A CHARACTERIZATION OF JACOBIANS BY THE EXISTENCE OF PICARD BUNDLES

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Abstract. Based on Matsusaka-Ran criterion we give a criterion to characterize when a principal polarized abelian variety is a Jacobian by the existence of Picard bundles.

1. Introduction

The problem of determining when an abelian variety is a Jacobian has been studied by many people along the years. Generalizing the classical criterion of Matsusaka, Ran gives in [12] a characterization of Jacobians by the existence of curves with minimal cohomology class in the abelian variety. This criterion is nowadays known as Matsusaka-Ran criterion.

More recently, G. Pareschi and M. Popa use the theory of Fourier-Mukai transforms as a useful tool in the study of the existence of subvarieties of a principal polarized abelian variety with minimal cohomology class. In this sense, they prove in [11] a cohomology criterion which claims that if \((A, \Theta)\) is an indecomposable principal polarized abelian variety and \(C\) is a geometrically non-degenerated reduced equidimensional curve in \(A\) such that the ideal sheaf \(\mathcal{I}_C(\Theta)\) is a GV-sheaf, then \((A, \Theta)\) is the Jacobian of \(C\) and \(C\) has minimal cohomology class. In the same paper they conjecture that if the Index Theorem with index 0 holds for \(\mathcal{I}_C(2\Theta)\), with respect to the Fourier-Mukai transform defined by the Poincaré bundle, then \(C\) has minimal cohomology class. Consequently, using Matsusaka-Ran criterion, this would give a different cohomological criterion for detecting Jacobians.

In this paper, we show the existence of such curves of minimal class using Picard bundles. Our main result is the following:

Theorem. Let \((A, \Theta)\) be an indecomposable p.p.a.v. of dimension \(g\). If there exists a WIT\(_g\) sheaf \(\mathcal{F}\) on \(A\) with Chern classes \(c_i(\mathcal{F}) = (-1)^i \frac{\theta^i}{i!}\), then \((A, \Theta)\) is a Jacobian and \(\mathcal{F}\) is a Picard bundle.

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Picard bundles were introduced by Schwarzenberger in [13] and have been used by many authors in the study of the geometry of abelian varieties (c.f [5], [6]). Mukai studied Picard bundles by means of Fourier-Mukai transforms in (c.f. [7]). We generalize its definition of Picard bundles and study some properties of these sheaves in Proposition 3.3.

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2. Fourier-Mukai transforms for abelian varieties

In this section, we recall some of the terminology of Fourier-Mukai functors and the results that we will need in the rest of the paper.

Let $A$ be an abelian variety of dimension $g$ and $\hat A = \text{Pic}^0(A)$ the dual abelian variety. $\hat A$ represents the Picard functor, so there exists a universal line bundle $\mathcal{P}$ on $A \times \hat A$, called the Poincaré bundle. Thus, if $\alpha \in \hat A$ corresponds to the line bundle $L$ on $A$, one has

$$\mathcal{P}_\alpha := \mathcal{P}|_{A \times \{\alpha\}} \simeq L$$

Analogously, if $x \in A$, we denote $\mathcal{P}_x := \mathcal{P}|_{\{x\} \times \hat A}$. The Poincaré bundle can be normalised by the condition that $\mathcal{P}|_{\{0\} \times \hat A}$ is the trivial line bundle on $\hat A$.

Denote $\pi_A: A \times \hat A \to A$ and $\pi_{\hat A}: A \times \hat A \to \hat A$ the natural projections. The following result was proved by Mukai.

**Theorem 2.1.** [7] The integral functor $\Phi: D^b(A) \longrightarrow D^b(\hat A)$ defined by $\mathcal{P}$

$$\Phi(E^\bullet) := R\pi_{\hat A\ast}(\pi_A^\ast(E^\bullet) \otimes \mathcal{P})$$

is a Fourier-Mukai transform, that is, an equivalence of categories. Its quasi-inverse is the integral functor defined by $\mathcal{P}^*[g]$ where $\mathcal{P}^*$ denotes the dual of $\mathcal{P}$.

