $\{x_1,x_N\}$ are the intersection points of C_1 with the other components and $\bar{z}$ is the singular point of E_1. Let us consider the sheaf on $X\times X$

$$\mathcal{E}=\mathcal{I}_\Delta^*\otimes\pi_1^*\mathcal{O}_X(-y_0)\,, \tag{2.5}$$

where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal immersion δ, π_1 is the canonical projection onto the first factor, and y_0 is a fixed smooth point of C_1. For every point $y\in X$ the restriction $\mathcal{E}_y$ of $\mathcal{E}$ to $X\times\{y\}$ is a semistable pure dimension one sheaf of multirank $(1,\dots,1)$ and degree 0 [23, Proposition 3.2]. By [2, Corollary 2.65], the Fourier-Mukai functor $\Phi^{\mathcal{E}}_{X\to X}$ induces a classifying morphism $\varphi\colon X\to\mathcal{M}((1,\dots,1),0)$. Moreover, for all points $y\in C_2\cup\dots\cup C_N$, the sheaves $\mathcal{E}_y$ are S-equivalent [24, Proposition 3.2], and then φ factors through a morphism

$$\bar{\varphi}\colon E_1\to\mathcal{M}((1,\dots,1),0)\,,$$

which one proves to be an isomorphism [24]. We shall give here a different proof (see Proposition 2.18).

The smooth points $y\in C_1$ are mapped to stable line bundles $\mathfrak{m}_y^*\otimes\mathcal{O}_X(-y_0)$ and all the remaining points are mapped to the strictly semistable sheaves, all of them S-equivalent to $\mathcal{O}_{C_1}(-1)\oplus\dots\oplus\mathcal{O}_{C_N}(-1)$.

Let us describe the action on $D^b(X)$ of the quasi-inverse $\Phi^{\mathcal{E}^*[1]}_{X\to X}$ of $\Phi^{\mathcal{E}}_{X\to X}$. Using the exact sequence $0\to\mathcal{I}_\Delta\to\mathcal{O}_{X\times X}\to\delta_*\mathcal{O}_X\to 0$ and flat cohomology base-change, we get for every complex $\mathcal{F}^\bullet$ in $D^b(X)$ an exact triangle

$$\Phi^{\mathcal{E}^*[1]}_{X\to X}(\mathcal{F}^\bullet) \to p^*(\mathbf{R}p_*(\mathcal{F}^\bullet \otimes \mathcal{O}_X(y_0))[1] \to \mathcal{F}^\bullet \otimes \mathcal{O}_X(y_0)[1] \to \Phi^{\mathcal{E}^*[1]}_{X\to X}(\mathcal{F}^\bullet)[1]\,,$$

where p is the projection of X onto one point. Applying this formula to the Jordan-Hölder factors $\mathcal{O}_{C_i}(-1)$ we have

$$\Phi^{\mathcal{E}^*[1]}_{X\to X}(\mathcal{O}_{C_1}(-1)) = \mathcal{I}_{C_1}[1]\,,\quad \mathcal{I}_{C_1} \text{ being the ideal sheaf of } C_1 \text{ in } X,$$
$$\Phi^{\mathcal{E}^*[1]}_{X\to X}(\mathcal{O}_{C_j}(-1)) = \mathcal{O}_{C_j}(-1)\,,\quad \text{for } j>1.$$

We now consider the integral functor $D^b(X) \to D^b(E_1)$ obtained as the composition of the quasi-inverse $\Phi^{\mathcal{E}^*[1]}_{X\to X}$ of $\Phi^{\mathcal{E}}_{X\to X}$ and the derived push-forward $\mathbf{R}\gamma_*\colon D^b(X) \to D^b(E_1)$. This is the integral functor with kernel

$$\mathcal{K}^\bullet = \mathbf{R}(1\times\gamma)_*\mathcal{E}^*[1]\,. \tag{2.6}$$

Lemma 2.17. *If $\mathcal{F}$ is a semistable sheaf on X of multirank $(1,\dots,1)$ and degree 0, then $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{F}) = \mathcal{O}_z$ for a uniquely determined point $z\in E_1$. Moreover z is the singular point of E_1 if and only if $\mathcal{F}$ is strictly semistable.*

Proof. If $\mathcal{F}$ is stable, then $\mathcal{F} = \Phi^{\mathcal{E}}_{X\to X}(\mathcal{O}_y)$ for a smooth point $y\in C_1$, so that $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{F}) = \mathcal{O}_{\gamma(y)}$. Let us now compute the image of the Jordan-Hölder factors. If $j>1$, the restriction of γ to C_j factors through the singular point $\bar z$ of E_1, so that

$$\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{O}_{C_j}(-1)) = \mathbf{R}\gamma_*(\mathcal{O}_{C_j}(-1)) = 0\,.$$

