FOURIER MUKAI TRANSFORMS FOR GORENSTEIN SCHEMES

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Abstract. We extend to singular schemes with Gorenstein singularities or fibered in schemes of that kind Bondal and Orlov's criterion for an integral functor to be fully faithful. We also prove that the original condition of characteristic zero cannot be removed by providing a counterexample in positive characteristic. We contemplate a criterion for equivalence as well. In addition, we prove that for locally projective Gorenstein morphisms, a relative integral functor is fully faithful if and only if its restriction to each fibre is also fully faithful. These results imply the invertibility of the usual relative Fourier-Mukai transform for an elliptic fibration as a direct corollary.

Contents

Introduction

Since its introduction by Mukai [31], the theory of integral functors and Fourier-Mukai transforms have been important tools in the study of the geometry of varieties and moduli spaces. At the first moment, integral functors were used mainly in connection with moduli spaces of sheaves and bundles, and provided new insights in the theory of Picard bundles on abelian varieties and in the theory of stable sheaves on

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abelian or K3 surfaces [33, 4, 5, 12]. In the relative version [32, 7, 11, 8, 23, 15, 18, 17] they have been also used in mirror symmetry and to produce new instances of stable sheaves on elliptic surfaces or elliptic Calabi-Yau threefolds. The reason is that the theory of integral functors is behind the spectral data constructions [20, 2, 24]; the irruption of the derived categories in string theory caused by homological mirror symmetry brought then a new interest to derived categories and integral functors (see [1, 3] for recent surveys of the subject and references therein).

Aside from their interest in Physics, derived categories are important geometric invariants of algebraic varieties. Much work is being done in this direction, particularly in the characterisation of all the algebraic varieties sharing the same derived category (also known as Mukai partners).

There are classic results like the theorem of Bondal and Orlov [10] which says that if X is a smooth projective variety whose canonical divisor is either ample or anti-ample, then X can be reconstructed from its derived category. Mukai proved [31] that there exist non isomorphic abelian varieties and non isomorphic K3 surfaces having equivalent derived categories. Orlov [35] proved that two complex K3 surfaces have equivalent derived categories if and only if the transcendental lattices of their cohomology spaces are Hodge-isometric, a result now called the derived Torelli theorem for K3 surfaces. After Mukai's work the problem of finding Fourier-Mukai partners has been contemplated by many people. Among them, we can cite Bridgeland-Maciocia [14] and Kawamata [26]; they have proved that if X is a smooth projective surface, then there is a finite number of surfaces Y (up to isomorphism) whose derived category is equivalent to the derived category of X. Kawamata proved that if X and Y are smooth projective varieties with equivalent derived categories, then $n = \dim X = \dim Y$ and if moreover $\kappa(X) = n$ (that is, X is of general type), then there exist birational morphisms $f\colon Z \to X$, $g\colon Z \to Y$ such that $f^*K_X \sim g^*K_Y$ (i.e. D-equivalence implies K-equivalence) [26]. Other important contributions are owed to Bridgeland [13], who proved that two crepant resolutions of a projective threefold with terminal singularities have equivalent derived categories; therefore, two birational Calabi-Yau threefolds have equivalent derived categories. The proof is based on a careful study of the behaviour of flips and flops under certain integral functors and the construction of the moduli space of perverse point sheaves.

All these results support the belief that derived categories and integral functors could be most useful in the understanding of the minimal model problem in higher dimensions. And this suggests that the knowledge of both the derived categories and the properties of integral functors for singular varieties could be of great relevance.

However, very little attention has been paid so far to singular varieties in the Fourier-Mukai literature. One of the reasons may be the fact that the fundamental results on integral functors are not easily generalised to the singular situation, because they rely deeply on properties inherent to smoothness.

We would like to mention two of the most important. One is Orlov's representation theorem [35] according to if X and Y are smooth projective varieties, any (exact) fully faithful functor between their derived categories is an integral functor. Particularly, any (exact) equivalence between their derived categories is an integral functor (integral functors that are equivalences are also known as Fourier-Mukai functors). Another is Bondal and Orlov's characterisation of those integral functors between the derived categories of two smooth varieties that are fully faithful [9].

Orlov's representation theorem has been generalised by Kawamata [28] to the smooth stack associated to a normal projective variety with only quotient singularities. Therefore D-equivalence also implies K-equivalence for those varieties when $\kappa(X)$ is maximal. In [38] Van den Bergh proves using non-commutative rings that Bridgeland's result about flopping contractions can be extended to quasi-projective varieties with only Gorenstein terminal singularities. The same result was proved by Chen [19]; the underlying idea is to embed such a threefold into a smooth fourfold and then use the essential smoothness. The author himself notices that his smoothing approach will not work for most general threefold flops because quotient singularities in dimension greater or equal to 3 are very rigid. In his paper, some general properties of the Fourier-Mukai transform on singular varieties can be found as well as the computation of a spanning class of the derived category of a normal projective variety with only isolated singularities. Finally, Kawamata [27] has obtained analogous results for some $\mathbb{Q}$-Gorenstein threefolds using algebraic stacks.

This paper is divided in two parts. In the first part, we give an extension of Bondal and Orlov's characterisation of fully faithful integral functors to proper varieties with (arbitrary) Gorenstein singularities. This is the precise statement.

Theorem (Theorem 1.22)**.** *Let X and Y be projective Gorenstein schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times Y)$ of finite projective dimension over X and over Y. Assume also that X is integral. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is strongly simple over X.*

One should notice that this Theorem may fail to be true in positive characteristic even in the smooth case. A counterexample is given in Remark 1.25.

In the Gorenstein case, strong simplicity (Definition 1.19) is defined in terms of locally complete intersection zero cycles instead of the structure sheaves of the closed points, as it happens in the smooth case. In the latter situation, our result improves the characterization of fully faithfulness of Bondal and Orlov.

As in the smooth case, when X is a Gorenstein variety the skyscraper sheaves $\mathcal{O}_x$ form a spanning class for the derived category $D^b_c(X)$. Nevertheless, due to the fact that one may has an infinite number of non-zero $\operatorname{Ext}^i_X(\mathcal{O}_x, \mathcal{O}_x)$ when x is a singular point, this spanning class does not allow to give an effective criterion characterising the fully faithfulness of integral functors. However, Bridgeland's criterion that characterises when a fully faithful integral functor is an equivalence is also valid in the Gorenstein case. Moreover, since for a Gorenstein variety one has a more natural spanning class given by the structure sheaves of locally complete intersection cycles supported on closed points, one also proves the following alternative result.

Theorem (Theorem 1.28)**.** *Let X, Y and $\mathcal{K}^\bullet$ be as in the previous theorem with Y connected. A fully faithful integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$ is an equivalence of categories if and only if for every closed point $x \in X$ there exists a locally complete intersection cycle Z_x supported on x such that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}) \simeq \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}) \otimes \omega_Y$.*

We also derive in the Gorenstein case some geometric consequences of the existence of Fourier-Mukai functors (Proposition 1.30) which are analogous to certain well-known properties of smooth schemes.

The second part of the paper is devoted to relative integral functors. As already mentioned, relative Fourier-Mukai transforms have been considered mainly in connection with elliptic fibrations. And besides some standard functorial properties, like compatibility with (some) base changes, more specific results or instances of Fourier-Mukai functors (equivalences of the derived categories) are known almost only for abelian schemes [32] or elliptic fibrations.

We prove a new result that characterises when a relative integral functor is fully faithful or an equivalence, and generalises [19, Prop. 6.2]:

Theorem (Theorem 2.4)**.** *Let $p\colon X \to S$ and $q\colon Y \to S$ be locally projective Gorenstein morphisms (the base field is algebraically closed of characteristic zero). Let $\mathcal{K}^\bullet \in D^b(X \times_S Y)$ be a kernel of finite projective dimension over both X and Y. The relative integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$ is fully faithful (respectively an equivalence) if and only if $\Phi^{\mathbf{L}j_s^*\mathcal{K}^\bullet}_{X_s\to Y_s}\colon D^b_c(X_s) \to D^b_c(Y_s)$ is fully faithful (respectively an equivalence) for every closed point $s \in S$, where j_s is the immersion of $X_s \times Y_s$ into $X \times_S Y$.*

Though this result is probably true in greater generality, our proof needs the Gorenstein condition in an essential way. The above theorem, together with the characterisation of fully faithful integral functors and of Fourier-Mukai functors in the absolute Gorenstein case (Theorems 1.22 and 1.28) gives a criterion to ascertain when a relative integral functor between the derived categories of the total spaces of two Gorenstein fibrations is an equivalence. We expect that this theorem could be applied to very general situations. As a first application we give here a very simple and short proof of the invertibility result for elliptic fibrations:

Theorem (Proposition 2.7)**.** *Let S be an algebraic scheme over an algebraically closed field of characteristic zero, $X \to S$ an elliptic fibration with integral fibres and a section, $\hat{X} \to S$ the dual fibration and $\mathcal{P}$ the relative Poincaré sheaf on $X \times_S \hat{X}$. The relative integral functor*

$$\Phi^{\mathcal{P}}_{X\to\hat{X}}\colon D^b_c(X) \to D^b_c(\hat{X})$$

is an equivalence of categories.

This result has been proved elsewhere in different ways. When the total spaces of the fibrations involved are smooth the theorem can be proved, even if the fibres are singular, by considering the relative integral functor as an absolute one (defined by the direct image of the relative Poincaré to the direct product) and then applying the known criteria in the smooth case [11, 15, 6] (see also [7]). When the total spaces are singular, there is a proof in [16, 17] that follows a completely different path and is much longer than ours.

In this paper, scheme means algebraic scheme (that is, a scheme of finite type) over an algebraically closed field k. By a Gorenstein morphism, we understand a flat morphism of schemes whose fibres are Gorenstein. For any scheme X we denote by $D(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves. This is the essential image of the derived category of quasi-coherent sheaves in the derived category of all $\mathcal{O}_X$-modules. Analogously $D^+(X)$, $D^-(X)$ and $D^b(X)$ will denote the derived categories of complexes which are respectively bounded below, bounded above and bounded on both sides, and have quasi-coherent cohomology sheaves. The subscript c will refer to the corresponding subcategories of complexes with coherent cohomology sheaves.

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1. Fourier-Mukai transform on Gorenstein schemes

1.1. **Preliminary results.** We first recall some basic formulas which will be used in the rest of the paper.

If X is a scheme, there is a functorial isomorphism (in the derived category)

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{H}^\bullet)) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{E}^\bullet, \mathcal{H}^\bullet) \tag{1.1}$$

where $\mathcal{F}^\bullet$, $\mathcal{E}^\bullet$ are in $D^-(X)$, $\mathcal{H}^\bullet$ is in $D^+(X)$, and all have coherent cohomology ([21]). One also has a functorial isomorphism

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet) \tag{1.2}$$

where $\mathcal{F}^\bullet$ is a bounded complex of $\mathcal{O}_X$-modules with coherent cohomology and either $\mathcal{F}^\bullet$ or $\mathcal{H}^\bullet$ is of finite homological dimension (i.e. locally isomorphic to a bounded complex of locally free sheaves of finite rank). The usual proof (see [21] or [6]) requires that $\mathcal{H}^\bullet$ is of finite homological dimension; however, it still works when both members are defined. If we denote by $\mathcal{F}^{\bullet\vee} = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{O}_X)$ the dual in the derived category, (1.2) implies that

$$\mathcal{F}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{H}^\bullet \xrightarrow{\sim} \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{H}^\bullet)\,. \tag{1.3}$$

Nevertheless this formula may fail to be true when neither $\mathcal{F}^\bullet$ nor $\mathcal{H}^\bullet$ have finite homological dimension as the following example shows.