Let us denote by $\tilde{\Phi}: D^b(\hat A) \longrightarrow D^b(A)$ the integral functor defined by $\mathcal{P}^*$. A straightforward consequence are the following isomorphisms:

$$\Phi \circ \tilde{\Phi} \simeq Id_{D^b(\hat A)}[-g] \quad \text{and} \quad \tilde{\Phi} \circ \Phi \simeq Id_{D^b(A)}[-g]$$

**Remark 2.2.** In his original paper [7], Mukai consider instead of $\tilde{\Phi}$ the integral functor $\mathcal{I}: D^b(\hat A) \longrightarrow D^b(A)$ defined by $\mathcal{P}$. The relation between both functor is given by

$$\mathcal{I} \simeq \tilde{\Phi} \circ (-1_{\hat A})^*.$$
For simplicity, we shall write $\Phi^j(\mathcal{F}^\bullet) = \mathcal{H}^j(\Phi(\mathcal{F}^\bullet))$ to denote the $j$-th cohomology sheaf of the complex $\Phi(\mathcal{F}^\bullet)$ and the same for the functor $\hat{\Phi}$.

**Definition 2.3.** A coherent sheaf $\mathcal{F}$ on $A$ is WIT $i$ with respect to $\Phi$ (WIT$_i$-$\Phi$ in short) if $\Phi^j(\mathcal{F}) = 0$ for all $j \neq i$, or equivalently if there exists a sheaf $\hat{\mathcal{F}}$ on $\hat{A}$ such that $\Phi(\mathcal{F}) \simeq \hat{\mathcal{F}}[-i]$. The sheaf $\hat{\mathcal{F}}$ is called the Fourier-Mukai transform of $\mathcal{F}$ with respect to $\Phi$. When in addition $\hat{\mathcal{F}}$ is locally free, we say that $\mathcal{F}$ is IT $i$ with respect to $\Phi$.

We have analogous definitions of WIT and IT with respect the dual Fourier-Mukai functor $\hat{\Phi}$.

The following proposition collect some easy properties about this special kind of sheaves.

**Proposition 2.4.** Let $\mathcal{F}$ be a coherent sheaf on $A$. Then, the following holds:

1. $\mathcal{F}$ is IT$_i$-$\Phi$ if and only if $H^j(A, \mathcal{F} \otimes \mathcal{P}_\alpha) = 0$ for all $j \neq i$ and for all $\alpha \in \hat{A}$.
2. $\mathcal{F}$ is IT$_0$-$\Phi$ if and only if $\mathcal{F}$ is WIT$_0$-$\Phi$.
3. If $\mathcal{F}$ is WIT$_i$-$\Phi$, then $\hat{\mathcal{F}}$ is WIT$_{-i}$-$\hat{\Phi}$ and $\hat{\mathcal{F}} \simeq \mathcal{F}$.
4. If $\mathcal{F}$ is WIT$_g$-$\Phi$, then it is a locally free sheaf.
5. If $\mathcal{F}$ is an ample line bundle, then it is IT$_0$-$\Phi$.

**Proof.** Since $\mathcal{P}$ is a locally free sheaf, 1) and 2) follow straightforwardly from Grauert’s cohomology base change theorem. Part 3) follows from the isomorphism $\hat{\Phi} \circ \Phi \simeq [-g]$ and part 4) is a consequence of 3) and the definition of IT. Part 5) is a direct consequence of the vanishing results for ample line bundles (see for instance [9]) and 1). □

The relationship between the Chern characters of a WIT sheaf and those of its Fourier-Mukai transform is given by the following formula.

**Mukai formula** ([8, Corollary 1.18]): If $\mathcal{E}$ is a WIT$_j$-$\Phi$ sheaf, then

$$\text{ch}_i(\hat{\mathcal{E}}) = (-1)^{i+j}PD(\text{ch}_{g-i}(\mathcal{E}))$$

where $PD$ denotes the Poincaré duality isomorphism.