In the case $j=1$, the ideal $\mathcal{I}_{C_1}$ is supported on $C_2\cup\dots\cup C_N$ and one has $H^0(X,\mathcal{I}_{C_1})=0$, $\dim H^1(X,\mathcal{I}_{C_1})=1$. As above, the restriction of γ to $C_2\cup\dots\cup C_N$ factors through the singular point $\bar z$ of E_1 and one has

$$\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{O}_{C_1}(-1)) = \mathbf{R}\gamma_*(\mathcal{I}_{C_1})[1] = \mathcal{O}_{\bar z}\,.$$

It follows that if $\mathcal{F}$ is strictly semistable, by applying $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}$ to a Jordan-Hölder filtration, one obtains $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{F}) = \mathcal{O}_{\bar z}$. □

Proposition 2.18. *The integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}$ induces a morphism*

$$\eta\colon \mathcal{M}((1,\dots,1),0)\to E_1\,,$$

which is the inverse of $\bar\varphi$.

Proof. The morphism η exists by Lemma 2.17 and [2, Corollary 2.65]. One checks directly that $\eta\circ\bar\varphi$ coincides with the identity on the closed points. Since E_1 is separated, the subscheme $Z\hookrightarrow E_1$ of coincidences of $\eta\circ\bar\varphi$ and the identity is closed; moreover the closed points are dense because E_1 is projective, and thus, Z is topologically equal to E_1. Since E_1 is reduced, Z is algebraically equal to E_1 as well, so that $\eta\circ\bar\varphi$ is equal to the identity. Taking into account that $\mathcal{M}((1,\dots,1),0)$ is projective and reduced, the same argument proves that $\bar\varphi\circ\eta$ is the identity as well. □

Our next aim is to find the relationship between the moduli space $\mathcal{M}((\bar r,\dots,\bar r),0)$ of semistable pure dimension one sheaves of multirank $(\bar r,\dots,\bar r)$ and degree 0 on a cycle E_N, and the symmetric product $Sym^{\bar r}E_1$ of the rational curve with a node.

The following result is known but we could not find a suitable reference:

Lemma 2.19. *The moduli space $\mathcal{M}((\bar r,\dots,\bar r),0)$ is reduced.* □

A proof of Lemma 2.19 in the case $\bar{r} = 2$ can be found in [31, Huitiême partie, Theorème 18]. Seshadri shows also in [31] that the case $\bar{r} > 2$ follows from a property of certain determinantal varieties, that was not proved at the time; however the property was established later by Strickland [35] and this completed the proof (cf. also [33]).

Theorem 2.20. *Assume $\bar{r} > 1$. There exists a scheme isomorphism*

$$\mathcal{M}((\bar{r},\dots,\bar{r}),0) \xrightarrow{\sim} Sym^{\bar{r}} E_1 \,.$$

Proof. Using a smooth point y_0 of C_1 as above, we construct an isomorphism $\bar{\varphi}\colon E_1 \to \mathcal{M}((1,\dots,1),0)$. Since the direct sum of $\bar{r}$ semistable sheaves of multirank $(1,\dots,1)$ and degree 0 is semistable of multirank $(\bar{r},\dots,\bar{r})$ and degree 0, we have a morphism

$$E_1 \times \overset{\bar{r}}{\overbrace{\cdots}} \times E_1 \to \mathcal{M}((1,\dots,1),0) \times \overset{\bar{r}}{\overbrace{\cdots}} \times \mathcal{M}((1,\dots,1),0) \xrightarrow{\oplus} \mathcal{M}((\bar{r},\dots,\bar{r}),0) \,.$$

This morphism factors through the $\bar{r}$-th symmetric product $Sym^{\bar{r}} E_1$ and then induces a morphism $\bar{\varphi}_{\bar{r}}\colon Sym^{\bar{r}} E_1 \to \mathcal{M}((\bar{r},\dots,\bar{r}),0)$, which is one-to-one on closed points by Corollary 2.13.