Example 1.1. Let X be a Gorenstein scheme of dimension n over a field k. Let $x \in X$ be a singular point and let $\mathcal{F}$ be any $\mathcal{O}_X$-module. Since $\mathcal{O}_x^\vee \simeq \mathcal{O}_x[-n]$, if one had

$$\mathcal{O}_x^\vee \overset{\mathbf{L}}{\otimes} \mathcal{F} \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{O}_x, \mathcal{F})\,,$$

then one would have $\operatorname{Tor}_{n-i}(\mathcal{O}_x, \mathcal{F}) \simeq \operatorname{Ext}^i(\mathcal{O}_x, \mathcal{F})$ for every $i \in \mathbb{Z}$. It follows that $\operatorname{Ext}^i(\mathcal{O}_x, \mathcal{F}) = 0$ for all $i > n$ and every $\mathcal{O}_X$-module $\mathcal{F}$ and this is impossible because $\mathcal{O}_x$ is not of finite homological dimension. △

The formula (1.3) implies that if $f\colon X \to Y$ is a morphism, there is an isomorphism

$$\mathbf{L}f^*(\mathcal{F}^{\bullet\vee}) \simeq (\mathbf{L}f^*\mathcal{F}^\bullet)^\vee \tag{1.4}$$

if either $\mathcal{F}^\bullet$ is of finite homological dimension or f is of finite Tor-dimension (in this paper we shall only need to consider the case when f is flat or is a regular closed immersion).

Some other formulas will be useful. When X is a Gorenstein scheme, every object $\mathcal{F}^\bullet$ in $D^b_c(X)$ is reflexive, that is, one has an isomorphism in the derived category [36, 1.17]:

$$\mathcal{F}^\bullet \simeq (\mathcal{F}^{\bullet\vee})^\vee\,. \tag{1.5}$$

Then, one has

$$\operatorname{Hom}_{D(X)}(\mathcal{H}^\bullet, \mathcal{F}^\bullet) \simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^{\bullet\vee}, \mathcal{H}^{\bullet\vee}) \tag{1.6}$$

for every bounded complex $\mathcal{F}^\bullet$ in $D^b_c(X)$ and any complex $\mathcal{H}^\bullet$.

Moreover, if X is a zero dimensional Gorenstein scheme, the sheaf $\mathcal{O}_X$ is injective so that

$$\mathcal{F}^{\bullet *} \simeq \mathcal{F}^{\bullet\vee} \quad \text{and} \quad \mathcal{H}^i(\mathcal{F}^{\bullet\vee}) \simeq (\mathcal{H}^{-i}(\mathcal{F}^\bullet))^* \tag{1.7}$$

for every object $\mathcal{F}^\bullet$ in $D^b_c(X)$, where $\mathcal{F}^{\bullet *} = \mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{O}_X)$ is the ordinary dual.

Let $f\colon X \to Y$ be a proper morphism of schemes. The relative Grothendieck duality states the existence of a functorial isomorphism in the derived category

$$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \mathbf{R}f_*\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)\,. \tag{1.8}$$

for $\mathcal{G}^\bullet$ in $D(Y)$ and $\mathcal{F}^\bullet$ in $D(X)$ (see for instance [34]). By applying the derived functor of the global section functor, we obtain the *global duality formula*

$$\operatorname{Hom}_{D(Y)}(\mathbf{R}f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)\,. \tag{1.9}$$

In other words, the direct image $\mathbf{R}f_*\colon D(X) \to D(Y)$ has a right adjoint $f^!\colon D(Y) \to D(X)$.

There is a natural map $f^*\mathcal{G}^\bullet \overset{\mathbf{L}}{\otimes} f^!\mathcal{O}_Y \to f^!\mathcal{G}^\bullet$, which is an isomorphism when either $\mathcal{G}^\bullet$ has finite homological dimension or $\mathcal{G}^\bullet$ is reflexive and $f^!\mathcal{O}_Y$ has finite homological dimension.

When f is a Gorenstein morphism of relative dimension n, the object $f^!\mathcal{O}_Y$ reduces to an invertible sheaf ω_f, called *the relative dualizing sheaf*, located at the place $-n$, $f^!\mathcal{O}_Y \simeq \omega_f[n]$.

Grothendieck duality is compatible with base-change. We state this result for simplicity only when f is Gorenstein. In this case, since f is flat, base-change compatibility means that if $g\colon Z \to Y$ is a morphism and $f_Z\colon Z \times_Y X \to Z$ is the induced morphism, then the relative dualizing sheaf for f_Z is $\omega_{f_Z} = g_X^*\omega_f$ where $g_X\colon Z \times_Y X \to X$ is the projection.

As it is customary, when f is the projection onto a point, we denote the dualizing sheaf by ω_X.

1.2. **Complexes of relative finite projective dimension.** In this subsection we shall prove a weaker version of (1.2) in some cases.

Lemma 1.2. *Let $\mathcal{E}^\bullet$ be an object in $D^b_c(X)$. The following conditions are equivalent:*

1. *$\mathcal{E}^\bullet$ is of finite homological dimension.*
2. *$\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{G}^\bullet$ is an object of $D^b(X)$ for every $\mathcal{G}^\bullet$ in $D^b(X)$.*
3. *$\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{G}^\bullet)$ is in $D^b(X)$ for every $\mathcal{G}^\bullet$ in $D^b(X)$.*

Proof. Since X is noetherian, the three conditions are local so that we can assume that X is affine. It is clear that (1) implies (2) and (3). Now let us see that (3) implies (1). Let us consider a quasi-isomorphism $\mathcal{L}^\bullet \to \mathcal{E}^\bullet$ where $\mathcal{L}^\bullet$ is a bounded above complex of finite free modules. If $\mathcal{K}^n$ is the kernel of the differential $\mathcal{L}^n \to \mathcal{L}^{n+1}$, then for n small enough the truncated complex $\mathcal{K}^n \to \mathcal{L}^n \to \dots$ is still quasi-isomorphic to $\mathcal{E}^\bullet$ because $\mathcal{E}^\bullet$ is an object of $D^b_c(X)$. Let x be a point and $\mathcal{O}_x$ its residual field. Since $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{O}_x)$ has bounded homology, one also has that $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{K}^n, \mathcal{O}_x) = 0$ for n small enough. For such n the module $\mathcal{K}^n$ is free in a neighbourhood of x and one concludes. To prove that (2) implies (1), one proceeds analogously replacing Ext^1 by Tor_1. $\square$

This lemma suggests the following definition.

Definition 1.3. Let $f\colon X \to Y$ be a morphism of schemes. An object $\mathcal{E}^\bullet$ in $D(X)$ is said to be of *finite homological dimension over* Y (resp. of *finite projective dimension over* Y), if $\mathcal{E}^\bullet \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathcal{G}^\bullet$ (resp. $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)$), is in $D^b(X)$ for any $\mathcal{G}^\bullet$ in $D^b(Y)$. $\triangle$

These notions are similar (though weaker) to the notions of finite Tor-amplitude and finite Ext-amplitude considered in [**?**].

In the absolute case (i.e. when f is the identity), finite projective dimension is equivalent to finite homological dimension by the previous lemma. To characterise complexes of finite projective dimension over Y when f is projective, we shall need the following result (c.f. [35, Lem. 2.13]).

Lemma 1.4. *Let A be a noetherian ring, $f\colon X \to Y = \operatorname{Spec} A$ a projective morphism and $\mathcal{O}_X(1)$ a relatively very ample line bundle.*

(1) *Let $\mathcal{M}^\bullet$ be an object of $D^-(X)$. Then $\mathcal{M}^\bullet = 0$ (resp. is an object of $D^b(X)$) if and only if $\mathbf{R}f_*(\mathcal{M}^\bullet(r)) = 0$ (resp. is an object of $D^b(Y)$) for every integer r.*
(2) *Let $g\colon \mathcal{M}^\bullet \to \mathcal{N}^\bullet$ be a morphism in $D^-(X)$. Then g is an isomorphism if and only if the induced morphism $\mathbf{R}f_*(\mathcal{M}^\bullet(r)) \to \mathbf{R}f_*(\mathcal{N}^\bullet(r))$ is an isomorphism in $D^-(Y)$ for every integer r.*

(As it is usual, we set $\mathcal{M}^\bullet(r) = \mathcal{M}^\bullet \otimes \mathcal{O}_X(r)$.)

Proof. Let $i\colon X \hookrightarrow \mathbb{P}^N_A$ be the closed immersion of A-schemes defined by $\mathcal{O}_X(1)$. Since $\mathcal{M}^\bullet = 0$ if and only if $i_*\mathcal{M}^\bullet = 0$ and $\mathcal{M}^\bullet$ has bounded cohomology if and only if $i_*\mathcal{M}^\bullet$ has bounded cohomology as well, we can assume that $X = \mathbb{P}^N_A$. Now one has an exact sequence (Beilinson's resolution of the diagonal)

$$0 \to \mathcal{E}_N \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{O}_\Delta \to 0$$

where $\mathcal{E}_j = \pi_1^*\mathcal{O}_{\mathbb{P}^N_A/A}(-j) \otimes \pi_2^*\Omega^j_{\mathbb{P}^N_A/A}(j)$, π_1 and π_2 being the projections of $\mathbb{P}^N_A \times \mathbb{P}^N_A$ onto its factors. Then $\mathcal{O}_\Delta$ is an object of the smallest triangulated subcategory of $D^b(\mathbb{P}^N_A \times \mathbb{P}^N_A)$ that contains the sheaves $\mathcal{E}_j$ for $0 \le j \le N$. Since $F(\mathcal{F}^\bullet) = \mathbf{R}\pi_{2*}(\pi_1^*(\mathcal{M}^\bullet) \overset{\mathbf{L}}{\otimes} \mathcal{F}^\bullet)$ is an exact functor $D^b(\mathbb{P}^N_A \times \mathbb{P}^N_A) \to D^-(\mathbb{P}^N_A)$, $\mathcal{M}^\bullet \simeq F(\mathcal{O}_\Delta)$ is an object of the smallest triangulated category generated by the objects $F(\mathcal{E}_j)$ for $0 \le j \le N$. Thus to prove (1) we have only to see that $F(\mathcal{E}_j) = 0$ (resp. have bounded homology) for all $0 \le j \le N$. This follows because we have

$$F(\mathcal{E}_j) \simeq \mathbf{R}\pi_{2*}(\pi_1^*(\mathcal{M}^\bullet(-j))) \overset{\mathbf{L}}{\otimes} \Omega^j_{\mathbb{P}^N_A/A}(j) \simeq f^*\mathbf{R}f_*(\mathcal{M}^\bullet(-j))) \otimes \Omega^j_{\mathbb{P}^N_A/A}(j)$$

by the projection formula [34, Prop. 5.3] and flat base-change.

By applying the first statement to the cone of g, the second statement follows. $\square$

One can also easily prove that $\mathcal{M}^\bullet = 0$ if and only if $\mathbf{R}f_*(\mathcal{M}^\bullet(r)) = 0$ for all r by using the spectral sequence $R^pf_*(\mathcal{H}^q(\mathcal{M}^\bullet(r))) \implies R^{p+q}f_*(\mathcal{M}^\bullet(r))$.

Lemma 1.5. *Let $f\colon X \to Y$ be a proper morphism and $\mathcal{E}^\bullet$ an object of $D^b_c(X)$. If $\mathcal{E}^\bullet$ is either of finite projective dimension or of finite homological dimension over Y, then $\mathbf{R}f_*\mathcal{E}^\bullet$ is of finite homological dimension.*

Proof. The duality isomorphism (1.8) together with Lemma 1.2 imply that $\mathbf{R}f_*\mathcal{E}^\bullet$ is of finite homological dimension when $\mathcal{E}^\bullet$ is of finite projective dimension over Y. If $\mathcal{E}^\bullet$ is of finite homological dimension over Y, we use the same lemma and the projection formula. □

Proposition 1.6. *Let $f\colon X \to Y$ be a projective morphism and $\mathcal{E}^\bullet$ an object of $D^b_c(X)$. The following conditions are equivalent:*

1. *$\mathcal{E}^\bullet$ is of finite projective dimension over Y.*
2. *$\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ is of finite homological dimension for every integer r.*
3. *$\mathcal{E}^\bullet$ is of finite homological dimension over Y.*

Thus, if f is locally projective, $\mathcal{E}^\bullet$ is of finite projective dimension over Y if and only if it is of finite homological dimension over Y.