**Definition 2.5.** A **principally polarized abelian variety** (p.p.a.v. in short) is an abelian variety $A$ endowed with an ample line bundle $\mathcal{L}$ such that $\chi(\mathcal{L}) = 1$.

**Remark 2.6.** If $A$ is an abelian variety and we denote by $\tau_x$ the translation morphism by a point $x \in X$, recall that $A$ is a p.p.a.v if and
only if there exists an ample line bundle $L$ on $A$ such that the morphism $\phi_L: A \to \hat{A}$ defined as $\phi_L(x) = \tau_x^L \otimes L^{-1}$ is an isomorphism. Moreover, by Proposition 2.4, the polarization $L$ is $\Pi_0$, and it satisfies
$$\phi_L^*(\hat{L}) \simeq L^{-1}.$$  

3. Picard Bundles on Jacobians

Let $C$ be a smooth curve of genus $g \geq 2$ and consider $J_d(C)$ the Picard scheme parameterizing line bundles of degree $d$ on $C$. This is a fine moduli space. Denote by $P_d$ the universal Poincaré line bundle on the direct product $C \times J_d(C)$ and $p: C \times J_d(C) \to C$ and $q: C \times J_d(C) \to J_d(C)$ the projections. Fixing a point $x_0 \in C$, it is normalized by imposing $P_d\mid_{\{x_0\} \times J_d} \simeq \mathcal{O}_{J_d}$. The higher direct images $R^i q_*(P_d)$ of $P_d$ on $J_d(C)$ are known in the literature as degree $d$ Picard sheaves.

Let us show how Picard sheaves can be seen in terms of the Fourier-Mukai transform. Let $J_0(C) = J(C)$ be the Jacobian of $C$, that is, the abelian variety that parametrizes the line bundles on $C$ with degree zero. The Riemann theta divisor $\Theta$ is a natural polarization on $J(C)$ that defines a structure of principally polarized abelian variety of dimension $g$ on $J(C)$. By Remark 2.2, this gives a natural identification between $J(C)$ and its dual abelian variety $\hat{J(C)}$. With this identification, if we denote by $a: C \hookrightarrow J(C)$ the Abel morphism, the normalized Poincaré bundle $P_0$ is precisely the restriction $(a \times 1)^* \mathcal{P}$ of the universal line bundle $\mathcal{P}$ on $J(C) \times J(C)$. On the other hand, the line bundle $P_d \otimes p^* \mathcal{O}_C(-dx_0)$ defines an isomorphism $\lambda_d: J_d(C) \simeq J(C)$ and by normalization of the Poincaré sheaves that we have considered, one has isomorphisms
$$P_d \simeq (1 \times \lambda_d)^* P_0 \otimes p^* \mathcal{O}_C(dx_0).$$

Using the base-change and the projection formulas, the Picard sheaf $R^i q_*(P_d)$ is
$$R^i q_*(P_d) \simeq \lambda_d^* \Phi^i(a_* \mathcal{O}_C(dx_0)).$$

Considering that all Jacobians are already identified and although the last isomorphism is no longer true for an arbitrary line bundle $L$ of degree $d$, the above discussion justifies following notion of Picard sheaves.

**Definition 3.1.** Let $L$ be a line bundle on $C$ of degree $d$. The sheaves $\Phi^i(a_* L)$ are called the degree $d$ Picard sheaves.
Remark 3.2. The use of Fourier-Mukai transforms in the study of Picard bundles is originally due to Mukai [7]. In this paper, he just considers the Picard sheaf $F_d = \Phi^1(a_*\mathcal{O}_C(dx_0))$ corresponding to the line bundle $\mathcal{O}_C(dx_0)$.