If $\mathcal{F}$ is a semistable sheaf of multirank $(\bar{r},\dots,\bar{r})$ and degree 0, by Corollary 2.13, its graded object with respect to a Jordan-Hölder filtration is $Gr(\mathcal{F}) = \mathcal{L}_1^{\oplus a_1} \oplus \cdots \oplus \mathcal{L}_m^{\oplus a_m} \oplus (\oplus_{j=1}^N \mathcal{O}_{C_j}(-1))^{\oplus(\bar{r}-u)}$, where $\mathcal{L}_i$ are stable line bundles of degree 0 and $u = a_1 + \cdots + a_m$. If $\mathcal{K}^\bullet$ is the kernel defined by Equation (2.6), using a Jordan-Hölder filtration of $\mathcal{F}$ as in the proof of Lemma 2.17, one sees that

$$\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{F}) = \mathcal{O}_{Z_1} \oplus \cdots \oplus \mathcal{O}_{Z_m} \oplus \mathcal{O}_{\bar{Z}} \,,$$

where Z_i is a zero dimensional closed subscheme of E_1 of length a_i supported at a smooth point z and $\bar{Z}$ is a zero dimensional closed subscheme of length $\bar{r} - u$ supported at the singular point $\bar{z}$. Then $\Phi^{\mathcal{K}^\bullet}_{X\to E_1}(\mathcal{F})$ is a skyscraper sheaf of length $\bar{r}$ on E_1 and by [2, Corollary 2.65] there exists a morphism

$$\eta_r\colon \mathcal{M}((\bar{r},\dots,\bar{r}),0) \to Sym^{\bar{r}} E_1 \,,$$

which is the inverse of $\bar{\varphi}_{\bar{r}}$ on closed points. Since $Sym^{\bar{r}} E_1$ is projective and reduced, proceeding as in the proof of Proposition 2.18 we see that $\eta_r \circ \bar{\varphi}_{\bar{r}}\colon Sym^{\bar{r}} E_1 \to Sym^{\bar{r}} E_1$ is the identity. Since $\mathcal{M}((\bar{r},\dots,\bar{r}),0)$ is reduced, a similar argument shows that $(\bar{\varphi}_{\bar{r}}) \circ (\eta_r)\colon \mathcal{M}((\bar{r},\dots,\bar{r}),0) \to \mathcal{M}((\bar{r},\dots,\bar{r}),0)$ is the identity as well. □

Remark 2.21. Arguing as in Proposition 1.9 and Corollary 1.25, one gets that the equivalences Φ and Ψ in Section 1 induce isomorphisms

$$\begin{aligned}
\mathcal{M}((\bar{r},\dots,\bar{r}),d) &\simeq \mathcal{M}((d-\bar{r},\dots,d-\bar{r}),-d) \qquad \text{for } d > \bar{r}\\
\mathcal{M}((\bar{r},\dots,\bar{r}),d) &\simeq \mathcal{M}((\bar{r}-d,\dots,\bar{r}-d),d) \qquad \text{for } d \le \bar{r}\\
\mathcal{M}((\bar{r},\dots,\bar{r}),d) &\simeq \mathcal{M}((\bar{r},\dots,\bar{r}),\bar{r}h+d) \,.
\end{aligned}$$

Thus, for any integers $\lambda, \mu \in \mathbb{Z}$, one gets that the moduli spaces $\mathcal{M}((\bar{r},\dots,\bar{r}),d)$ where $d = \lambda h \bar{r}$ and $\mathcal{M}((r_0,\dots,r_0),d_0)$ where $r_0 = \mu\lambda\bar{r} \pm \bar{r}$ and $d_0 = \pm\lambda h \bar{r}$ are also isomorphic to the $\bar{r}$-th symmetric product $Sym^{\bar{r}} E_1$ of the nodal curve. △

Using the results obtained so far, we give a complete description of all moduli spaces $\mathcal{M}_X(r,d)$ of semistable sheaves on the curve E_2 (cf. Figure 1) with respect to a polarization of the minimum possible degree $h = 2$. Note that if $h = 2$, the case $2r_0/h \le d_0 < r_0$ in Corollary 1.25 is not possible. One then has:

Corollary 2.22. *Let X be a curve of type E_2 with a polarization H of degree $h = 2$, and let (r, d) be a pair of integers with $r \geq 0$. The moduli space $\mathcal{M}_X(r, d)$ of semistable sheaves with Hilbert polynomial $P(s) = rs + d$ on X is isomorphic either to the d_0-th symmetric power $Sym^{d_0}(X)$ of the curve or to $\mathcal{M}(r_0, 0)$. Moreover, if r_0 is even, then the biggest connected component of $\mathcal{M}(r_0, 0)$ is isomorphic to the symmetric power $Sym^{r_0/2}E_1$ of the nodal curve E_1.* □

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