Proof. If $\mathcal{E}^\bullet$ is of finite projective dimension (resp. of finite homological dimension) over Y, so is $\mathcal{E}^\bullet(r)$ for every r, and then $\mathbf{R}f_*(\mathcal{E}^\bullet(r))$ is of finite homological dimension by Lemma 1.5. Assume that (2) is satisfied. Then (1) is a consequence of the duality isomorphism $\mathbf{R}f_*(\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)(r)) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathbf{R}f_*(\mathcal{E}^\bullet(-r)), \mathcal{G}^\bullet)$ and Lemma 1.2, whilst (3) follows from the same lemma and the projection formula. □

Corollary 1.7. *Let $f\colon X \to Y$ be a projective morphism and $\mathcal{E}^\bullet$ an object of $D^b_c(X)$. If $\mathcal{E}^\bullet$ is of finite projective dimension over Y, then $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{O}_Y)$ is also of finite projective dimension over Y. In particular, if f is Gorenstein, $\mathcal{E}^{\bullet\vee}$ is of finite projective dimension over Y.*

Proof. Let us write $\mathcal{N}^\bullet = \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{O}_Y)$. By Proposition 1.6, it suffices to see that $\mathbf{R}f_*(\mathcal{N}^\bullet(r))$ is of finite homological dimension for every r. This follows again by Proposition 1.6, due to the isomorphism $\mathbf{R}f_*(\mathcal{N}^\bullet(r)) \simeq [\mathbf{R}f_*(\mathcal{E}^\bullet(-r))]^\vee$. □

Proposition 1.8. *Let $f\colon X \to Y$ be a locally projective Gorenstein morphism of schemes and $\mathcal{E}^\bullet$ an object of $D_c(X)$ of finite projective dimension over Y. One has*

$$\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} f^*\mathcal{G}^\bullet \otimes \omega_f[n] \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)$$

for $\mathcal{G}^\bullet$ in $D^b_c(Y)$. Moreover, if Y is Gorenstein, then

$$\mathcal{E}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} f^*\mathcal{G}^\bullet \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^*\mathcal{G}^\bullet)\,.$$

Proof. One has natural morphisms

$$\begin{aligned}\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{O}_X) \overset{\mathbf{L}}{\otimes} f^*\mathcal{G}^\bullet \otimes \omega_f[n] &\to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^*\mathcal{G}^\bullet \otimes \omega_f[n]) \\ &\to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, f^!\mathcal{G}^\bullet)\,.\end{aligned} \tag{1.10}$$

We have to prove that the composition is an isomorphism. This is a local question on Y, so that we can assume that $Y = \operatorname{Spec} A$.

By Lemma 1.4 we have to prove that the induced morphism

(1.11)

$$\mathbf{R}f_*(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet,\mathcal{O}_X)\overset{\mathbf{L}}{\otimes} f^*\mathcal{G}^\bullet\otimes\omega_f[n]\otimes\mathcal{O}_X(r))\to\mathbf{R}f_*(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet,f^!\mathcal{G}^\bullet)\otimes\mathcal{O}_X(r))$$

is an isomorphism in $D^-(Y)$ for any integer r. The first member is isomorphic to

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{R}f_*(\mathcal{E}^\bullet(-r)),\mathcal{O}_Y)\overset{\mathbf{L}}{\otimes}\mathcal{G}^\bullet$$

by the projection formula and relative duality; the second one is isomorphic to

$$\mathbf{R}f_*(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet,f^!\mathcal{G}^\bullet)\otimes\mathcal{O}_X(r))\simeq\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{R}f_*(\mathcal{E}^\bullet(-r)),\mathcal{G}^\bullet)\,. \tag{1.12}$$

Thus, we have to prove that the natural morphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{R}f_*\mathcal{E}^\bullet(-r),\mathcal{O}_Y)\overset{\mathbf{L}}{\otimes}\mathcal{G}^\bullet\to\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathbf{R}f_*\mathcal{E}^\bullet(-r),\mathcal{G}^\bullet)\,, \tag{1.13}$$

is an isomorphism. Since $\mathbf{R}f_*\mathcal{E}^\bullet(-r)$ is of finite homological dimension by Proposition 1.6, one concludes by (1.2). □

1.3. **Depth and local properties of Cohen-Macaulay and Gorenstein schemes.** Here we state some preliminary results about depth on singular schemes and local properties of Cohen-Macaulay and Gorenstein schemes. We first recall a local property of Cohen-Macaulay schemes.

Lemma 1.9. [37, Prop. 6.2.4] *Let A be a noetherian local ring. A is Cohen-Macaulay if and only if there is an ideal I of A with $\dim A/I=0$ and such that A/I has finite homological dimension.*

Proof. Let n be the dimension of the ring A. If A is Cohen-Macaulay, $\operatorname{depth}(A)=n$. Then there is a regular sequence $(a_1,\dots,a_n)$ in A and taking $I=(a_1,\dots,a_n)$ we conclude. Conversely, if I is an ideal satisfying $\dim A/I=0$ and $\operatorname{hdim}(A/I)=s<\infty$, then the Auslander and Buchsbaum's formula $\operatorname{depth}(A/I)+\operatorname{hdim}(A/I)=\operatorname{depth}(A)$ [29, Thm. 19.1] proves that $s=\operatorname{depth}(A)$; then $s\le n$. Moreover, if $\mathcal{M}^\bullet$ is a free resolution of A/I of length s, one has $s\ge n$ by the intersection theorem [37, 6.2.2.]. Thus A is Cohen-Macaulay. □

Let $\mathcal{F}$ be a coherent sheaf on a scheme X of dimension n. We write n_x for the dimension of the local ring $\mathcal{O}_{X,x}$ of X at a point $x\in X$ and $\mathcal{F}_x$ for the stalk of $\mathcal{F}$ at x. $\mathcal{F}_x$ is a $\mathcal{O}_{X,x}$-module. The integer number $\operatorname{codepth}(\mathcal{F}_x)=n_x-\operatorname{depth}(\mathcal{F}_x)$ is called the codepth of $\mathcal{F}$ at x. For any integer $m\in\mathbb{Z}$, the *m-th singularity set* of $\mathcal{F}$ is defined to be

$$S_m(\mathcal{F})=\{x\in X\mid \operatorname{codepth}(\mathcal{F}_x)\ge n-m\}\,.$$

Then, if X is equidimensional, a closed point x is in $S_m(\mathcal{F})$ if and only if $\operatorname{depth}(\mathcal{F}_x)\le m$. If x is a point of X (not necessarily closed) the zero cycles Z_x of $\operatorname{Spec}\mathcal{O}_{X,x}$ supported on the closed point x of $\operatorname{Spec}\mathcal{O}_{X,x}$ will be called *zero cycles (of X) supported on x* by abuse of language.

Since $\operatorname{depth}(\mathcal{F}_x)$ is the first integer i such that either

- $\operatorname{Ext}^i(\mathcal{O}_x,\mathcal{F})\neq 0$ or
- $H^i_x(\operatorname{Spec}\mathcal{O}_{X,x},\mathcal{F}_x)\neq 0$ or
- $\operatorname{Ext}^i(\mathcal{O}_Z,\mathcal{F}_x)\neq 0$ for some zero cycle Z supported on x or
- $\operatorname{Ext}^i(\mathcal{O}_Z,\mathcal{F}_x)\neq 0$ for every zero cycle Z supported on x

(see for instance [22]), we have alternative descriptions of $S_m(\mathcal{F})$:

$$\begin{aligned} S_m(\mathcal{F}) &= \{x\in X \mid H^i_x(\operatorname{Spec}\mathcal{O}_{X,x},\mathcal{F}_x)\neq 0 \text{ for some } i\leq m+n_x-n\}\\ &=\{x\in X \mid \operatorname{Ext}^i(\mathcal{O}_Z,\mathcal{F}_x)\neq 0 \text{ for some } i\leq m+n_x-n \text{ and some}\\ &\qquad \text{zero cycle } Z \text{ supported on } x\}\\ &=\{x\in X \mid \operatorname{Ext}^i(\mathcal{O}_Z,\mathcal{F}_x)\neq 0 \text{ for some } i\leq m+n_x-n \text{ and any}\\ &\qquad \text{zero cycle } Z \text{ supported on } x\} \end{aligned} \tag{1.14}$$

Lemma 1.10. *If X is smooth, then the m-th singularity set of $\mathcal{F}$ can be described as*

$$S_m(\mathcal{F})=\cup_{p\geq n-m}\{x\in X \mid L_p j_x^*\mathcal{F}\neq 0\}\,,$$

where j_x is the immersion of the point x.

Proof. Let $x\in X$ be a point and $\mathcal{L}^\bullet$ the Koszul complex associated locally to a regular sequence of generators of the maximal ideal of $\mathcal{O}_{X,x}$. Since $\mathcal{L}^{\bullet\vee}\simeq\mathcal{L}^\bullet[-n_x]$, one has an isomorphism $\operatorname{Ext}^i(\mathcal{O}_x,\mathcal{F}_x)\simeq L_{n_x-i}j_x^*\mathcal{F}$ which proves the result. □

In the singular case, this characterization of $S_m(\mathcal{F})$ is not true. However, there is a similar interpretation for Cohen-Macaulay schemes as we shall see now. By Lemma 1.9, if X is Cohen-Macaulay, for every point x there exist zero cycles supported on x defined locally by a regular sequence; we refer to them as *locally complete intersection* or *l.c.i.* cycles. If $Z\hookrightarrow X$ is such a l.c.i. cycle, by the Koszul complex theory the structure sheaf $\mathcal{O}_Z$ has *finite homological dimension* as an $\mathcal{O}_X$-module.

We denote by j_Z the immersion of Z in X. Recall that for every object $\mathcal{K}^\bullet$ in $D^b(X)$, $L_ij_Z^*\mathcal{K}^\bullet$ denotes the cohomology sheaf $\mathcal{H}^{-i}(j_Z^*\mathcal{L}^\bullet)$ where $\mathcal{L}^\bullet$ is a bounded above complex of locally free sheaves quasi-isomorphic to $\mathcal{K}^\bullet$.

Lemma 1.11. *If X is Cohen-Macaulay, then the m-th singularity set $S_m(\mathcal{F})$ can be described as*

$$\begin{aligned} S_m(\mathcal{F})=\{x\in X \mid &\textit{ there is an integer } i\geq n-m \textit{ with } L_ij_{Z_x}^*\mathcal{F}\neq 0\\ &\textit{for any l.c.i zero cycle } Z_x \textit{ supported on } x\}\,. \end{aligned}$$

Proof. Let Z_x be a l.c.i. zero cycle supported on x and $\mathcal{L}^\bullet$ the Koszul complex associated locally to a regular sequence of generators of the ideal of Z_x. As in the smooth case, we have that $\mathcal{L}^{\bullet\vee}\simeq\mathcal{L}^\bullet[-n_x]$ and then an isomorphism $\operatorname{Ext}^i(\mathcal{O}_{Z_x},\mathcal{F})\simeq L_{n_x-i}j_{Z_x}^*\mathcal{F}$. The result follows from (1.14). □

Lemma 1.12. *If $j\colon X\hookrightarrow W$ is a closed immersion and $\mathcal{F}$ is a coherent sheaf on X, then $S_m(\mathcal{F})=S_m(j_*\mathcal{F})$.*

Proof. Since $H^i_x(\operatorname{Spec}\mathcal{O}_{X,x},\mathcal{F}_x)=H^i_x(\operatorname{Spec}\mathcal{O}_{W,x},(j_*\mathcal{F})_x)$, the result follows from (1.14). □

Proposition 1.13. *Let X be an equidimensional scheme of dimension n and $\mathcal{F}$ a coherent sheaf on X.*

(1) *$S_m(\mathcal{F})$ is a closed subscheme of X and* $\operatorname{codim} S_m(\mathcal{F})\geq n-m$.
(2) *If Z is an irreducible component of the support of $\mathcal{F}$ and c is the codimension of Z in X, then* $\operatorname{codim} S_{n-c}(\mathcal{F})=c$ *and Z is also an irreducible component of $S_{n-c}(\mathcal{F})$.*

Proof. All questions are local and then, by Lemma 1.12, we can assume that X is affine and smooth. By Lemma 1.10, $S_m(\mathcal{F}) = \cup_{p\geq n-m} X_p(\mathcal{F})$, where $X_p(\mathcal{F}) = \{x \in X \mid L_p j_x^*\mathcal{F} \neq 0\}$. To prove (1), we have only to see that $X_p(\mathcal{F})$ is closed of codimension greater or equal than p. This can be seen by induction on p. If $p = 0$, then $X_0(\mathcal{F})$ is the support of $\mathcal{F}$ and the statement is clear. For $p = 1$, $X_1(\mathcal{F})$ is the locus of points where $\mathcal{F}$ is not locally free, which is closed of codimension greater or equal than 1, since $\mathcal{F}$ is always free at the generic point. If $p > 1$, let us consider an exact sequence $0 \to \mathcal{N} \to \mathcal{L} \to \mathcal{F} \to 0$ where $\mathcal{L}$ is free and finitely generated. Then $L_p j_x^*\mathcal{F} \simeq L_{p-1} j_x^*\mathcal{N}$ so that $X_p(\mathcal{F}) = X_{p-1}(\mathcal{N})$ which is closed by induction. Moreover, if $x \in X_p(\mathcal{F})$, then $L_p j_x^*\mathcal{F} \neq 0$, so that $p \leq \dim \mathcal{O}_{X,x}$ because $\mathcal{O}_{X,x}$ is a regular ring. It follows that $\operatorname{codim} X_p(\mathcal{F}) = \max_{x\in X_p(\mathcal{F})}\{\dim \mathcal{O}_{X,x}\} \geq p$.