Let $\Delta: D^b(J(C)) \to D^b(J(C))$ be the dualizing functor defined by $\Delta(F^\bullet) = R\text{Hom}(F^\bullet, \mathcal{O}_{J(C)})[g]$. From Grothendieck duality, there is an isomorphism of functors

$$\Delta \circ \Phi \simeq ((-1)^* \circ \Phi \circ \Delta)[g].$$

Taking into account that if $L$ is a line bundle on $C$, its derived dual is $\Delta(a_* L) \simeq a_*(L^* \otimes \omega_C)[1]$ where $\omega_C$ is dualizing sheaf of $C$, one has an isomorphism

$$R\text{Hom}((\Phi(a_* L), \mathcal{O}_{J(C)})) \simeq (-1)^* \Phi(a_*(L^* \otimes \omega_C))[1].$$

which, in some cases, gives a duality relation between degree $d$ and degree $2g - 2 - d$ Picard bundles.

Applying the theory of Fourier-Mukai transforms, we get some properties of Picard sheaves that we summarize in the following proposition. Compare it with Theorem 4.2 and Proposition 4.3 in [7] where Mukai studies its $F_d = \Phi^1(a_*\mathcal{O}_C(dx_0))$.

Proposition 3.3. The following holds:

1. $\Phi^i(a_* L)$ are zero for $i \neq 0, 1$.
2. For $d < 0$, $\Phi^0(a_* L) = 0$ and $\Phi^1(a_* L)$ is simple locally free of rank $g - d - 1$. There is an isomorphism

$$\Phi^1(a_* L) \simeq (-1)^* R\text{Hom}(\Phi^0(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}).$$

3. For $0 \leq d < g - 1$, $\Phi^1(a_* L)$ is supported on $J(C)$.
4. For $g - 1 \leq d < 2g - 1$, $\Phi^0(a_* L)$ and $\Phi^1(a_* L)$ are both non-zero.
5. For $d \geq 2g - 1$, $\Phi^0(a_* L)$ is a simple locally free sheaf of rank $d + 1 - g$ and $\Phi^1(a_* L) = 0$. There is an isomorphism

$$\Phi^0(a_* L) \simeq (-1)^* R\text{Hom}(\Phi^1(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}).$$

Proof. The first part is because the support of $L$ has dimension 1. If $d < 0$, from Grauert’s base-change theorem $\Phi^0(a_* L) = 0$ and $\Phi^1(a_* L)$ is a locally free sheaf of rank $g - d - 1$. Since $\Phi$ is an equivalence of categories and $L$ is simple, $\Phi(a_* L)[1] = \Phi^1(a_* L)$ is simple as well. Analogously, one gets the corresponding statements in 5). In both cases, the duality relation between degree $d$ and degree $2g - 2 - d$ Picard bundles is a consequence of the equation (3). Let us show 3). By cohomology base-change, one has that $\Phi^1(a_* L)_\alpha \simeq H^1(C, L \otimes \mathcal{P}_\alpha)$ which, being $L \otimes \mathcal{P}_\alpha$ of degree $d$, is non-zero for every $\alpha \in J(C)$ because
\(\chi(L \otimes P_\alpha) < 0\) by Riemman-Roch theorem. Now we prove 4). By the equation (3), there is an isomorphism
\[
\Phi^0(a_*L) \simeq (-1)^*\text{Hom}^{-1}(\Phi(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}).
\]
From 3), the sheaf \(\Phi^1(a_*(L^* \otimes \omega_C))\) is supported on \(J(C)\), and then \(\text{Hom}^0(\Phi^1(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)})\) is non-zero. From the spectral sequence for local homomorphisms
\[
E^{p,q}_2 = \text{Hom}^p(\Phi^{-q}(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}) \Rightarrow E^{p+q}_\infty = \text{Hom}^{p+q}(\Phi(a_*(L^* \otimes \omega_C)), \mathcal{O}_{J(C)}),
\]
and the above isomorphism we easily conclude that \(\Phi^0(a_*L)\) is non-zero. Finally, to show that \(\Phi^1(a_*L)\) is also non-zero, consider the line bundle \(L \otimes \mathcal{O}_C(-dx_0)\) where \(x_0\) is the point of \(C\) that we have fixed to normalize the Poincaré bundle. This is a line bundle of degree zero and then \(L \otimes \mathcal{O}_C(-dx_0) \simeq P_\alpha\) for some \(\alpha \in J(C)\). By Theorem 4.2 in [7], there is a point \(\kappa \in J(C)\) in the support of the sheaf \(\Phi^1(\mathcal{O}_C(dx_0))\).
Using again cohomology base-change, one obtains that
\[
H^1(C, \mathcal{O}_C(dx_0) \otimes P_\kappa) \simeq H^1(C, L \otimes P_{\kappa - \alpha})
\]
is non-zero. Hence, the point \(\kappa - \alpha\) belongs to the support of \(\Phi^1(a_*L)\) and we have the result. \(\square\)