We finally prove (2). By [29, Thm. 6.5], the prime ideal of Z is also a minimal associated prime to $\mathcal{F}$. Thus, if x is the generic point of Z, the maximal ideal of the local ring $\mathcal{O}_{X,x}$ is a prime associated to $\mathcal{F}_x$, and then $\operatorname{Hom}(\mathcal{O}_x, \mathcal{F}_x) \neq 0$. This proves that $x \in S_{n-c}(\mathcal{F})$ and then $Z \subseteq S_{n-c}(\mathcal{F})$. The result follows. □

Corollary 1.14. *Let X be a Cohen-Macaulay scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $h\colon Y \hookrightarrow X$ be an irreducible component of the support of $\mathcal{F}$ and c the codimension of Y in X. There is a non-empty open subset U of Y such that for any l.c.i. zero cycle Z_x supported on $x \in U$ one has*

$$\begin{aligned} L_c j_{Z_x}^*\mathcal{F} &\neq 0 \\ L_{c+i} j_{Z_x}^*\mathcal{F} &= 0\,, \qquad \textit{for every } i > 0. \end{aligned}$$

Proof. By Lemma 1.11 the locus of the points that verify the conditions is $U = Y \cap (S_{n-c}(\mathcal{F}) - S_{n-c-1}(\mathcal{F}))$, which is open in Y by Proposition 1.13. Proving that U is not empty is a local question, and we can then assume that Y is the support of $\mathcal{F}$. Now $Y = S_{n-c}(\mathcal{F})$ by (2) of Proposition 1.13 and $U = S_{n-c}(\mathcal{F}) - S_{n-c-1}(\mathcal{F})$ is non-empty because the codimension of $S_{n-c-1}(\mathcal{F})$ in X is greater or equal than $c + 1$ again by Proposition 1.13. □

The following proposition characterises objects of the derived category supported on a closed subscheme.

Proposition 1.15. [9, Prop. 1.5]. *Let $j\colon Y \hookrightarrow X$ be a closed immersion of codimension d of irreducible Cohen-Macaulay schemes and $\mathcal{K}^\bullet$ an object of $D^b_c(X)$. Assume that*

1. *If $x \in X - Y$ is a closed point, then $\mathbf{L} j_{Z_x}^*\mathcal{K}^\bullet = 0$ for some l.c.i. zero cycle Z_x supported on x.*
2. *If $x \in Y$ is a closed point, then $L_i j_{Z_x}^*\mathcal{K}^\bullet = 0$ for some l.c.i. zero cycle Z_x supported on x when either $i < 0$ or $i > d$.*

Then there is a sheaf $\mathcal{K}$ on X whose topological support is contained in Y and such that $\mathcal{K}^\bullet \simeq \mathcal{K}$ in $D^b_c(X)$. Moreover, this topological support coincides with Y unless $\mathcal{K}^\bullet = 0$.

Proof. Let us write $\mathcal{H}^q = \mathcal{H}^q(\mathcal{K}^\bullet)$. For every zero cycle Z_x in X there is a spectral sequence

$$E_2^{-p,q} = L_p j_{Z_x}^*\mathcal{H}^q \implies E_\infty^{-p+q} = L_{p-q} j_{Z_x}^*\mathcal{K}^\bullet$$

Let q_0 be the maximum of the q's with $\mathcal{H}^q \neq 0$. If $x \in \operatorname{supp}(\mathcal{H}^{q_0})$, one has $j_{Z_x}^*\mathcal{H}^{q_0} \neq 0$ for every l.c.i. zero cycle Z_x supported on x. A nonzero element in $j_{Z_x}^*\mathcal{H}^{q_0}$ survives up to infinity in the spectral sequence. Since there is a l.c.i. zero cycle Z_x such that $E^q_\infty = L_{-q}j^*_{Z_x}\mathcal{K}^\bullet = 0$ for every $q > 0$ by hypothesis, one has $q_0 \le 0$. A similar argument shows that the topological support of all the sheaves $\mathcal{H}^q$ is contained in Y: assume that this is not true and let us consider the maximum q_1 of the q's such that $j^*_x\mathcal{H}^q \neq 0$ for a certain point $x \in X - Y$; then $j^*_{Z_x}\mathcal{H}^{q_1} \neq 0$ and a nonzero element in $j^*_{Z_x}\mathcal{H}^{q_1}$ survives up to infinity in the spectral sequence, which is impossible since $\mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet = 0$.

Let $q_2 \le q_0$ be the minimum of the q's with $\mathcal{H}^q \neq 0$. We know that $\mathcal{H}^{q_2}$ is topologically supported on a closed subset of Y. Take a component $Y' \subseteq Y$ of the support. If $c \ge d$ is the codimension of Y', then there is a non-empty open subset U of Y' such that $L_cj^*_{Z_x}\mathcal{H}^{q_2} \neq 0$ for any closed point $x \in U$ and any l.c.i. zero cycle Z_x supported on x, by Corollary 1.14. Elements in $L_cj^*_{Z_x}\mathcal{H}^{q_2}$ would be killed in the spectral sequence by $L_pj^*_{Z_x}\mathcal{H}^{q_2+1}$ with $p \ge c+2$. By Lemma 1.11 the set

$$\{x \in X \mid L_ij^*_{Z_x}\mathcal{H}^{q_2+1} \neq 0 \text{ for some } i \ge c+2 \text{ and any l.c.i. cycle } Z_x\}$$

is equal to $S_{n-(c+2)}(\mathcal{H}^{q_2+1})$ and then has codimension greater or equal than $c+2$ by Proposition 1.13. Thus there is a point $x \in Y'$ such that any nonzero element in $L_cj^*_{Z_x}\mathcal{H}^{q_2}$ survives up to the infinity in the spectral sequence. Therefore, $L_{c-q_2}j^*_{Z_x}\mathcal{K}^\bullet \neq 0$ for any l.c.i. zero cycle Z_x supported on x. Thus $c - q_2 \le d$ which leads to $q_2 \ge c-d \ge 0$ and then $q_2 = q_0 = 0$. So $\mathcal{K}^\bullet = \mathcal{H}^0$ in $D^b(X)$ and the topological support of $\mathcal{K} = \mathcal{H}^0$ is contained in Y. Actually, if $\mathcal{K}^\bullet \neq 0$, then this support is the whole of Y: if this was not true, since Y is irreducible, the support would have a component $Y' \subset Y$ of codimension $c > d$ and one could find, reasoning as above, a non-empty subset U of Y' such that $L_cj^*_{Z_x}\mathcal{K}^\bullet \neq 0$ for all $x \in U$ and all l.c.i. zero cycle Z_x supported on x. This would imply that $c \le d$, which is impossible. □

Taking into account that $\mathcal{O}^\vee_{Z_x} = \mathcal{O}_{Z_x}[-n]$ where $n = \dim X$, Proposition 1.15 may be reformulated as follows:

Proposition 1.16. *Let $j\colon Y \hookrightarrow X$ be a closed immersion of codimension d of irreducible Cohen-Macaulay schemes of dimensions m and n respectively, and let $\mathcal{K}^\bullet$ be an object of $D^b_c(X)$. Assume that for any closed point $x \in X$ there is a l.c.i. zero cycle Z_x supported on x such that*

$$\operatorname{Hom}^i_{D(X)}(\mathcal{O}_{Z_x}, \mathcal{K}^\bullet) = 0\,,$$

unless $x \in Y$ and $m \le i \le n$. Then there is a sheaf $\mathcal{K}$ on X whose topological support is contained in Y and such that $\mathcal{K}^\bullet \simeq \mathcal{K}$ in $D^b_c(X)$. Moreover, the topological support is Y unless $\mathcal{K}^\bullet = 0$. □

1.4. **Integral functors.** Let X and Y be proper schemes. We denote the projections of the direct product $X \times Y$ to X and Y by π_X and π_Y.

Let $\mathcal{K}^\bullet$ be an object in $D^b(X \times Y)$. The integral functor defined by $\mathcal{K}^\bullet$ is the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^-(X) \to D^-(Y)$ given by

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{F}^\bullet) = \mathbf{R}\pi_{Y*}(\pi^*_X\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)\,.$$

If the kernel $\mathcal{K}^\bullet \in D^b_c(X \times Y)$ is of finite homological dimension over X, then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is defined over the whole $D(X)$ and maps $D^b_c(X)$ to $D^b_c(Y)$.

If Z is a third proper scheme and $\mathcal{L}^\bullet$ is an object of $D^b_c(Y \times Z)$, arguing exactly as in the smooth case, we prove that there is an isomorphism of functors

$$\Phi^{\mathcal{L}^\bullet}_{Y\to Z} \circ \Phi^{\mathcal{K}^\bullet}_{X\to Y} \simeq \Phi^{\mathcal{L}^\bullet * \mathcal{K}^\bullet}_{X\to Z}$$

where

$$\mathcal{L}^\bullet * \mathcal{K}^\bullet = \mathbf{R}\pi_{X,Z*}(\pi_{X,Y}^*\mathcal{K}^\bullet \overset{\mathbf{L}}{\otimes} \pi_{Y,Z}^*\mathcal{L}^\bullet)\,. \tag{1.15}$$

If either $\mathcal{K}^\bullet$ or $\mathcal{L}^\bullet$ is of finite homological dimension over Y, then $\mathcal{L}^\bullet * \mathcal{K}^\bullet$ in bounded.

1.5. **Adjoints.** We can describe nicely the adjoints to an integral functor when we work with Gorenstein schemes. In this subsection X and Y are projective Gorenstein schemes.

Proposition 1.17. *Let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times Y)$ of finite projective dimension over X and Y.*

(1) *The functor $\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}\colon D^b_c(Y)\to D^b_c(X)$ is a left adjoint to the functor $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$.*

(2) *The functor $\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}\colon D^b_c(Y)\to D^b_c(X)$ is a right adjoint to the functor $\Phi_{X\to Y}^{\mathcal{K}^\bullet}$.*

(here $m=\dim X$ and $n=\dim Y$)

Proof. We shall freely use (1.4) for the projections π_X and π_Y.

(1) We first notice that one has

$$(\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)^\vee \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi_X^*\mathcal{F}^{\bullet\vee}) \simeq \mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_X^*\mathcal{F}^{\bullet\vee}\,, \tag{1.16}$$

for $\mathcal{F}^\bullet$ in $D^b_c(X)$ by (1.1) and Proposition 1.8. The latter applies because π_X is a projective morphism and $\mathcal{K}^\bullet$ is of finite projective dimension over X. Now, if $\mathcal{G}^\bullet$ is an object of $D^b_c(Y)$ there is a chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{D(Y)}(\mathcal{G}^\bullet, \Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{F}^\bullet)) &\simeq \operatorname{Hom}_{D(X\times Y)}(\pi_Y^*\mathcal{G}^\bullet, \pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)\\
&\simeq \operatorname{Hom}_{D(X\times Y)}((\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)^\vee, (\pi_Y^*\mathcal{G}^\bullet)^\vee)\\
&\simeq \operatorname{Hom}_{D(X\times Y)}(\pi_X^*\mathcal{F}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \mathcal{K}^{\bullet\vee}, \pi_Y^*\mathcal{G}^{\bullet\vee})\\
&\simeq \operatorname{Hom}_{D(X\times Y)}(\pi_X^*\mathcal{F}^{\bullet\vee}, \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times Y}}(\mathcal{K}^{\bullet\vee}, \pi_Y^*\mathcal{G}^{\bullet\vee}))\\
&\simeq \operatorname{Hom}_{D(X\times Y)}(\pi_X^*\mathcal{F}^{\bullet\vee}, (\mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{G}^\bullet)^\vee)\\
&\simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^{\bullet\vee}, \mathbf{R}\pi_{X*}((\mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{G}^\bullet)^\vee))\\
&\simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^{\bullet\vee}, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbf{R}\pi_{X*}(\mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_Y^*\mathcal{G}^\bullet \otimes \pi_Y^*\omega_Y[n]), \mathcal{O}_X))\\
&\simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^{\bullet\vee}, [\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}(\mathcal{G}^\bullet)]^\vee)\\
&\simeq \operatorname{Hom}_{D(X)}(\Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}(\mathcal{G}^\bullet), \mathcal{F}^\bullet)\,.
\end{aligned}$$

where the second follows from (1.6) which applies because $\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet$ is bounded, the third is (1.16), the forth and the fifth are (1.1), the seventh is relative duality and the ninth is again (1.6).

(2) The adjunction between the direct and inverse images and relative duality proves that the functor

$$H(\mathcal{G}^\bullet) = \mathbf{R}\pi_{X,*}(\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi_Y^!\mathcal{G}^\bullet))$$

satisfies

$$\operatorname{Hom}_{D(Y)}(\Phi_{X\to Y}^{\mathcal{K}^\bullet}(\mathcal{F}^\bullet), \mathcal{G}^\bullet) \simeq \operatorname{Hom}_{D(X)}(\mathcal{F}^\bullet, H(\mathcal{G}^\bullet))\,. \tag{1.17}$$

Then we conclude by Proposition 1.8 since π_Y is a projective morphism. □

We shall need some basic results about adjoints and fully faithfulness which we state without proof.

Proposition 1.18. *Let $\Phi\colon \mathcal{A}\to\mathcal{B}$ a functor and $G\colon\mathcal{B}\to\mathcal{A}$ a left adjoint (resp. $H\colon\mathcal{B}\to\mathcal{A}$ a right adjoint). Then the following conditions are equivalent:*

1. *Φ is fully faithful.*
2. *$G\circ\Phi$ is fully faithful (resp. $H\circ\Phi$ is fully faithful).*
3. *The counit morphism $G\circ\Phi\to\operatorname{Id}$ is an isomorphism (resp. the unit morphism $\operatorname{Id}\to H\circ\Phi$ is an isomorphism).*

Moreover, Φ is an equivalence if and only if Φ and G (resp. Φ and H) are fully faithful. □

1.6. **Strongly simple objects.** Let X and Y be proper Gorenstein schemes. In this situation, the notion of strong simplicity is the following.

Definition 1.19. An object $\mathcal{K}^\bullet$ in $D^b_c(X\times Y)$ is *strongly simple* over X if it satisfies the following conditions:

1. For every closed point $x\in X$ there is a l.c.i. zero cycle Z_x supported on x such that
$$\operatorname{Hom}^i_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_{x_1}}),\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_{x_2}}))=0$$
unless $x_1=x_2$ and $0\le i\le\dim X$.
2. $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x),\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x))=k$ for every closed point $x\in X$.

△

The last condition can be written as $\operatorname{Hom}^0_{D(Y)}(\mathbf{L}j_x^*\mathcal{K}^\bullet,\mathbf{L}j_x^*\mathcal{K}^\bullet)=k$, because the restriction $\mathbf{L}j_x^*\mathcal{K}^\bullet$ of $\mathcal{K}^\bullet$ to the fibre $j_x\colon Y\simeq\{x\}\times Y\hookrightarrow X\times Y$ can also be computed as $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$.

In order to fix some notation, for any zero-cycle Z_x of X and any scheme S, we shall denote by j_{Z_x} the immersion $Z_x\times S\hookrightarrow X\times S$.

Proposition 1.20. *Assume that Y is projective, and let $\mathcal{K}^\bullet$ be a kernel in $D^b(X\times Y)$ of finite projective dimension over X. If $\mathcal{K}^\bullet$ is strongly simple over X, its dual $\mathcal{K}^{\bullet\vee}$ is strongly simple over X as well.*

Proof. If Z_x is a l.c.i. zero cycle supported on x, one has that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x})=p_{2*}\mathbf{L}j_{Z_x}^*\mathcal{K}^\bullet$, with $p_2\colon Z_x\times Y\to Y$ the second projection. Since $\omega_{p_2}\simeq\mathcal{O}_{Z_x\times Y}$ because Z_x is zero dimensional and Gorenstein, one obtains
$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x})^\vee\simeq p_{2*}(\mathbf{L}j_{Z_x}^*\mathcal{K}^\bullet)^\vee\,.$$
Moreover, $(\mathbf{L}j_{Z_x}^*\mathcal{K}^\bullet)^\vee\simeq\mathbf{L}j_{Z_x}^*(\mathcal{K}^{\bullet\vee})$ by (1.4) since j_{Z_x} is a regular closed immersion. Then,
$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x})^\vee\simeq\Phi^{\mathcal{K}^{\bullet\vee}}_{X\to Y}(\mathcal{O}_{Z_x})\,.$$
It follows that $\mathcal{K}^{\bullet\vee}$ satisfies condition (1) of Definition 1.19. To see that it also fulfils condition (1), we have to prove that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)^\vee\simeq\Phi^{\mathcal{K}^{\bullet\vee}}_{X\to Y}(\mathcal{O}_x)$, and this is equivalent to the base change formula $\mathbf{L}j_x^*(\mathcal{K}^{\bullet\vee})\simeq(\mathbf{L}j_x^*\mathcal{K}^\bullet)^\vee$. Since we cannot longer use (1.4) because j_x may fail to be of finite Tor-dimension, we proceed in a different way. To see that the natural morphism $\mathbf{L}j_x^*(\mathcal{K}^{\bullet\vee})\to(\mathbf{L}j_x^*\mathcal{K}^\bullet)^\vee$ is an isomorphism, it suffices to

check that $j_{x*}\mathbf{L}j_x^*(\mathcal{K}^{\bullet\vee}) \simeq j_{x*}[(\mathbf{L}j_x^*\mathcal{K}^\bullet)^\vee]$ since j_x is a closed embedding. On the one hand, we have

$$j_{x*}\mathbf{L}j_x^*(\mathcal{K}^{\bullet\vee}) \simeq \mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} j_{x*}\mathcal{O}_Y \simeq \mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_X^*\mathcal{O}_x\,,$$

whilst on the other hand,

$$\begin{aligned} j_{x*}[(\mathbf{L}j_x^*\mathcal{K}^\bullet)^\vee] &= j_{x*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_Y}(\mathbf{L}j_x^*\mathcal{K}^\bullet, \mathcal{O}_Y) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, j_{x*}\mathcal{O}_Y) \\ &\simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times Y}}(\mathcal{K}^\bullet, \pi_X^*\mathcal{O}_x)\,. \end{aligned}$$

We conclude by Proposition 1.8.

□

Remark 1.21. When X and Y are smooth, strong simplicity is usually defined by the following conditions (see [6]):

(1) $\operatorname{Hom}^i_{D(Y)}(\mathbf{L}j_{x_1}^*\mathcal{K}^\bullet, \mathbf{L}j_{x_2}^*\mathcal{K}^\bullet) = 0$ unless $x_1 = x_2$ and $0 \le i \le \dim X$;
(2) $\operatorname{Hom}^0_{D(Y)}(\mathbf{L}j_x^*\mathcal{K}^\bullet, \mathbf{L}j_x^*\mathcal{K}^\bullet) = k$ for every closed point x.

Since our definition is weaker, Theorem 1.22 improves Bondal and Orlov's result [9, Thm. 1.1].

We now give the criterion for an integral functor between derived categories of Gorenstein proper schemes to be fully faithful.

Theorem 1.22. *Let X and Y be projective Gorenstein schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times Y)$ of finite projective dimension over X and over Y. Assume also that X is integral. Then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X)\to D^b_c(Y)$ is fully faithful if and only if the kernel $\mathcal{K}^\bullet$ is strongly simple over X.*

Proof. If the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is fully faithful, then $\mathcal{K}^\bullet$ is strongly simple over X.

Let us prove the converse. Before starting, we fix some notation: we denote by π_i the projections of $X\times X$ onto its factors and by U the smooth locus of X, which is not empty because X is integral. We also denote $m=\dim X$, $n=\dim Y$ and $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y}$.

By Proposition 1.17, Φ has a left adjoint $G = \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}_{Y\to X}$ and a right adjoint $H = \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}_{Y\to X}$. By Proposition 1.18 it suffices to show that $G\circ\Phi$ is fully faithful. We know that $H\circ\Phi \simeq \Phi^{\mathcal{M}^\bullet}_{X\to X}$, and $G\circ\Phi\simeq\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}$, with $\mathcal{M}^\bullet$ and $\widetilde{\mathcal{M}^\bullet}$ given by (1.15). Notice that since $\mathcal{K}^\bullet$ is of finite projective dimension over X and Y, $\mathcal{M}^\bullet$ and $\widetilde{\mathcal{M}^\bullet}$ are bounded.

The strategy of the proof is as follows: we first show that both $\mathcal{M}^\bullet$ and $\widetilde{\mathcal{M}^\bullet}$ are single sheaves supported topologically on the image Δ of the diagonal morphism $\delta\colon X\hookrightarrow X\times X$; then we prove that $\widetilde{\mathcal{M}^\bullet}$ is actually schematically supported on the diagonal, that is, $\widetilde{\mathcal{M}^\bullet}=\delta_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on X and finally that $\mathcal{N}$ is a line bundle; this will imply that $\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}$ is the twist by $\mathcal{N}$ which is an equivalence of categories, in particular fully faithful.

a) *$\mathcal{M}^\bullet$ and $\widetilde{\mathcal{M}^\bullet}$ are single sheaves topologically supported on the diagonal.*

Let us fix a closed point $(x_1,x_2)\in X\times X$ and consider the l.c.i. zero cycles Z_{x_1} and Z_{x_2} of the first condition of the definition of strongly simple object. One has

$$\operatorname{Hom}^i_{D(X)}(\mathcal{O}_{Z_{x_1}}, \Phi^{\mathcal{M}^\bullet}_{X\to X}(\mathcal{O}_{Z_{x_2}})) \simeq \operatorname{Hom}^i_{D(Y)}(\Phi(\mathcal{O}_{Z_{x_1}}), \Phi(\mathcal{O}_{Z_{x_2}}))\,,$$

which is zero unless $x_1 = x_2$ and $0 \le i \le m$ because $\mathcal{K}^\bullet$ is strongly simple. Applying Proposition 1.16 to the immersion $\{x_2\} \hookrightarrow X$ we have that $\Phi^{\mathcal{M}^\bullet}_{X\to X}(\mathcal{O}_{Z_{x_2}})$ reduces to a coherent sheaf topologically supported at x_2. Since $\Phi^{\mathcal{M}^\bullet}_{X\to X}(\mathcal{O}_{Z_{x_2}}) \simeq p_{2*}\mathbf{L}j^*_{Z_{x_2}}\mathcal{M}^\bullet$, where $p_2\colon Z_{x_2}\times X \to X$ is the second projection, the complex $\mathbf{L}j^*_{Z_{x_2}}\mathcal{M}^\bullet$ is isomorphic to a single coherent sheaf $\mathcal{F}$ topologically supported at (x_2, x_2). If we denote by $i_{Z_{x_1}} : Z_{x_2} \times Z_{x_1} \hookrightarrow Z_{x_2}\times X$ and $j_{Z_{x_2}\times Z_{x_1}} : Z_{x_2}\times Z_{x_1} \hookrightarrow X \times X$ the natural immersions, we have

$$\mathbf{L}j^*_{Z_{x_2}\times Z_{x_1}}\mathcal{M}^\bullet \simeq \mathbf{L}i^*_{Z_{x_1}}\mathbf{L}j^*_{Z_{x_2}}\mathcal{M}^\bullet \simeq \mathbf{L}i^*_{Z_{x_1}}\mathcal{F}\,.$$

Thus, $L_p j^*_{Z_{x_2}\times Z_{x_1}}\mathcal{M}^\bullet = 0$ unless $x_1 = x_2$ and $0 \le p \le m$. Applying now Proposition 1.15 to δ, we obtain that $\mathcal{M}^\bullet$ reduces to a coherent sheaf $\mathcal{M}$ supported topologically on the diagonal as claimed.