Consider now the line bundle \(L = \mathcal{O}_C(2\Theta) \in J_{2g}(C)\). By the last proposition, the Picard sheaf \(\Phi^0(a_*\mathcal{O}_C(2\Theta)) = a_*\mathcal{O}_C(2\Theta)\) is a vector bundle on \(J(C)\).

The aim of this section is to show some of the properties that this Picard bundle has. Namely, if \(\mathcal{F} = a_*\mathcal{O}_C(2\Theta)\), then
1. \(\mathcal{F}\) is a quotient of \(\mathcal{O}_{J(C)}(2\Theta)\).
2. \(\mathcal{F}\) is WIT\(g\)-\(\Phi\).
3. \(c_i(\mathcal{F}) = (-1)^i\frac{\theta^i}{i!}\).
4. \(\mathcal{F}\) is simple.

Let us consider the exact sequence
\[
0 \longrightarrow \mathcal{I}_C(2\Theta) \longrightarrow \mathcal{O}_{J(C)}(2\Theta) \longrightarrow a_*\mathcal{O}_C(2\Theta) \longrightarrow 0 \quad (4)
\]
Since \(\Theta\) is an ample divisor, the line bundle \(\mathcal{O}_{J(C)}(2\Theta) \otimes P_\alpha\) is also ample for any \(\alpha \in J(C)\). Thus, by applying the vanishing results for ample line bundles (see for instance [9]), we get that
\[
H^i(J(C), \mathcal{O}_{J(C)}(2\Theta) \otimes P_\alpha) = 0 \text{ for all } i > 0 \text{ and all } \alpha \in J(C)
\]
and, by Proposition 2.4, one concludes that \(\mathcal{O}_{J(C)}(2\Theta)\) is IT\(g\)-\(\Phi\). 

On the other hand, Theorem 4.1 in [10] proves that the sheaf $\mathcal{I}_C(2\Theta)$ is also $\text{IT}_0-\Phi$. In this particular situation, this can be proved directly as follows. If $g = 2$, $C$ is an effective divisor on $J(C)$ and $[C] = \Theta$. Then $\mathcal{I}_C(2\Theta) \simeq \mathcal{O}_{J(C)}(\Theta)$ and we conclude as above. Suppose now that $g > 2$. It suffices to show that $H^1(J(C), \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha) = 0$ for every $\alpha \in J(C)$. Since

$$H^1(J(C), \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha) \simeq \text{Ext}^1(\mathcal{O}_{J(C)}, \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha)$$

it is enough to prove that any extension

$$0 \rightarrow \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{J(C)} \rightarrow 0 \quad (5)$$

is trivial. Since $C$ and $J(C)$ are smooth, the Abel morphism is a regular embedding and then a standard local computation using the Koszul complex yields $\text{Ext}^i(\mathcal{O}_C, \mathcal{O}_{J(C)}) = 0$ for all $i \neq g - 1$. Thus $\mathcal{I}_C \simeq \mathcal{O}_{J(C)}$. Dualizing twice the extension (5), one has the commutative diagram

$$
\begin{array}{c}
0 \longrightarrow \mathcal{I}_C(2\Theta) \otimes \mathcal{P}_\alpha \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{J(C)} \longrightarrow 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 \longrightarrow \mathcal{O}_{J(C)}(2\Theta) \otimes \mathcal{P}_\alpha \longrightarrow \mathcal{E}^{**} \longrightarrow \mathcal{O}_{J(C)} \longrightarrow 0
\end{array}
$$

and, using the snake lemma, an exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{E}^{**} \rightarrow a_*\mathcal{O}_C(2\Theta) \otimes \mathcal{P}_\alpha \rightarrow 0.$$

Since $\mathcal{O}_{J(C)}(2\Theta)$ is $\text{IT}_0$, $\mathcal{E}^{**}$ is the trivial extension and so is $\mathcal{E}$ which proves the statement in this case.