For $\widetilde{\mathcal{M}^\bullet}$, we proceed as follows. We have

$$\begin{aligned}\mathcal{H}^i(\mathbf{L}j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{Z_{x_1}})^\vee) &\simeq \operatorname{Hom}^i_{D(Z_{x_2})}(\mathbf{L}j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{Z_{x_1}}), \mathcal{O}_{Z_{x_2}})\\ &\simeq \operatorname{Hom}^i_{D(X)}(\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{Z_{x_1}}), \mathcal{O}_{Z_{x_2}}) \simeq \operatorname{Hom}^i_{D(Y)}(\Phi(\mathcal{O}_{Z_{x_1}}), \Phi(\mathcal{O}_{Z_{x_2}})),\end{aligned}$$

which is zero unless $x_1 = x_2$ and $0 \le i \le m$ because $\mathcal{K}^\bullet$ is strongly simple. Since Z_{x_2} is a zero dimensional Gorenstein scheme, (1.7) implies that $L_i j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{Z_{x_1}}) = 0$ again unless $x_1 = x_2$ and $0 \le i \le m$. By Proposition 1.15 for the immersion $\{x_1\} \hookrightarrow X$, one has that $\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{Z_{x_1}})$ is a sheaf supported topologically at x_1. Now, a similar argument to the one used for $\mathcal{M}^\bullet$ proves that $\widetilde{\mathcal{M}^\bullet}$ reduces to a coherent sheaf $\widetilde{\mathcal{M}}$ supported topologically on the diagonal.

b) *$\widetilde{\mathcal{M}}$ is schematically supported on the diagonal, that is, $\widetilde{\mathcal{M}} = \delta_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on X; moreover $\mathcal{N}$ is a line bundle.*

It might happen that the schematic support is an infinitesimal neighborhood of the diagonal; we shall see that this is not the case. Let us denote by $\bar\delta\colon W \hookrightarrow X\times X$ the schematic support of $\widetilde{\mathcal{M}}$ so that $\widetilde{\mathcal{M}} = \bar\delta_*\mathcal{N}$ for a coherent sheaf $\mathcal{N}$ on W. The diagonal embedding δ factors through a closed immersion $\tau\colon X \hookrightarrow W$ which topologically is a homeomorphism.

b1) $\pi_{2*}(\widetilde{\mathcal{M}})$ *is locally free.*

To see this, we shall prove that $\operatorname{Hom}^1_{D(X)}(\pi_{2*}(\widetilde{\mathcal{M}}), \mathcal{O}_x) = 0$ for every closed point $x \in X$. Since $\widetilde{\mathcal{M}}$ is topologically supported on the diagonal, we have that $\pi_{2*}(\widetilde{\mathcal{M}}) \simeq \mathbf{R}\pi_{2*}(\widetilde{\mathcal{M}}) \simeq \Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_X)$. We have

$$\operatorname{Hom}^1_{D(X)}(\pi_{2*}(\widetilde{\mathcal{M}}), \mathcal{O}_x) \simeq \operatorname{Hom}^1_{D(X)}(\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_X), \mathcal{O}_x) \simeq \operatorname{Hom}^1_{D(X)}(\mathcal{O}_X, \Phi^{\mathcal{M}^\bullet}_{X\to X}(\mathcal{O}_x))$$

for every closed point $x \in X$, because $H\circ\Phi$ is a right adjoint to $G\circ\Phi$. Since $\Phi^{\mathcal{M}^\bullet}_{X\to X}(\mathcal{O}_x) \simeq \mathbf{L}j^*_x\mathcal{M}$ has only negative cohomology sheaves and all of them are supported at x, one has that $\operatorname{Hom}^1_{D(X)}(\pi_{2*}(\widetilde{\mathcal{M}}), \mathcal{O}_x) = 0$ and $\pi_{2*}(\widetilde{\mathcal{M}})$ is locally free.

b2) $\pi_{1*}(\widetilde{\mathcal{M}})$ *is a line bundle on the smooth locus U of X.*

We know that $\Phi^{\mathcal{M}}_{X\to X}(\mathcal{O}_{Z_{x_2}})$ reduces to a single sheaf supported at x_2. Then, for every point $x_1 \in U$ one has

$$\begin{aligned}\mathcal{H}^i(\mathbf{L}j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{x_1})^\vee) &\simeq \operatorname{Hom}^i_{D(Z_{x_2})}(\mathbf{L}j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{x_1}), \mathcal{O}_{Z_{x_2}})\\ &\simeq \operatorname{Hom}^i_{D(X)}(\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{x_1}), \mathcal{O}_{Z_{x_2}}) \simeq \operatorname{Hom}^i_{D(X)}(\mathcal{O}_{x_1}, \Phi^{\mathcal{M}}_{X\to X}(\mathcal{O}_{Z_{x_2}}))\end{aligned}$$

which is zero unless $x_2 = x_1$ and $0 \le i \le m$ because x_1 is a smooth point. Since Z_{x_2} is a zero dimensional Gorenstein scheme, (1.7) implies that whenever x_1 is a smooth point, then $L_i j^*_{Z_{x_2}}\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{x_1}) = 0$ unless $x_2 = x_1$ and $0 \le i \le m$. By Proposition 1.15, $\Phi^{\widetilde{\mathcal{M}^\bullet}}_{X\to X}(\mathcal{O}_{x_1})$ reduces to a single sheaf supported at x_1. In particular $\mathbf{L}j^*_x(\widetilde{\mathcal{M}}) \simeq j^*_x(\widetilde{\mathcal{M}})$ for every smooth point x, and thus the restriction of $\widetilde{\mathcal{M}}$ to $U \times X$ is flat over U. Moreover, for every point $x \in U$, we have that

$$\operatorname{Hom}_X(j^*_x\widetilde{\mathcal{M}}, \mathcal{O}_x) \simeq \operatorname{Hom}^0_{D(X)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \simeq k\,.$$

By [12, Lemmas 5.2 and 5.3] there is a point x_0 in U such that the Kodaira-Spencer map for the family $\widetilde{\mathcal{M}}_{|U\times X}$ is injective at x_0. We now proceed as in the proof of [12, Thm. 5.1]: the morphism $\operatorname{Hom}^1_{D(X)}(\mathcal{O}_{x_0}, \mathcal{O}_{x_0}) \to \operatorname{Hom}^1_{D(X)}(G\circ\Phi(\mathcal{O}_{x_0}), G\circ\Phi(\mathcal{O}_{x_0}))$ is injective so that the morphism $\operatorname{Hom}^1_{D(X)}(\mathcal{O}_{x_0}, \mathcal{O}_{x_0}) \to \operatorname{Hom}^1_{D(Y)}(\Phi(\mathcal{O}_{x_0}), \Phi(\mathcal{O}_{x_0}))$ is injective as well and then the counit morphism $j^*_{x_0}\widetilde{\mathcal{M}} \simeq G\circ\Phi(\mathcal{O}_{x_0}) \to \mathcal{O}_{x_0}$ is an isomorphism. Thus, the rank of $\pi_{1*}(\widetilde{\mathcal{M}})$ at the point x_0 is one, and then it is one everywhere in U.

b3) $\tau_{|U}\colon U \hookrightarrow W_U = W\cap(U\times X)$ *is an isomorphism and* $\mathcal{N}' = \mathcal{N}_{|U}$ *is a line bundle.*

We proceed locally. We then write $U = \operatorname{Spec} A$, $W_U = \operatorname{Spec} B$ so that τ corresponds to a surjective ring morphism $B \to A \to 0$ and the projection $q_1 = \pi_{1|U}\colon W_U \to U$ to an immersion $A \hookrightarrow B$. Now $\mathcal{N}'$ is a B-module which is isomorphic to A as an A-module, $\mathcal{N}' \simeq e\cdot A$, because $q_{1*}(\mathcal{N}') = \pi_{1*}(\widetilde{\mathcal{M}})_{|U}$ is a line bundle. It follows that $\mathcal{N}'$ is also generated by e as a B-module. The kernel of $B \to \mathcal{N}' \simeq e\cdot B \to 0$ is the annihilator of $\mathcal{N}'$ and then it is zero by the very definition of W. It follows that $B \simeq A$ as an A-module and then the morphism $B \to A \to 0$ is an isomorphism. Hence, $W_U \simeq U$, q_1 is the identity map, and $\mathcal{N}' \simeq q_{1*}(\mathcal{N}')$ is a line bundle.

b4) $\tau\colon X \hookrightarrow W$ *is an isomorphism and* $\mathcal{N}$ *is a line bundle.*

Since $U \simeq W_U$, $\pi_{2*}\widetilde{\mathcal{M}}_{|U} \simeq \mathcal{N}_{|U} \simeq \pi_{1*}\widetilde{\mathcal{M}}_{|U}$, which is a line bundle on U. Then, the locally free sheaf $\pi_{2*}\widetilde{\mathcal{M}}$ has to be a line bundle. Then the same argument used in b3) proves the remaining statement. □

Corollary 1.23. *An object $\mathcal{K}^\bullet$ in $D^b_c(X\times Y)$ satisfying the conditions of Theorem 1.22 is strongly simple over X if and only if*

1. $\operatorname{Hom}^i_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_{x_1}}), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_{x_2}})) = 0$ *for any pair Z_{x_1} and Z_{x_2} of l.c.i. zero cycles (supported on x_1, x_2 respectively) unless $x_1 = x_2$ and $0 \le i \le \dim X$;*
2. $\operatorname{Hom}^0_{D(Y)}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)) = k$ *for every point $x \in X$.*

From Propositions 1.20 and 1.18 and Corollary 1.7, we obtain:

Corollary 1.24. *Let X and Y be projective integral Gorenstein schemes over an algebraically closed field of characteristic zero, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times Y)$ of finite* *projective dimension over both factors. The integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is an equivalence if and only if $\mathcal{K}^\bullet$ is strongly simple over both factors.*

Remark 1.25. Theorem 1.22 is false in positive characteristic even in the smooth case. Let X be a smooth projective scheme of dimension m over a field k of characteristic $p>0$, and $F\colon X\to X^{(p)}$ the *relative Frobenius morphism* [25, 3.1], which is topologically a homeomorphism. Let $\Gamma\hookrightarrow X\times X^{(p)}$ be the graph of F, whose associated integral functor is the direct image $F_*\colon D^b_c(X)\to D^b_c(X^{(p)})$. Since $F_*(\mathcal{O}_x)\simeq\mathcal{O}_{F(x)}$, one easily sees that Γ is strongly simple over X. However, $F_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_{X^{(p)}}$-module of rank p^m [25, 3.2], so that $\operatorname{Hom}^0_{D(X^{(p)})}(F_*(\mathcal{O}_X),\mathcal{O}_{F(x)})\simeq k^{p^m}$ whereas $\operatorname{Hom}^0_{D(X)}(\mathcal{O}_X,\mathcal{O}_x)\simeq k$; thus F_* is not fully faithful. △

1.7. **A criterion for equivalence.** The usual Bridgeland criterion [12, Thm. 5.1] that characterises when an integral functor over the derived category of a smooth variety is an equivalence (or a Fourier-Mukai functor) also works in the Gorenstein case. The original proof is based on the fact that if X is smooth, the skyscraper sheaves $\mathcal{O}_x$ form a spanning class for the derived category $D^b_c(X)$ [12]. This is also true for Gorenstein varieties. Moreover in this case there is a more natural spanning class (see [19] for a similar statement), that allows to give a similar criterion.

Lemma 1.26. *If X is a Gorenstein scheme, then the following sets are spanning classes for $D^b_c(X)$:*

(1) $\Omega_1=\{\mathcal{O}_x\}$ *for all closed points $x\in X$.*
(2) $\Omega_2=\{\mathcal{O}_{Z_x}\}$ *for all closed points $x\in X$ and all l.c.i. zero cycles Z_x supported on x.*

Proof. (1) Arguing as in [12, Lemma 2.2], one proves that if $\operatorname{Hom}^i(\mathcal{E}^\bullet,\mathcal{O}_x)=0$ for every i and every $x\in X$, then $\mathcal{E}^\bullet=0$. Suppose now that $\operatorname{Hom}^i(\mathcal{O}_x,\mathcal{E}^\bullet)=0$ for every i and every $x\in X$. By (1.6), $\operatorname{Hom}^i(\mathcal{O}_x,\mathcal{E}^\bullet)\simeq\operatorname{Hom}^i(\mathcal{E}^{\bullet\vee},\mathcal{O}_x^\vee)$ and since $\mathcal{O}_x^\vee\simeq\mathcal{O}_x[-m]$ where $m=\dim X$ because X is Gorenstein, we have that $\operatorname{Hom}^{i-m}(\mathcal{E}^{\bullet\vee},\mathcal{O}_x)=0$ for every i and every $x\in X$. Then $\mathcal{E}^{\bullet\vee}=0$ and from (1.5), one concludes that $\mathcal{E}^\bullet=0$.

(2) By Proposition 1.16 with $Y=\emptyset$, if $\operatorname{Hom}^i_{D(X)}(\mathcal{O}_{Z_x},\mathcal{E}^\bullet)=0$ for every i and every Z_x, then $\mathcal{E}^\bullet=0$. On the other hand, since $\mathcal{O}_{Z_x}$ is of finite homological dimension, Serre duality can be applied to get an isomorphism

$$\operatorname{Hom}^i(\mathcal{E}^\bullet,\mathcal{O}_{Z_x})^*\simeq\operatorname{Hom}^i(\mathcal{E}^\bullet,\mathcal{O}_{Z_x}\otimes\omega_X)^*\simeq\operatorname{Hom}^{m-i}(\mathcal{O}_{Z_x},\mathcal{E}^\bullet)$$

where $m=\dim X$. By the first part, if $\mathcal{E}^\bullet$ is a non-zero object in $D^b_c(X)$ the second member is non-zero for some i and we finish. □

Theorem 1.27. *Let X and Y be projective Gorenstein schemes over an algebraically closed field of characteristic zero. Assume also that X is integral and Y is connected. If $\mathcal{K}^\bullet$ is an object in $D^b_c(X\times Y)$ of finite projective dimension over both X and Y, then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X)\to D^b_c(Y)$ is an equivalence of categories if and only if one has*

(1) *$\mathcal{K}^\bullet$ is strongly simple over X.*
(2) *For every closed point $x\in X$, $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)\simeq\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)\otimes\omega_Y$.*

Proof. By Proposition 1.17, the functor $H=\Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}_{Y\to X}$ is a right adjoint to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ while $G=\Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}_{Y\to X}$ is a left adjoint to it. If $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is an equivalence, there is an isomorphism of functors $H \simeq G$ and then the left adjoints are also isomorphic, that is $\Phi^{\mathcal{K}^\bullet}_{X\to Y} \simeq \Phi^{\mathcal{K}^\bullet\otimes\pi_Y^*\omega_Y^{-1}\otimes\pi_X^*\omega_X}_{X\to Y}$. Applying this to $\mathcal{O}_x$ we get $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)\otimes\omega_Y \simeq \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x)$.

For the converse, notice first that the derived category $D^b(Y)$ is indecomposable because Y is connected [12, Ex. 3.2]. Then we have to prove that for any object $\mathcal{E}^\bullet$ in $D^b_c(Y)$ the condition $H(\mathcal{E}^\bullet)=0$ implies that $G(\mathcal{E}^\bullet)=0$ [12, Thm. 3.3]. Since for every object $\mathcal{E}^\bullet$ in $D^b_c(Y)$ one has a functorial isomorphism

$$G(\mathcal{E}^\bullet) \simeq H(\mathcal{E}^\bullet\otimes\omega_Y[n])\otimes\omega_X^{-1}[-m]\,, \tag{1.18}$$

it is enough to prove that $H(\mathcal{E}^\bullet\otimes\omega_Y[n])=0$. We have

$$\begin{aligned}\operatorname{Hom}^i(\mathcal{O}_x, H(\mathcal{E}^\bullet\otimes\omega_Y[n])) &\simeq \operatorname{Hom}^i(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \mathcal{E}^\bullet\otimes\omega_Y[n])\\ &\simeq \operatorname{Hom}^{n+i}(\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_x), \mathcal{E}^\bullet)\\ &\simeq \operatorname{Hom}^{n+i}(\mathcal{O}_x, H(\mathcal{E}^\bullet))=0\end{aligned}$$

and one concludes by Lemma 1.26. □

Using now the second part of Lemma 1.26, we prove analogously the following:

Theorem 1.28. *Let X and Y be projective Gorenstein schemes over an algebraically closed field of characteristic zero. Assume also that X is integral and Y is connected. If $\mathcal{K}^\bullet$ is an object in $D^b_c(X\times Y)$ of finite projective dimension over both X and Y, then the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X)\to D^b_c(Y)$ is an equivalence of categories if and only if one has*

1. *$\mathcal{K}^\bullet$ is strongly simple over X.*
2. *For every closed point $x\in X$ there is a l.c.i. cycle Z_x such that $\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x}) \simeq \Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{O}_{Z_x})\otimes\omega_Y$.*

Remark 1.29. The second condition in the above lemma can be also written in either the form $p_{2*}(\mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet)\simeq p_{2*}(\mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet)\otimes\omega_Y$ or the form $\mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet \simeq \mathbf{L}j^*_{Z_x}\mathcal{K}^\bullet\otimes p_2^*\omega_Y$, where $p_2\colon Z_x\times Y\to Y$ is the projection. △

1.8. **Geometric applications of Fourier-Mukai functors.** As in the smooth case, the existence of a Fourier-Mukai functor between the derived categories of two Gorenstein schemes has important geometrical consequences. In the following proposition, we list some of them.

Proposition 1.30. *Let X and Y be projective Gorenstein schemes and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times Y)$ of finite projective dimension over both X and Y. If the integral functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X)\to D^b_c(Y)$ is a Fourier-Mukai functor, the following statements hold:*

1. *The right and the left adjoints to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ are functorially isomorphic*
$$\Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_X^*\omega_X[m]}_{Y\to X} \simeq \Phi^{\mathcal{K}^{\bullet\vee}\otimes\pi_Y^*\omega_Y[n]}_{Y\to X}$$
and they are quasi-inverses to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$.
2. *X and Y have the same dimension, that is, $m=n$.*
3. *ω_X^r is trivial for an integer r if and only if ω_Y^r is trivial. Particularly, ω_X is trivial if and only if ω_Y is trivial. In this case, the functor $\Phi^{\mathcal{K}^{\bullet\vee}}_{Y\to X}$ is a quasi-inverse to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$.*

Proof. (1) Since $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ is an equivalence, its quasi-inverse is a right and a left adjoint. The statement follows from Proposition 1.17 using the uniqueness of the adjoints.

(2) Applying the above isomorphism to $\mathcal{O}_{Z_y}$ where Z_y is a l.c.i. zero cycle supported on y, one obtains $p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee})[n] \simeq p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee}) \otimes \omega_X[m]$ where $p_X \colon X \times Z_y \to X$ is the projection. Since the two functors are equivalences, these are non-zero objects in $D^b_c(X)$. Let q_0 be the minimum (resp. maximum) of the q's with $\mathcal{H}^q(p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee})) \neq 0$. Since $\mathcal{H}^{q_0}(p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee})) \simeq \mathcal{H}^{q_0+m-n}(p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee}))\otimes\omega_X$ one has $\mathcal{H}^{q_0+m-n}(p_{X*}(\mathbf{L}j^*_{Z_y}\mathcal{K}^{\bullet\vee}))\otimes\omega_X \neq 0$ which contradicts the minimality (resp. maximality) if $m-n<0$ (resp. >0). Thus, $n=m$.

(3) If we denote by H the right adjoint to $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$, thanks to (1) and (1.18) we have that $H(\mathcal{E}^\bullet)\otimes\omega_X^r \simeq H(\mathcal{E}^\bullet\otimes\omega_Y^r)$ for every $\mathcal{E}^\bullet\in D^b_c(Y)$ and every integer r. If $\omega_X^r\simeq\mathcal{O}_X$, taking $\mathcal{E}^\bullet=\mathcal{O}_Y$ we have $H(\mathcal{O}_Y)\simeq H(\omega_Y^r)$ and applying the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$ to this isomorphism we get $\omega_Y^r\simeq\mathcal{O}_Y$. The converse is similar. □

2. Relative Fourier-Mukai transforms for Gorenstein morphisms

2.1. **Generalities and base change properties.** Let S be a scheme and let $p\colon X\to S$ and $q\colon Y\to S$ be proper morphisms. We denote by π_X and π_Y the projections of the fibre product $X\times_S Y$ onto its factors and by $\rho = p\circ\pi_X = q\circ\pi_Y$ the projection of $X\times_S Y$ onto the base scheme S so that we have the following cartesian diagram

$$\begin{tikzcd} & X\times_S Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] \arrow[dd, "\rho"] & \\ X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Let $\mathcal{K}^\bullet$ be an object in $D^b(X\times_S Y)$. The relative integral functor defined by $\mathcal{K}^\bullet$ is the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^-(X)\to D^-(Y)$ given by

$$\Phi^{\mathcal{K}^\bullet}_{X\to Y}(\mathcal{F}^\bullet) = \mathbf{R}\pi_{Y*}(\mathbf{L}\pi_X^*\mathcal{F}^\bullet \overset{\mathbf{L}}{\otimes} \mathcal{K}^\bullet)\,.$$

We shall denote this functor by Φ from now on.

Let $s\in S$ be a closed point. Let us denote $X_s=p^{-1}(s)$, $Y_s=q^{-1}(s)$, and $\Phi_s\colon D^-(X_s)\to D^-(Y_s)$ the integral functor defined by $\mathcal{K}^\bullet{}_s=\mathbf{L}j_s^*\mathcal{K}^\bullet$, with $j_s\colon X_s\times Y_s\hookrightarrow X\times_S Y$ the natural embedding.

When the kernel $\mathcal{K}^\bullet\in D^b_c(X\times_S Y)$ is of finite homological dimension over X, the functor Φ is defined over the whole $D(X)$ and it maps $D^b_c(X)$ into $D^b_c(Y)$. If moreover $q\colon Y\to S$ is flat, then $\mathcal{K}^\bullet{}_s$ is of finite homological dimension over X_s for any $s\in S$.

If $p\colon X\to S$ and $q\colon Y\to S$ are flat morphisms, from the base-change formula we obtain that

$$\mathbf{L}j_s^*\Phi(\mathcal{F}^\bullet)\simeq\Phi_s(\mathbf{L}j_s^*\mathcal{F}^\bullet) \tag{2.1}$$

for every $\mathcal{F}^\bullet\in D(X)$, where $j_s\colon X_s\hookrightarrow X$ and $j_s\colon Y_s\hookrightarrow Y$ are the natural embeddings. In this situation, base change formula also gives that

$$j_{s*}\Phi_s(\mathcal{G}^\bullet)\simeq\Phi(j_{s*}\mathcal{G}^\bullet) \tag{2.2}$$

for every $\mathcal{G}^\bullet\in D(X_s)$.

Proposition 1.8 allows us to obtain the following result.

Lemma 2.1. *Let $p\colon X \to S$ and $q\colon Y \to S$ be locally projective Gorenstein morphisms, and let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times_S Y)$ of finite projective dimension over both X and Y. Then the functor*

$$H = \Phi_{Y\to X}^{\mathcal{K}^{\bullet\vee} \overset{\mathbf{L}}{\otimes} \pi_X^* \omega_{X/S}[m]}\colon D^b_c(Y) \to D^b_c(X)$$

is a right adjoint to the functor $\Phi^{\mathcal{K}^\bullet}_{X\to Y}$.

2.2. **Criteria for fully faithfulness and equivalence in the relative setting.** In this subsection we work over an algebraically closed field of characteristic zero.

In the relative situation the notion of strongly simple object is the following.