By applying the Fourier-Mukai transform $\Phi$ to exact sequence (4) we then obtain

$$0 \longrightarrow \hat{\mathcal{I}}_C(2\Theta) \longrightarrow \hat{\mathcal{O}}_{J(C)}(2\Theta) \longrightarrow a_*\hat{\mathcal{O}}_C(2\Theta) \longrightarrow 0 \quad (6)$$

The next step is to compute the Chern classes of $a_*\hat{\mathcal{O}}_C(2\Theta)$. In fact, this is an old computation originally due to Schwarzenberger [13]. Here we are going to deduce it in an easy way using the Fourier-Mukai transform.

The Chern characters of $a_*\mathcal{O}_C(2\Theta)$ can be obtained using Grothendieck Riemann Roch theorem for the Abel morphism $C \overset{\alpha}{\rightarrow} J(C)$

$$\text{ch}(a_*\mathcal{O}_C(2\Theta)) \cdot \text{td}(J(C)) = a_*\text{ch}(\mathcal{O}_C(2\Theta|_C)) \cdot \text{td}(C)$$

Remember that $C$ has minimal cohomology class, that is,$$
[C] = \frac{\Theta^{g-1}}{(g-1)!}.$$
and the Todd class of $J(C)$ is trivial because it is an abelian variety. Then, one gets

$$\text{ch}_j(a_* \mathcal{O}_C(2\Theta)) = \begin{cases} 0 & j < g - 1 \\ \frac{g^{g-1}}{(g-1)!} & j = g - 1 \\ g+1 & j = g \end{cases} \quad (7)$$

By applying Mukai formula (1) we may compute the Chern characters of Fourier-Mukai transform of $a_* \mathcal{O}_C(2\Theta)$. Thus we get

$$\text{ch}_j(\hat{a}_* \mathcal{O}_C(2\Theta)) = \begin{cases} g+1 & j = 0 \\ -\Theta & j = 1 \\ 0 & j > 1 \end{cases} \quad (8)$$

Lemma 3.4. Let $\mathcal{E}$ be a vector bundle on a smooth variety $X$. The following are equivalent:

a) $\text{ch}_j(\mathcal{E}) = 0$ for all $j \geq 2$.

b) $c_i(\mathcal{E}) = \frac{c_1(\mathcal{E})^i}{i!}$ for all $i$.

Proof. By definition the total Chern class of $\mathcal{E}$ is

$$c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \cdots + c_r(\mathcal{E})t^r = \prod_{i=1}^r (1 + a_it)$$

and the Chern character is

$$\text{ch}(\mathcal{E}) = \sum_{i=1}^r e^{a_it} = \sum \left(1 + a_it + \frac{(a_it)^2}{2!} + \cdots \right) = r + \text{ch}_1(\mathcal{E})t + \cdots$$

So, if $c_i(\mathcal{E}) = \frac{c_1(\mathcal{E})^i}{i!}$ we get that $c_t(\mathcal{E}) = e^{c_1(\mathcal{E})t}$. Then

$$c_1(\mathcal{E})t = \log(c_t(\mathcal{E})) = \sum \log(1 + a_it) = \sum (a_it - \frac{(a_it)^2}{2} + \frac{(a_it)^3}{3} + \cdots) = \text{ch}_1(\mathcal{E})t - \text{ch}_2(\mathcal{E})t^2 + 2 \text{ch}_3(\mathcal{E})t^3 - 3 \text{ch}_4(\mathcal{E})t^4 + \cdots$$

Hence $\text{ch}_j(\mathcal{E}) = 0$ for any $j \geq 2$.

Conversely, if we assume that $\text{ch}_j(\mathcal{E}) = 0$ for all $j \geq 2$, then one obtains

$$\log(\prod_{i=1}^r (1 + a_it)) = c_1(\mathcal{E})t$$

which implies the condition b).
Using Equation (8) and Lemma 3.4, one obtains that
\[ c_i(a_\ast \mathcal{O}_C(2\Theta)) = (-1)^i \frac{\theta^i}{i!} \]  
Finally, notice that the sheaf \( \mathcal{F} \) is simple because
\[ \text{Hom}_{D(J(C))}(a_\ast \mathcal{O}_C(2\Theta), a_\ast \mathcal{O}_C(2\Theta)) \simeq \text{Hom}_C(\mathcal{O}_C(2\Theta), \mathcal{O}_C(2\Theta)), \]
in particular, it is an indecomposable sheaf.

We summarize everything up in the following:

**Proposition 3.5.** Let \( C \) be a smooth curve of genus \( g \geq 2 \), and \((J(C), \Theta)\) its Jacobian. Then, there exits a Picard bundle \( \mathcal{F} \) on the abelian variety \( J(C) \) such that the following holds:
1. \( \mathcal{F} \) is WIT\(_g\)-\( \hat{\Phi} \).
2. \( c_i(\mathcal{F}) = (-1)^i \frac{\theta^i}{i!} \).
3. \( \mathcal{F} \) is a quotient of \( \mathcal{O}_{J(C)}(-2\Theta) \).
4. \( \mathcal{F} \) is simple.

4. A Characterization of Jacobians via Picard Bundles

In this section we shall use Matsusaka-Ran criterion to prove that the existence of a sheaf satisfying the two first properties of the Picard bundle in Proposition 3.5 is enough to ensure that an indecomposable p.p.v.a. is the Jacobian of a curve.

Let us introduce some necessary notions and recall Matsusaka-Ran criterion.

**Definition 4.1.** A curve \( C \) on an abelian variety \( A \) is said to generate \( A \), if \( A \) is the smallest abelian variety containing \( C \). More generally, an effective algebraic 1-cycle \( \sum n_i C_i \) on \( A \), with \( n_i > 0 \) for all \( n_i \), generates \( A \), if the union of the curves \( C_i \) generates \( A \).

The following result is the criterion of Matsusaka-Ran criterion \([12]\). This precise statement can be found in \([1]\).

**Theorem 4.2** (Matsusaka-Ran criterion). Suppose \((A, \Theta)\) is a polarized abelian variety of dimension \( g \) and \( C = \sum_{i=1}^{r} n_i C_i \) is an effective 1-cycle generating \( A \) with \([C] \cdot \Theta = g \). Then \( n_i = 1 \) for all \( 1 \leq i \leq r \), the curves \( C_i \) are smooth, and \((A, \Theta)\) is isomorphic to the product of the canonically polarized Jacobians of the \( C_i \)'s:
\[ (A, \Theta) \simeq (J(C_1), \Theta_1) \times \cdots \times (J(C_r), \Theta_r) \]
In particular, if \( C \) is an irreducible curve generating \( A \) with \([C] \cdot \Theta = g\), then \( C \) is smooth and \((A, \Theta)\) is the Jacobian of \( C \).

Thus, the criterion that characterizes Jacobians by the existence of Picard bundles is the following

**Theorem 4.3.** Let \((A, \Theta)\) be an indecomposable p.p.a.v. of dimension \( g \). Suppose that there exists a sheaf \( \mathcal{F} \) on \( A \) such that satisfies the following conditions:

1. \( \mathcal{F} \) is WIT\(_g\)-\( \hat{\Phi} \).
2. \( c_i(\mathcal{F}) = (-1)^i \frac{\theta^i}{i!} \).

Then there exists a smooth curve \( C \) in \( A \) such that \((A, \Theta) \cong (\text{J}(C), \Theta)\). Moreover, if the sheaf \( \mathcal{F} \) is indecomposable, then it is a simple degree \( \text{rk}(\mathcal{F}) + g - 1 \) Picard bundle with \( \text{rk}(\mathcal{F}) \geq g \).