Definition 2.2. Let $p\colon X \to S$ and $q\colon Y \to S$ be proper Gorenstein morphisms. An object $\mathcal{K}^\bullet \in D^b_c(X \times_S Y)$ is *relatively strongly simple* over X if $\mathcal{K}^\bullet{}_s$ is bounded and strongly simple over X_s for every closed point $s \in S$. △

Lemma 2.3. *Let $Z \to S$ be a proper morphism and $\mathcal{E}^\bullet$ be an object of $D^b_c(Z)$ such that $\mathbf{L}j_s^*\mathcal{E}^\bullet = 0$ in $D^b_c(Z_s)$ for every closed point s in S, where $j_s\colon Z_s \hookrightarrow Z$ is the immersion of the fibre. Then $\mathcal{E}^\bullet = 0$.*

Proof. For every closed point s in S there is a spectral sequence $E_2^{-p,q} = L_p j_s^* \mathcal{H}^q(\mathcal{E}^\bullet)$ converging to $E_\infty^{p+q} = \mathcal{H}^{p+q}(\mathbf{L}j_s^*\mathcal{E}^\bullet) = 0$. Assume that $\mathcal{E}^\bullet \neq 0$ and let q_0 be the maximum of the integers q such that $\mathcal{H}^q(\mathcal{E}^\bullet) \neq 0$. If s is a point in the image of the support of $\mathcal{H}^{q_0}(\mathcal{E}^\bullet)$, one has that $j_s^*\mathcal{H}^{q_0}(\mathcal{E}^\bullet) \neq 0$ and every non-zero element in $E_2^{0,q_0} = j_s^*\mathcal{H}^{q_0}(\mathcal{E}^\bullet)$ survives to infinity. Then $E_\infty^{q_0} \neq 0$ and this is impossible. □

Theorem 2.4. *Let $p\colon X \to S$ and $q\colon Y \to S$ be locally projective Gorenstein morphisms. Let $\mathcal{K}^\bullet$ be an object in $D^b_c(X \times_S Y)$ of finite projective dimension over both X and Y. The relative integral functor $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$ is fully faithful (resp. an equivalence) if and only if $\Phi_s\colon D^b_c(X_s) \to D^b_c(Y_s)$ is fully faithful (resp. an equivalence) for every closed point $s \in S$.*

Proof. By Proposition 1.18, if Φ is fully faithful the unit morphism

$$\mathrm{Id} \to H \circ \Phi$$

is an isomorphism (where H is the right adjoint given at Lemma 2.1). Then, given a closed point $s \in S$ and $\mathcal{G}^\bullet \in D^b_c(X_s)$, one has an isomorphism $j_{s*}\mathcal{G}^\bullet \to (H \circ \Phi)(j_{s*}\mathcal{G}^\bullet)$. Since $(H \circ \Phi)(j_{s*}\mathcal{G}^\bullet) \simeq j_{s*}(H_s \circ \Phi_s)(\mathcal{G}^\bullet)$ by (2.2) and j_s is a closed immersion, the unit morphism $\mathcal{G}^\bullet \to (H_s \circ \Phi_s)(\mathcal{G}^\bullet)$ is an isomorphism; this proves that Φ_s is fully faithful.

Now assume that Φ_s is fully faithful for any closed point $s \in S$. Let us see that the unit morphism $\eta\colon \mathrm{Id} \to H \circ \Phi$ is an isomorphism. For each $\mathcal{F}^\bullet \in D^b_c(X)$we have an exact triangle

$$\mathcal{F}^\bullet \xrightarrow{\eta(\mathcal{F}^\bullet)} (H \circ \Phi)(\mathcal{F}^\bullet) \to \mathrm{Cone}(\eta(\mathcal{F}^\bullet)) \to \mathcal{F}^\bullet[1]\,.$$

Then, by (2.1), for every closed point s in S we have an exact triangle

$$\mathbf{L}j_s^*\mathcal{F}^\bullet \to (H_s \circ \Phi_s)(\mathbf{L}j_s^*\mathcal{F}^\bullet) \to \mathbf{L}j_s^*\,\mathrm{Cone}(\eta(\mathcal{F}^\bullet)) \to \mathbf{L}j_s^*\mathcal{F}^\bullet[1]\,.$$

so that $\mathbf{L}j_s^*[\mathrm{Cone}(\eta(\mathcal{F}^\bullet))] \simeq \mathrm{Cone}(\eta_s(\mathbf{L}j_s^*\mathcal{F}^\bullet)) \simeq 0$ because $\eta_s\colon \mathrm{Id} \to H_s \circ \Phi_s$. We finish by Lemma 2.3.

A similar argument gives the statement about equivalence. □

As a corollary of the previous theorem and Theorem 1.22, we obtain the following result.

Theorem 2.5. *Let $p\colon X \to S$ and $q\colon Y \to S$ be locally projective Gorenstein morphisms with integral fibres. Let $\mathcal{K}^\bullet$ be an object in $D^b_c(X\times_S Y)$ of finite projective dimension over each factor. The kernel $\mathcal{K}^\bullet$ is relatively strongly simple over X (resp. over X and Y) if and only if the functor $\Phi = \Phi^{\mathcal{K}^\bullet}_{X\to Y}\colon D^b_c(X) \to D^b_c(Y)$ is fully faithful (resp. an equivalence).*

2.3. **Application to Weierstrass elliptic fibrations.** In this subsection we work over an algebraically closed field of characteristic zero.

Let $p\colon X \to S$ be a relatively integral elliptic fibration, that is, a proper flat morphism whose fibres are integral Gorenstein curves with arithmetic genus 1. Generic fibres of p are smooth elliptic curves, and the degenerated fibers are rational curves with one node or one cusp. If $\hat{p}\colon \hat{X} \to S$ denotes the dual elliptic fibration, defined as the relative moduli space of torsion free rank 1 sheaves of relative degree 0, it is known that for every closed point $s \in S$ there is an isomorphism $\hat{X}_s \simeq X_s$ between the fibers of both fibrations. If we assume that the original fibration $p\colon X \to S$ has a section $\sigma\colon S \hookrightarrow X$ taking values in the smooth locus of p, then p and $\hat{p}$ are globally isomorphic. Let us identify from now on X and $\hat{X}$ and consider the commutative diagram

$$\begin{tikzcd} & X\times_S X \arrow[dl, "\pi_1"'] \arrow[dd, "\rho"] \arrow[dr, "\pi_2"] & \\ X \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

The relative Poincaré sheaf is

$$\mathcal{P} = \mathcal{I}_\Delta \otimes \pi_1^*\mathcal{O}_X(H) \otimes \pi_2^*\mathcal{O}_X(H) \otimes \rho^*\omega^{-1}\,,$$

where $H = \sigma(S)$ is the image of the section and $\omega = R^1p_*\mathcal{O}_X \simeq (p_*\omega_{X/S})^{-1}$.

Relatively integral elliptic fibrations have a Weierstrass form [30, Lemma II.4.3]: The line bundle $\mathcal{O}_X(3H)$ is relatively very ample and if $\mathcal{E} = p_*\mathcal{O}_X(3H) \simeq \mathcal{O}_S \oplus \omega^{\otimes 2} \oplus \omega^{\otimes 3}$ and $\bar{p}\colon \mathbb{P}(\mathcal{E}^*) = \operatorname{Proj}(S^\bullet(\mathcal{E})) \to S$ is the associated projective bundle, there is a closed immersion $j\colon X \hookrightarrow \mathbb{P}(\mathcal{E}^*)$ of S-schemes such that $j^*\mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1) = \mathcal{O}_X(3H)$. In particular, p is a projective morphism.

Lemma 2.6. *The relative Poincaré sheaf $\mathcal{P}$ is of finite projective dimension and relatively strongly simple over both factors.*

Proof. By the symmetry of the expression of $\mathcal{P}$ it is enough to prove that $\mathcal{P}$ is of finite projective dimension and strongly simple over the first factor. For the first claim, it is enough to prove that $\mathcal{I}_\Delta$ has finite projective dimension. Let us consider the exact sequence

$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{X\times_S X} \to \delta_*\mathcal{O}_X \to 0$$

where $\delta\colon X \hookrightarrow X\times_S X$ is the diagonal morphism. It suffices to see that $\delta_*\mathcal{O}_X$ has finite projective dimension. We have to prove that for any $\mathcal{N}^\bullet \in D^b(X)$, the complex $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S X}}(\delta_*\mathcal{O}_X, \pi_1^!\mathcal{N}^\bullet)$ is bounded. This is a complex supported at the diagonal, so that it suffices to see that $\mathbf{R}\pi_{1*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S X}}(\delta_*\mathcal{O}_X, \pi_1^!\mathcal{N}^\bullet)$ is bounded. This follows from the following formulas.

$$\mathbf{R}\pi_{1*}\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_{X\times_S X}}(\delta_*\mathcal{O}_X, \pi_1^!\mathcal{N}^\bullet) \simeq \mathbf{R}\mathcal{H}om^\bullet_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{N}^\bullet) \simeq \mathcal{N}^\bullet\,.$$

Let us prove that $\mathcal{P}$ is strongly simple over the first factor. Fix a closed point $s \in S$ and consider two different points x_1 and x_2 in the fiber X_s. If both are non-singular, then

$$\operatorname{Hom}^i_{D(X_s)}(\Phi^{\mathcal{P}_s}_{X_s \to X_s}(\mathcal{O}_{x_1}), \Phi^{\mathcal{P}_s}_{X_s \to X_s}(\mathcal{O}_{x_2})) \simeq H^i(X_s, \mathcal{O}_{X_s}(x_1 - x_2)) = 0 \quad \text{for every } i$$

because $\mathcal{O}_{X_s}(x_2 - x_1)$ is a non-trivial line bundle of degree zero. Assume that x_2 is singular and x_1 is not, the other case being similar. Let Z_{x_2} be a l.c.i zero cycle supported on x_2. We have

$$\operatorname{Hom}^i_{D(X_s)}(\Phi^{\mathcal{P}_s}_{X_s \to X_s}(\mathcal{O}_{x_1}), \Phi^{\mathcal{P}_s}_{X_s \to X_s}(\mathcal{O}_{Z_{x_2}})) = H^i(X_s, \mathcal{J}_{Z_{x_2}} \otimes \mathcal{O}_{X_s}(x_1))$$

where $\mathcal{J}_{Z_{x_2}}$ denotes the direct image by the finite morphism $Z_{x_2} \times X_s \to X_s$ of the ideal sheaf of the graph $Z_{x_2} \hookrightarrow Z_{x_2} \times X_s$ of the immersion $Z_{x_2} \hookrightarrow X_s$.

Let us consider the exact sequences of $\mathcal{O}_{X_s}$-modules

$$0 \to \mathcal{J}_{Z_{x_2}} \to \mathcal{O}_{Z_{x_2}} \otimes_k \mathcal{O}_{X_s} \to \mathcal{O}_{Z_{x_2}} \to 0$$
$$0 \to \mathcal{J}_{Z_{x_2}}(x_1) \to \mathcal{O}_{Z_{x_2}} \otimes_k \mathcal{O}_{X_s}(x_1) \to \mathcal{O}_{Z_{x_2}} \to 0$$

Since $H^0(X_s, \mathcal{O}_{X_s}) \simeq k$ the morphism $\mathcal{O}_{Z_{x_2}} \otimes_k H^0(X_s, \mathcal{O}_{X_s}) \to \mathcal{O}_{Z_{x_2}}$ of global sections induced by the first sequence is an isomorphism. Moreover, $H^0(X_s, \mathcal{O}_{X_s}) \simeq H^0(X_s, \mathcal{O}_{X_s}(x_1))$ and then we also have an isomorphism of global sections $\mathcal{O}_{Z_{x_2}} \otimes_k H^0(X_s, \mathcal{O}_{X_s}) \xrightarrow{\sim} \mathcal{O}_{Z_{x_2}} \otimes_k H^0(X_s, \mathcal{O}_{X_s}(x_1))$. Thus, $\mathcal{O}_{Z_{x_2}} \otimes_k H^0(X_s, \mathcal{O}_{X_s}(x_1)) \xrightarrow{\sim} \mathcal{O}_{Z_{x_2}}$ so that $H^i(X_s, \mathcal{J}_{Z_{x_2}}(x_1)) = 0$ for $i = 0, 1$.

Finally, since $\operatorname{Hom}^0_{D(X_s)}(\mathcal{P}_x, \mathcal{P}_x) = k$ for every point $x \in X_s$, we conclude that $\mathcal{P}_s$ is strongly simple over X_s. □

Now by Corollary 1.24 we have

Proposition 2.7. *The relative integral functor*

$$\Phi^{\mathcal{P}}_{X \to X} \colon D^b_c(X) \to D^b_c(X)$$

defined by the Poincaré sheaf is an equivalence of categories.

Notice that the proof of this result does not use spanning classes.

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