**Proof.** Consider \( \hat{\mathcal{F}} \) the Fourier-Mukai transform of \( \mathcal{F} \). Denote by \( Z = \text{supp}(\hat{\mathcal{F}}) \) the support of \( \hat{\mathcal{F}} \) and \( i: Z \hookrightarrow A \) the natural inclusion. Then, \( \hat{\mathcal{F}} \cong i_* \mathcal{G} \) for some sheaf \( \mathcal{G} \) on \( Z \).

Using Lemma 3.4 and Mukai formula (1), we compute Chern characters of \( \hat{\mathcal{F}} \) getting that

\[
\text{ch}_j(\hat{\mathcal{F}}) = \begin{cases} 
0 & j < g - 1 \\
\frac{\theta^{g-1}}{(g-1)!} & j = g - 1 \\
\text{rk}(\mathcal{F}) & j = g 
\end{cases}
\] (10)

This proves that \( Z \) is a subscheme of codimension \( g - 1 \). Define now the 1-cycle

\[
\text{Z}_1(\hat{\mathcal{F}}) = \sum_{\text{dim } V = 1} l_V(\hat{\mathcal{F}})[V]
\]

where the sum is over all 1-dimensional subvarieties in \( Z \) and \( l_V(\hat{\mathcal{F}}) \) is the length of the stalk of \( \hat{\mathcal{F}} \). As a consequence of Grothendieck-Riemman-Roch theorem (c.f. [4, Theorem 18.3, Example 18.3.11]), it is known that

\[
\text{ch}(\hat{\mathcal{F}}) = \text{Z}_1(\hat{\mathcal{F}}) + \text{higher degree terms}.
\]

Hence the effective 1-cycle \( \text{Z}_1(\hat{\mathcal{F}}) \) on \( A \) satisfies that \( \text{Z}_1(\hat{\mathcal{F}}) = \frac{\theta^{g-1}}{(g-1)!} \).

This implies that this cycle generates \( A \), by Corollary II.2 and Corollary II.3 in [12]. Finally, since \((A, \Theta)\) is indecomposable, then \([Z] = \frac{\theta^{g-1}}{(g-1)!}\) is irreducible via the Poincaré duality. The Matsusaka-Ran criterion...
allows us to conclude that the abelian variety \((A, \Theta)\) is the Jacobian of a smooth curve which proves the first part of the theorem.

Assume now that the sheaf \(\mathcal{F}\) is indecomposable. According to the previous discussion the support \(Z = C \sqcup W\) where \(C\) is the smooth curve and \(W\) is a 0-dimensional closed subscheme. Since \(\hat{\mathcal{F}} \simeq i_*\mathcal{G}\) is also indecomposable, \(Z = C\) the inclusion \(i\) is \(\pm a: C \hookrightarrow J(C)\) where \(a\) is the Abel morphism, and \(\mathcal{G}\) is a torsion free sheaf on \(C\). By applying Grothendieck-Riemann-Roch theorem to \(\hat{\mathcal{F}} \simeq i_*\mathcal{G}\), we get

\[
i_*(\text{rk}(\mathcal{G})) = \frac{g-1}{(g-1)!} \quad \text{and} \quad i_*(c_1(\mathcal{G}) - \frac{1}{2}\text{rk}(\mathcal{G}) K_C) = \text{rk}(\mathcal{F})
\]

\(K_C\) being a canonical divisor of \(C\). Thus, \(i = a\) and \(\mathcal{G}\) is a line bundle on \(C\) of degree \(\text{rk}(\mathcal{F}) + g - 1\). From Proposition 2.4 \(a_*\mathcal{G}\) is \(\text{IT}_0-\Phi\) and then \(\mathcal{F}\) is simple and \(\text{rk}(\mathcal{F}) \geq g\) by Proposition 3.3. \(\square\)

Remark 4.4. The same proof shows that when \((A, \Theta)\) is decomposable, then it is isomorphic to the direct product of the Jacobians of the irreducible components of \(Z_1(\hat{\mathcal{F}})\).

References

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